This chain rule for paths includes the special case

$$\vec{r}(t) = \vec{x} + t\cdot\vec{v} \qquad (\text{maybe with } \|\vec{v}\| = 1)$$

Then we retrieve the directional derivative

b/c $\frac{d}{dt} f(\vec{x}+t\vec{v}) = \nabla f(\vec{x}+t\vec{v})\cdot\vec{v}$

at $t=0$: $D_{\vec{v}} f(\vec{x}) = \nabla f(\vec{x})\cdot\vec{v}$

But now we see, thanks to this chain rule, that (eg) when sampling a temperature field by driving through it in order to "measure" the directional derivative, all that is relevant is that $\vec{v}$ at the point of interest is the desired direction vector, but there is no need for the entire path to be a straight line.

The multi-variable chain rule

A fct f of several var's (say x, y, z), but each of these depends on other variables (s,t)

$$g(s,t) = f(x(s,t), y(s,t), z(s,t))$$

eg f temperature in space, but we only care for a sphere, and s,t are coords on the sphere.

mathematicians' convention: these are different fcts, so use different symbols for them!

physicists' convention: if this "f" aka "g" is temperature call both of them "T"

$$T(s,t) = T(x(s,t), y(s,t), z(s,t))$$

$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}(x(s,t), y(s,t), z(s,t)) \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}(\ldots)\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}(\ldots)\frac{\partial z}{\partial s}$$

$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}(\ldots)\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}(\ldots)\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}(\ldots)\frac{\partial z}{\partial t}$$

(same as chain rules for paths; merely either s or t is frozen and then we calc the partial as a SV derivative)

Example:

$$f(x,y,z) = xy + z$$

$$x(s,t) = s^2$$

$$y(s,t) = st$$

$$z(s,t) = t^2$$

$$g(s,t) = f(x(s,t), y(s,t), z(s,t))$$

$$\frac{\partial g}{\partial s} = \underbrace{f_x}_{y}\ \underbrace{\frac{\partial x}{\partial s}}_{2s} + \underbrace{f_y}_{x}\ \underbrace{\frac{\partial y}{\partial s}}_{t} + \underbrace{f_z}_{1}\ \underbrace{\frac{\partial z}{\partial s}}_{0}$$

$$= 2sy + tx$$ (so far that's a half-ass job...) y should be $y(s,t)$; $x \to x(s,t)$

$$= 2s \cdot st + t \cdot s^2$$

$$= 3s^2 t$$ now it's done.

$$\frac{\partial g}{\partial t} = f_x \frac{\partial x}{\partial t} + f_y \frac{\partial y}{\partial t} + f_z \frac{\partial z}{\partial t}$$

$$= y(s,t) \cdot 0 + x(s,t) \cdot s + \cancel{f_z}\, 1 \cdot 2t$$

$$= 0 + s^2 \cdot s + 2t$$

$$= s^3 + 2t$$

Compare: you could have dodged the MV chain rule

b/c all fcts have explicit formulas

$$g(s,t) = x(s,t)\, y(s,t) + z(s,t)$$
$$= s^2 \cdot st + t^2$$
$$= s^3 t + t^2$$

$$\frac{\partial g}{\partial s} = 3s^2 t$$

$$\frac{\partial g}{\partial t} = s^3 + 2t$$

But in the absence of explicit formulas you may really need it!

Implicit differentiation

$$F(x,y,z) = x^3 + 2x + y^3 - y + z^3 + z^2 - 9$$

the point $(1,2,-1)$ happens to be on the level set

$F = 0$

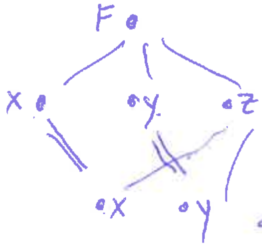
We hope/expect/pretend that this level set

can be described as a function $z = z(x,y)$

$z(1,2) = -1$

Would like $\frac{\partial z}{\partial x}(1,2)$, $\frac{\partial z}{\partial y}(1,2)$

$$G(x,y) = F\,(x, y, z(x,y)) = 0$$



If you hate using the same variable @ two levels, make it

$x = s$
$y = t$
$z = z(s,t)$

then call these s,t

$$\frac{\partial G}{\partial x} = F_x \underbrace{\frac{\partial x}{\partial x}}_{1} + F_y \underbrace{\frac{\partial y}{\partial x}}_{0} + F_z \frac{\partial z}{\partial x} = 0 \qquad \text{b/c } G \equiv 0$$

want to solve for this one!

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\partial F/\partial x}{\partial F/\partial z}$$

[(This bit finished up after class:)

So $\dfrac{\partial z}{\partial x} = -\dfrac{3x^2+2}{3z^2+2z}$

where z is still the poorly known implicit fct z(x,y) – can't help this

But $\dfrac{\partial z}{\partial x}(1,2) = -\dfrac{3\cdot 1^2+2}{3(-1)^2+2(-1)} = -\dfrac{5}{1} = \underline{\underline{-5}}$

Similarly $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{3y^2-1}{3z^2+2z}$

$\dfrac{\partial z}{\partial y}(1,2) = -\dfrac{3\cdot 2^2-1}{3(-1)^2+2(-1)} = \underline{\underline{-11}}$

Thus, even without a formula for z, we have found the partial derivative