$\vec{\nabla} f(\vec{x}_*)$ is orthogonal to all such $\vec{r}\,'(0)$

but $\nabla g(\vec{x}_*)$ is also orthogonal to all such $\vec{r}\,'(0)$

Since they are both $\perp$ to the same plane they must be parallel.

(~~To be continued next~~ class)

In using the LM method, we have one extra unknown λ, called the Lagrange multiplier.

eg if variables are x, y, z we have four unknowns: x, y, z, λ

and we also have four equations:

$\nabla f = \lambda \nabla g$ gives 3 eqns (one each component)

$g = 0$ is the 4th eqn.

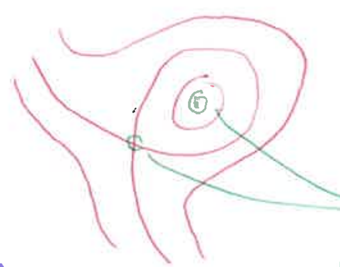
We'll have to eliminate among these eqns to solve for x, y, z, λ

As an aside: A theorem from advanced calculus, called the Implicit Function theorem, guarantees for g with continuous derivatives, that the level set $\{\vec{x} : g(\vec{x}) = 0\}$ looks like a ~~nice~~

nice — curve in case of 2 var's / surface — " — 3 var's — near a point $\vec{x}$, whenever

$\nabla g(\vec{x}) \neq 0$

Eg:



if these are level curves

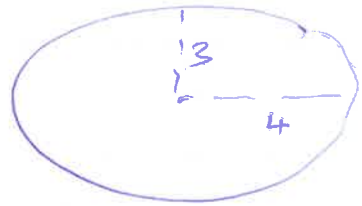
at these pts the gradient will vanish

(Level set might still be nice curve even if $\nabla g = 0$ but then it's not guaranteed)

Example:

Find min and max of $f(x,y) = 2x+5y$

on the ellipse $g(x,y) = (\frac{x}{4})^2 + (\frac{y}{3})^2 - 1 = 0$



LM says: If (x,y) is such a min or such a max

then there is some number λ st

$\nabla f(x,y) = \lambda \nabla g(x,y)$

$$\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \lambda \begin{bmatrix} 2x/16 \\ 2y/9 \end{bmatrix}$$

$$2 = 2\lambda x/16 \qquad (1)$$

$$5 = 2\lambda y/9 \qquad (2)$$

$$\frac{x^2}{16} + \frac{y^2}{9} = 1 \qquad (3)$$

From (1) get $x = \frac{16}{\lambda}$

(2) $y = \frac{45}{2\lambda}$

may divide by λ b/c (1), (2) clearly mandate that λ cannot be 0.

$\downarrow$ plug into (3)

$$\frac{16}{\lambda^2} + \frac{45^2}{9 \cdot 2^2 \lambda^2} = 1$$

$$\underbrace{16 + \frac{15^2}{4}}_{\frac{64+225}{4} = \frac{289}{4}} = \lambda^2$$

So $\lambda = \pm \frac{17}{2}$

$x = \pm \frac{32}{17}$ $\qquad$ $y = \pm \frac{45}{17}$

either choose + all the time or − all the time

$$f\left(\pm \frac{32}{17}, \pm \frac{45}{17}\right) = \pm 17$$

How do we know which is max, which is min (if any)

f is continuous and the set $\{g=0\}$ is closed & bounded

A simple way of guaranteeing that a set is closed is:

If the set is described by equations and possibly non-strict inequalities ($\leq, \geq$; NOT $<, >$) with all expressions entering being continuous, then the set is closed.

So we know there is a global min and global max.

Then we can reason that 17 is the max
-17 is the min

HW will be posted <u>with</u> hints (due Thu after break)

<u>Note (added after class)</u>: We do not have a 2^{nd} derivative test for constrained min/max problems.

If you ever feel the need to have such a test, look up "bordered Hessian" in the textbook "Vector Calculus" by Marsden & Tromba.

Don't look it up right now, b/c you wouldn't have the prerequ's to understand it. Once you master matrix algebra and view the 2^{d} derivative test as on page 56 ($D^2_{\vec{v}} f(\vec{a}) \gtrless 0$ for all $\vec{v}$ style) you may understand that test.

Needless to say, you won't be asked to do a 2^{nd} deriv test on Lagrange multiplier problems in this class