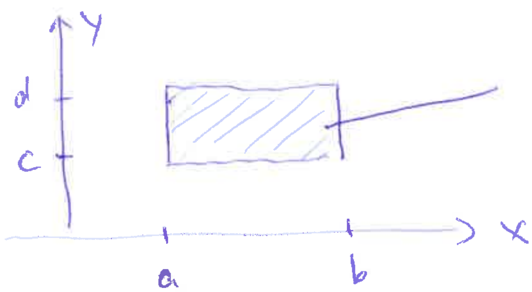
Integrals in $\mathbb{R}^2$ (and $\mathbb{R}^3$ etc)

Definite integrals over domains, in $\mathbb{R}^2$
↳ "nice sets"

in $\mathbb{R}$ (SVC) we'd have intervals here

SVC: $\displaystyle\int_{[a,b]} f(x)\,dx = \int_a^b f(x)\,dx$ (if $a<b$)

The nicest sets in $\mathbb{R}^2$ will be rectangles $[a,b]\times[c,d]$

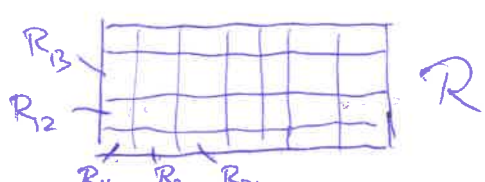


call this rectangle $\mathcal{R}$

Consider a 2-var. fct $f(x,y)$ def'd (at least) on $\mathcal{R}$

$\displaystyle\iint_{\mathcal{R}} f(x,y)\,dA$:
- what does this mean / how is it defined?
- how do we calc' it practically?

Definition of this integral as a MV Riemann integral is analogous to the case in SVC:



divide up $\mathcal{R}$ into smaller rectangles R_{ij} (get a partition)

write a doublesum $\displaystyle\sum_{i,j}\sum$ ~~f(x,y)~~ $f((x,y)_{ij})\,\text{area}(R_{ij})$

called a Riemann sum. The point $(x,y)_{ij}$ is taken anywhere in R_{ij}

The "norm" or "mesh size" of this partition is the largest of the $\Delta x_i, \Delta y_j$ making up the partition (the longest side of any of the rectangles in the partition)

Notation: $\|P\|$ is the meshsize (aka "norm") of the partition P.

If the $\lim\limits_{\|P\|\to 0} \sum\limits_{i}\sum\limits_{j} f((x,y)_{ij})\,\text{area}(R_{ij})$ exists

(and is independent of which points $(x,y)_{ij}$ you sampled from R_{ij})

then we call this limit the Riemann integral

$$\iint\limits_R f(x,y)\,dA$$

Thm: If f is continuous on R then $\iint\limits_R f(x,y)\,dA$ exists

Notation varies: Some write $\int\limits_R$ instead of $\iint\limits_R$

others prefer $dA(x,y)$ instead of dA

Physicists may write $\iint\limits_R f(\vec{x})\,\underbrace{d^2\vec{x}}_{\text{instead of } dA}$

or $\iint\limits_R f(x,y)\,d(x,y)$

BTW: A more precise explanation of $\lim\limits_{\|P\|\to 0}$ is found in Ch. 15.1 of book

Key theorems: (1) Linearity $\iint\limits_R (f(x,y)+g(x,y))\,dA = \iint\limits_R f(x,y)\,dA + \iint\limits_R g(x,y)\,dA$

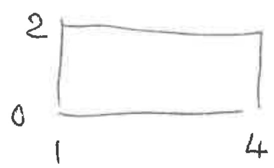
$\iint\limits_R c\,f(x,y)\,dA = c\iint f(x,y)\,dA$ (c a constant)

(2) Monotonicity: if $f(x,y) \le g(x,y)$ in R then $\iint\limits_R f\,dA \le \iint\limits_R g\,dA$

(3) Fubini's theorem: If f is continuous fct on R where $R = [a,b] \times [c,d]$

then $$\iint_R f(x,y)\,dA = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$
$$= \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

Example

$R =$ 

$f(x,y) = x^2 + x \cdot y + y^2$

$$\iint_R (x^2+xy+y^2)\,dA = \int_0^2 \left(\underbrace{\int_1^4 (x^2+y^2+xy)\,dx}_{} \right) dy$$
$$= \int_0^2 \left(\left[\frac{x^3}{3} + xy^2 + \frac{x^2}{2} \cdot y \right]_{x=1}^{4} \right) dy$$
$$= \int_0^2 \left(\frac{4^3-1^3}{3} + (4-1)y^2 + \frac{4^2-1^2}{2}\,y \right) dy$$
$$= \int_0^2 \left(21 + 3y^2 + \frac{15}{2}\,y \right) dy$$
$$= 42 + \left[y^3 \right]_0^2 + \left[\frac{15}{2}\,\frac{y^2}{2} \right]_0^2$$
$$= 42 + 8 + \frac{15}{2} \cdot \frac{2^2}{2}$$
$$= \underline{\underline{65}}$$

often omit parentheses:

$$\iint_R (x^2+xy+y^2)\,dA = \int_0^2 \int_1^4 (x^2+xy+y^2)\,dx\,dy$$

$$= \int_1^4 \int_0^2 (x^2+xy+y^2)\,dy\,dx$$