If 2^d derivative test is too tedious, can we dodge it?

Popular WRONG idea: calc' all crit. points, evaluate the values at all of them, the one with the largest value is a max, the one with the smallest value is a min

This reasoning is flawed.

Flaw in reasoning: Don't know if max / min even exists!

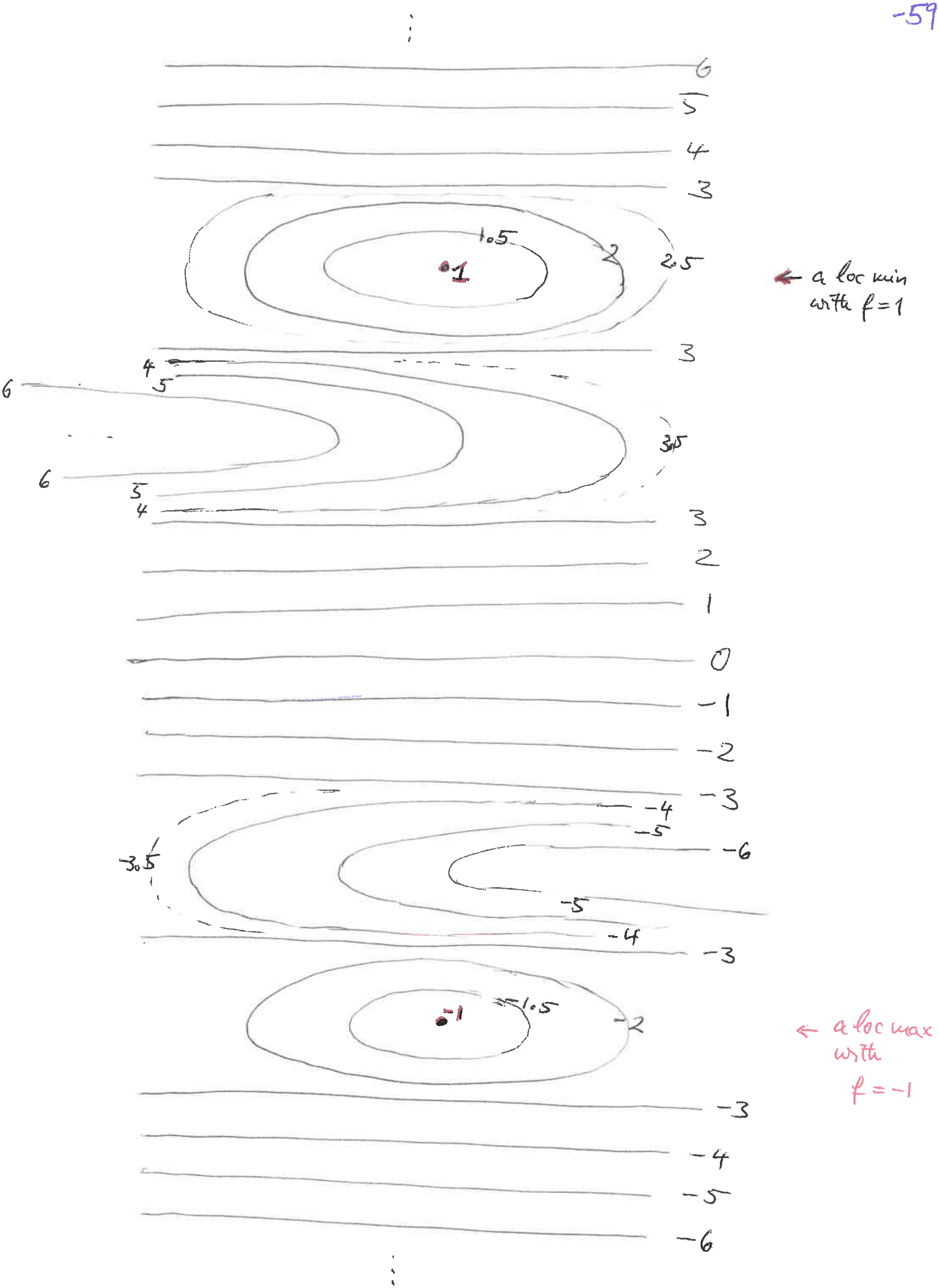
It is even possible that the only critical point is a loc. min but is not a global min. (In 2 or more variables)

eg $f(x,y) = y^2 + 3(y+e^x-1)^2 + 2(y+e^x-1)^3$

(there is no global min. try $x=0$, $y=-$"huge")

Can even construct an example (more transparently by contour plot than formula) where there are only two crit. points, a loc min and a loc max but the value @ loc min is larger than value at loc. max.

