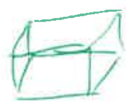
often omit parentheses:

$$\iint_R (x^2+xy+y^2)\,dA = \int_0^2 \int_1^4 (x^2+xy+y^2)\,dx\,dy$$

$$= \int_1^4 \int_0^2 (x^2+xy+y^2)\,dy\,dx$$

Note: Analogously, we can handle integrals over boxes in $\mathbb{R}^3$



$R$

$$\iiint_R f(x,y,z)\,dV$$

Riemann sums would be made up in terms of a partition into smaller boxes $B_i$:

$$\sum_{\substack{\text{all boxes}\\ B_i}} f(\cancel{x_i}\ \underbrace{(x,y,z)_i}_{\text{sample point in } B_i})\ \mathrm{vol}\,(B_i)$$

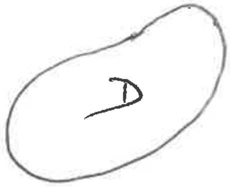
Fubini also applies

if $R = \{(x,y,z) : a \le x \le b,\ c \le y \le d,\ e \le z \le f\}$

$$\iiint_R f(x,y,z)\,dV = \int_e^f \left( \int_c^d \left( \int_a^b f(x,y,z)\,dx \right) dy \right) dz$$

or any other order of the variables

What about other domains than rectangles?




What do we mean by $\iint_D f(x,y)\,dA$ ?

A radically simple trick:

Define a function $\tilde{f}$ as follows: $\tilde{f}(x,y) = \begin{cases} f(x,y) & \text{if } (x,y)\in D \\ 0 & \text{otherwise} \end{cases}$

find a big rectangle $R$ containing $D$ (if $D$ is bounded; other cases won't be discussed)

Then $\iint_D f(x,y)\,dA := \iint_R \tilde{f}(x,y)\,dA$ (does not depend how big $R$ is as long as $R$ contains $D$)

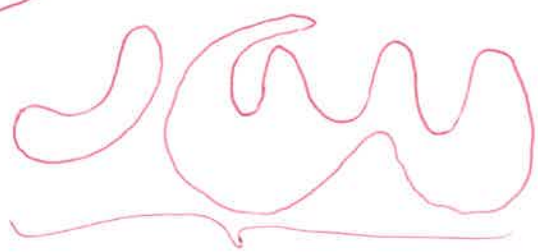
Usually $\tilde{f}$ is not continuous

But the following theorem is still true:

Thm: If $D$ is a closed and bounded domain whose boundary consists of one or several simple closed curves that are piecewise smooth → smooth except for finitely many corners.

no start & end point (rather ends where it starts) and has no self-intersections

a non-closed curve; here is another one



these are simple closed

← these three are closed curves

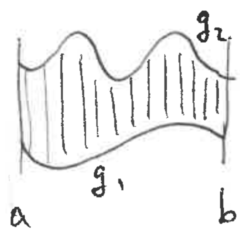
and if $f$ is continuous on $D$, then $\iint_D f(x,y)\,dA$ still exists.

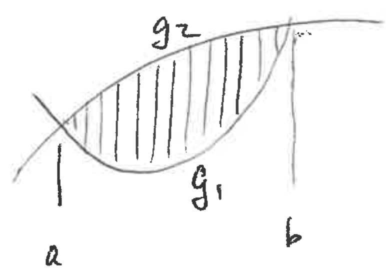
How to calculate such integral practically?

1st (easier) case assume that $D$ is (horizontally or vertically) simple:

$D$ is vertically simple iff it can be written as

$$D = \{(x,y) : a \le x \le b \ , \ g_1(x) \le y \le g_2(x)\}$$ with cont. fcts $g_1, g_2$ st $g_1 \le g_2$



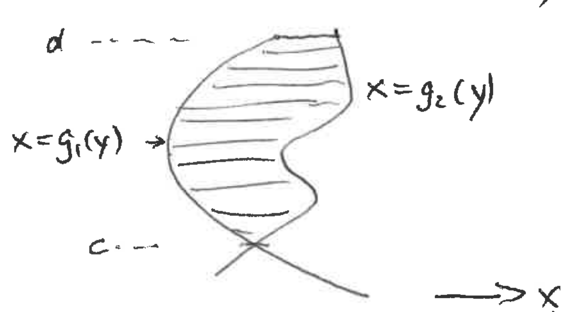




$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

---

$D$ is horizontally simple if it can be written as

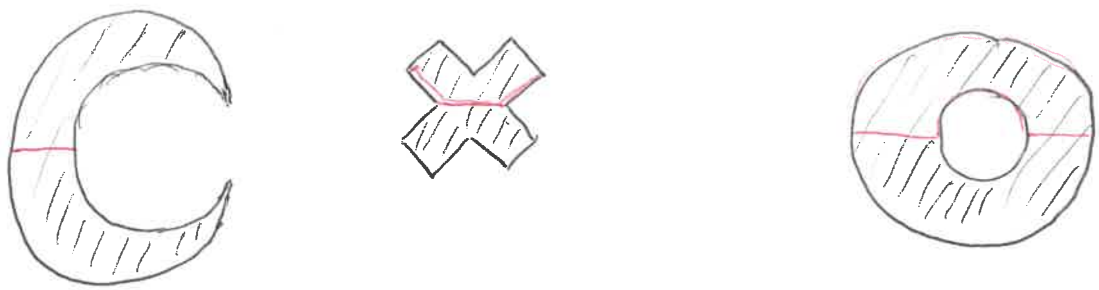
$$D = \{(x,y) : c \le y \le d \ , \ g_1(y) \le x \le g_2(y)\}$$ with cont fcts $g_1, g_2$ st. $g_1 \le g_2$




$$\iint_D f(x,y)\,dA = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy$$

$2^d$ (less easy) case:

What if $D$ is neither horizontally nor vertically simple?



In most cases (in all cases you care for), such domains can be decomposed into pieses that are horizontally and/or vertically simple

Then if $D = D_1 \cup D_2$ (with $D_1 \cap D_2$ only on boundary curves)

$$\iint_D f\,dA = \iint_{D_1} f\,dA + \iint_{D_2} f\,dA \qquad \text{(or more pieces)}$$

In some cases $D$ can be written as a difference of domains

like ring = big disc minus small disc

(makes sense only if $f$ is defined in big disk, not merely $D$)

then $\displaystyle\int_{\text{ring}} f\,dA = \int_{\text{big disk}} f\,dA - \int_{\text{small disk}} f\,dA$