Second Derivative Tests

Recall in SVC:

Assume f is twice differentiable and $f'(a) = 0$ and a in the interior of the domain. Then:

- If $f''(a) > 0$ then f has a local minimum at a
- If $f''(a) < 0$ —"— maximum at a
- If $f''(a) = 0$ the test is inconclusive: a could be min or max or neither

Conversely: If f has a local min at a, then $f''(a) \geq 0$

If f has a local max at a, then $f''(a) \leq 0$

A similar result is available in MVC, but in the presence of several 2nd order derivatives (f_{xx}, f_{yy}, f_{xy}, ...) ~~the~~ it's not obvious to guess how the rule may look like

Here is the mysterious 2nd derivative test for functions of (only) two variables:

Assume f is twice differentiable with all 2nd partial deriv's continuous in a neighborhood of (a,b) and $\nabla f(a,b) = \vec{0}$ with (a,b) is in the interior of the domain.

Concoct the quantity $D := f_{xx}(a,b)\, f_{yy}(a,b) - f_{xy}(a,b)^2$

Then:
- If $D > 0$ and $f_{xx}(a,b) > 0$ then f has a loc min at (a,b)
- If $D > 0$ and $f_{xx}(a,b) < 0$ —"— max at (a,b)
- If $D < 0$ then f has a saddle pt at (a,b)
- If $D = 0$ the test is inconclusive (any of the above could happen)

might just as well have written $f_{yy}(a,b)$ here

Conversely, if ~~(a,b)~~ f has a loc. min at (a,b) then

$D \geq 0$ and $f_{xx}(a,b) \geq 0$ and $f_{yy}(a,b) \geq 0$;

if f has a loc max at (a,b) then

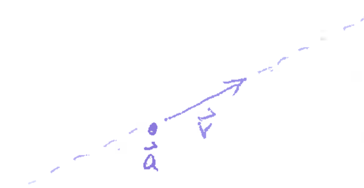
$D \geq 0$ and $f_{xx}(a,b) \leq 0$, $f_{yy}(a,b) \leq 0$

Here is a more intelligible way of understanding the 2^d deriv test (and it naturally generalizes to any # of variables:

Assume f is twice differentiable with continuous 2^d partial derivs in a nbhd of $\vec{a}$ and $\nabla f(\vec{a}) = \vec{0}$ (and $\vec{a}$ in the interior of the domain of f).

For any vector $\vec{v}$ define the 2^d deriv in direction $\vec{v}$ at $\vec{a}$:

$$D^2_{\vec{v}} f(\vec{a}) = \frac{d^2}{dt^2} f(\vec{a} + t\vec{v}) \Big|_{t=0}$$



in other words, let $g(t) = f(\vec{a} + t\vec{v})$ and calc $g''(0)$

Then :

- If $D^2_{\vec{v}} f(\vec{a}) > 0$ for all $\vec{v} \neq \vec{0}$, then f has a loc. min at $\vec{a}$
- If $D^2_{\vec{v}} f(\vec{a}) < 0$ for all $\vec{v} \neq \vec{0}$, then f has a loc max at $\vec{a}$
- If $D^2_{\vec{v}} f(\vec{a})$ is > 0 for some $\vec{v}$ and < 0 for some other $\vec{v}$, then f has a saddle pt at $\vec{a}$.
- In all other cases (eg $D^2_{\vec{v}} f(\vec{a}) \geq 0$ for all $\vec{v} \neq \vec{0}$ and $= 0$ for some $\vec{v} \neq \vec{0}$) the test is inconclusive (any of the above could happen)

Converse: If f has a loc. min at $\vec{a}$, the $D^2_{\vec{v}} f(\vec{a}) \geq 0$ for all $\vec{v}$

" max " ≤ 0 "

How does this 2^d directional derivative test lead to the "mystery formula" given first? (In two variables)

$$g(t) = f(\vec{a}+\vec{v}t) \qquad \vec{v} = \begin{bmatrix} h \\ k \end{bmatrix} \qquad \vec{a} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= f(a+ht,\ b+kt)$$

$$g'(t) = f_x(\dots)\cdot h + f_y(\dots)k$$

$$g''(t) = (f_{xx}(\dots)h + f_{xy}(\dots)k)h + (f_{yx}(\dots)h + f_{yy}(\dots)k)k$$

$$g''(0) = f_{xx}(a,b)h^2 + 2f_{xy}(a,b)hk + f_{yy}(a,b)k^2$$

$$= \underbrace{\Big(f_{xx}(a,b) + 2f_{xy}(a,b)\underbrace{\tfrac{k}{h}}_{s} + f_{yy}(a,b)\underbrace{\left(\tfrac{k}{h}\right)^2}_{s^2}\Big)}_{\text{for this to be } >0 \text{ for all } s,}h^2 \qquad (*)$$

for this to be >0 for all s,
the quadratic formula should not give real zeros
in other words the stuff under $\sqrt{\ }$ should be <0

Write this out: it's $-4D$ under $\sqrt{\ }$

[added after class:] Thus, if $D>0$, the $\sqrt{\ }$ is imaginary, thus there are no zeros of the quadratic: it ~~change~~ does not change the sign, thus, if $f_{xx}>0$, the (quadratic) will be >0 everywhere and thus $D^2_{\vec{v}}f>0$ for all $\vec{v}\neq\vec{0}$, which implies that f has a loc min at $\vec{a}$. Similarly can argue with max.

If $D<0$, then the quadratic $(*)$ has real zeros and changes sign. In some directions the 2^d deriv looks as if $\vec{a}$ were a loc max; in others as if it were a loc min so $\vec{a}$ is a saddle point