→ see figure next page


6
5
4
3
1.5
2
2.5
1
a loc min with f=1
3
4
5
6
3.5
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-3.5
-5
-4
-3
-1.5
-1
-2
a loc max with f=-1
-3
-4
-5
-6

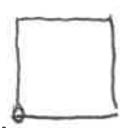
Thm: A continuous function on a domain that is both **bounded** and **closed** has a global min and a global max.

bounded → fits in some huge ball or huge box

closed → all ~~bd~~ boundary points are included

Ex: "open" square $\{(x,y): 0 < x < 1,\ 0 < y < 1\}$ is <u>not</u> closed

also $\{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1,\ \text{but } (x,y) \neq (0,0)\}$ is <u>not</u> closed (it is not open either) [sets <u>may</u> be <u>neither</u> open <u>nor</u> closed]



disallowed

 $\{(x,y): 0 \le x \le 1,\ 0 \le y \le 1\}$ <u>is</u> closed

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Once we have the assurance that glob min & glob max do exist, we can try to find them by 1st derivative test. However, in doing so: crit point (1st derivative) test only deals with interior points. Boundary pts have to be studied separately! (→ Hwk #35 in Ch 14.7)

Constrained minimax problems

Given a fct f and a fct g and we ask

Among all those $\vec{x}$ on level surface / level line $g(\vec{x})=0$ find locations $\vec{x}_*$ where f has a "loc" min or "loc" max (subject to the constraint), ie. $f(\vec{x}_*)$ is smallest (resp largest) among all those competitors $\vec{x}$ in some neighborhood that also lie on the surface $g(\vec{x})=0$. An example would be: f temperature function; $g(x,y,z) = x^2+y^2+z^2-6000^2$ (we ask for the hottest & coldest temp. on the surface of the earth, not in the universe).

One idea to solve such a pblm: try to solve $g(x,y,z)=0$ for z : $z=z(x,y)$
Then $f(x,y,z(x,y))$ is a two-var. fct & you try to minimize it by setting $\frac{\partial}{\partial x}(-''-)=0$, $\frac{\partial}{\partial y}(-''-)=0$. But eliminating to solve for z may be impractical. $\longrightarrow$ so here comes another method:

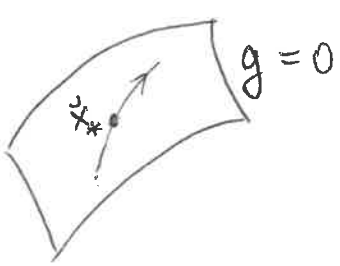
Method of Lagrange Multipliers

If $\vec{x}_*$ is a loc constrained min (or max) of f, subject to the constraint $g(\vec{x})=0$, then either

$$\vec{\nabla} f(\vec{x}_*) = \lambda \vec{\nabla} g(\vec{x}_*) \qquad \text{for some number } \lambda$$

or $\vec{\nabla} g(\vec{x}_*) = 0$

How come?



Supp. $\vec{r}(t)$ is a curve in $\underbrace{\text{level set } g=0}_{\text{with } \vec{r}(0)=\vec{x}_*}$

Consider $f(\vec{r}(t))$

$$\frac{d}{dt} f(\vec{r}(t)) = 0 = \nabla f(\vec{x}_*) \cdot \vec{r}\,'(0)$$

$\vec{\nabla} f(\vec{x}_*)$ is orthogonal to all such $\vec{r}'(0)$

but $\nabla g(\vec{x}_*)$ is also orthogonal to all such $\vec{r}'(0)$

Since they are both $\perp$ to the same plane they must be parallel.

(To be cont'd next class)