There are a few lessons to learn on the side of this example:

(1) If you were to grab the formula $\frac{\partial z}{\partial x} = \frac{-\partial F/\partial x}{\partial F/\partial z}$ and started symbolically cancelling curly-∂-expressions, you'd end up with the false statement $\frac{\partial z}{\partial x} = -\frac{\partial z}{\partial x}$

In reality, there is NO such rule that would allow such a symbolic cancellation; and part of the wisdom of using curly-∂'s is that they serve as a reminder of this distinction.

(2) More important is that there is a context to the formula $\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}$, namely "$z$ is an implicit function of x (and maybe further unnamed variables) and is determined from the equation $F(x,\dots,z)=0$."

Without this context, the formula $\partial z/\partial x = -\frac{\partial F/\partial x}{\partial F/\partial z}$ is merely intelligent-sounding gibberish.

(3) A notation like $\frac{\partial F}{\partial x}$, or $\frac{\partial z}{\partial x}$, tells which quantities vary (rate of change of F as x varies, or r.o.c. of z as x varies) but it does not tell which quantities are held fixed. [The answer "all others except x" may well be ambiguous!] So if you manipulate symbols without being clear what you are doing, you can run into paradoxes or confusion caused by ambiguity.

Above, I wrote $G(x,y)$ with a new name G for the composite function $(x,y) \mapsto F(x,y,z(x,y))$, and I used x,y as variables in two roles: (x,y) are two of the three variables in F, and they are also the only two variables in G. This way I could distinguish

$\frac{\partial G}{\partial x}$ (x varies, y is fixed) from

$\frac{\partial F}{\partial x}$ (x varies, y and z are fixed)

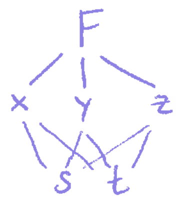
Now if I had followed the physicists' convention and <u>not</u> renamed the function, like $F(x,y) := F(x,y,z(x,y))$

then I'd be in trouble b/c which of the two quantities is $\frac{\partial F}{\partial x}$ to denote?

Is it (on the left) $\frac{\partial F}{\partial x}$ with y fixed (and no z anywhere in the picture)

or is it (on the right) $\frac{\partial F}{\partial x}$ with y and z fixed?

Now I <u>could</u> have re-used the F to humor the physicists if I had renamed the independent variables:



$$x = s \qquad y = t \qquad z = z(s,t)$$

$$F(s,t) := F(x(s,t),\, y(s,t),\, z(s,t))$$

then on the left, we'd get $\frac{\partial F}{\partial s}$, $\frac{\partial F}{\partial t}$; on the right we'd get $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$

along with inner derivatives $\frac{\partial x}{\partial s}$, $\frac{\partial x}{\partial t}$, $\frac{\partial y}{\partial s}$ etc.

Different "cultures" (ie different STEM areas) have devised different notations, but when they get mixed in calculus, watch out for ambiguities

MV composite functions are the usual situation where you need to be on the alert.

Here is a notation encountered in Thermodynamics:

A gas contained in a chamber (whose volume can be compressed with a piston) can be described by three quantities:

pressure p, temperature T, volume V.

But only two are independent, the third may follow (eg if you keep V fixed and raise T, then p will rise as well).

Some quantity (let's call it S) can be expressed in terms of any two of the three variables, and they will always call it S according to the physicist's convention:

Now $\frac{\partial S}{\partial p}$ would be ambiguous: is it the partial deriv of S (as a fct of p, T) wrt. p (hence T is fixed and V follows changes in p)

or is it the partial deriv' of S as a function of p, V (hence V is fixed and T follows changes of p).

This is why physicists invented the notation

$$\left(\frac{\partial S}{\partial p}\right)_T$$

partial wrt p while T is the fixed 2^{d} variable

vs $\left(\frac{\partial S}{\partial p}\right)_V$

partial wrt p while V is the fixed 2^{d} variable.

Minimax problems in MV.

Def: A function f defined on a domain D is said to have a <u>local minimum</u> at (a,b) if $f(x,y) \geq f(a,b)$ for all (x,y) in a neighborhood of (a,b)

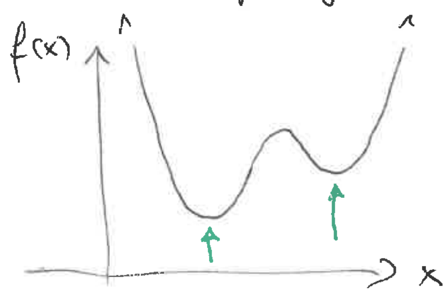
the set of those (x,y) or (x,y,z) that are admissible inputs into f

[or analogously for more than 2 variables]

some small disk around (a,b) as much as is in D

f is said to have a <u>local maximum</u> at (a,b) if $f(x,y) \leq f(a,b)$ for all (x,y) in a neighborhood of (a,b),

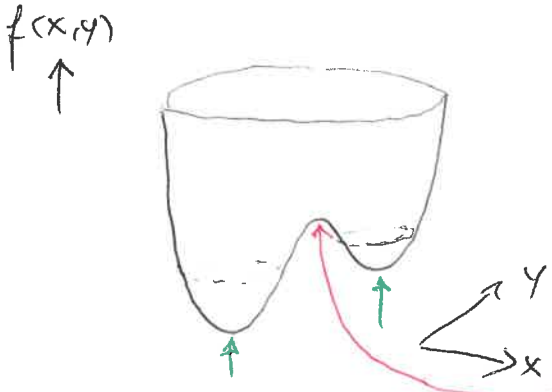
Def: A fct f on a domain D is said to have a global minimum at (a,b) if
$$f(x,y) \geq f(a,b) \text{ for all } (x,y) \text{ in } D$$
Analogous for global max.



this fct has two local min's (one of them global) and one local max (no global max)

~~to~~ global min implies in particular local min (but not vice versa)

MV example:



two local mins

no local max

this is what we call a saddle point.

Derivative tests always apply to local min/max

(But since global min/max are in particular also local min/max, derivative tests still apply, but they'll only talk about the local part of the info).

We call (a,b) in the interior of the domain D of f

a <u>critical point</u> if $\frac{\partial f}{\partial x}(a,b)$ is 0 or DNE

and $\frac{\partial f}{\partial y}(a,b)$ is 0 or DNE

(*) <u>Thm:</u> If (a,b) is a loc min or a loc max of f and is in the interior of the domain, then (a,b) is a critical point.

We say (a,b) is in the interior of D if an entire neighborhood of (a,b) still is ~~inside~~ contained in D

<u>How come?</u> Just refer to SVC applying the reasoning to the sliced fcts $x \mapsto f(x,b)$, $y \mapsto f(a,y)$

2d derivative tests can distinguish loc min & loc max & saddle points (details follow)

<u>Warning:</u> Thm (*) starts with <u>If</u> (a,b) is a loc min or loc max

It does not guarantee that there is one even if $\left\{ \begin{array}{l} \frac{\partial f}{\partial x}=0, \\ \frac{\partial f}{\partial y}=0 \end{array} \right\}$ has a ~~[illegible]~~ solution