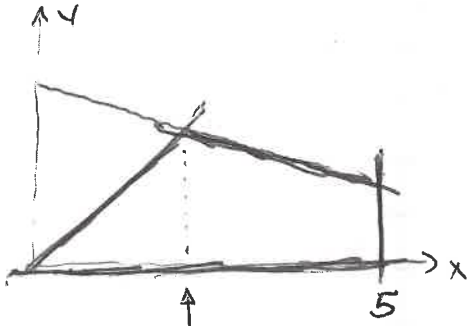
Example: Suppose D is bounded by $y=x$, $y=3-\frac{x}{2}$, $y=0$, $x=5$



$x=2$ calc'd where $y=x$, $y=3-\frac{x}{2}$ intersect

$$\iint_D f(x,y)\,dA = \int_0^5 \int_0^{\circledast} f(x,y)\,dy\,dx$$

$$\circledast = \begin{cases} x & \text{if } x \le 2 \\ 3-\frac{x}{2} & \text{if } x \ge 2 \end{cases}$$

$$\int_0^5 \begin{Bmatrix} \text{some expr} & \text{if } x \le 2 \\ \text{another} & \text{if } x \ge 2 \end{Bmatrix} dx = \int_0^2 \quad dx + \int_2^5 \quad dx$$

$$\iint_D f(x,y)\,dA = \int_0^2 \int_0^x f(x,y)\,dy\,dx + \int_2^5 \int_0^{3-\frac{x}{2}} f(x,y)\,dy\,dx$$

Change of Variables (15.4 & 6)

Recall subst. rule from SVC (Calc 2)

$$\int_{v=a}^{v=b} f(\underbrace{g(v)}_{=u})\,\underbrace{g'(v)}_{\frac{du}{dv}}\,dv = \int_{u=g(a)}^{u=g(b)} f(u)\,du$$

Features: get a derivative factor from the differentials $\left(g'(v) = \frac{du}{dv}\right)$

domain of integration changes : $[a,b]$ vs. $[g(a), g(b)]$

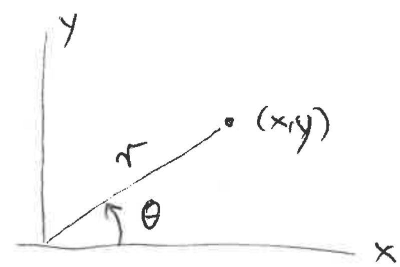
Sneak preview from 15.6:

$$\iint_D f(u_1,u_2)\,du_1\,du_2 = \underbrace{\iint}_{\substack{\text{(appropriate}\\ \text{limits for } D\\ \text{in } v \text{ coords)}}} f\big(g_1(v_1,v_2),\, g_2(v_1,v_2)\big)\cdot \left| \det \begin{bmatrix} \frac{\partial u_1}{\partial v_1} & \frac{\partial u_1}{\partial v_2} \\ \frac{\partial u_2}{\partial v_1} & \frac{\partial u_2}{\partial v_2} \end{bmatrix} \right| dv_1\,dv_2$$

where $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ (Formula will look mysterious now, but will be explained later)

Special cases that occur with paramount frequency are discussed first (Ch 15.4):

(1) <u>polar coordinates:</u>



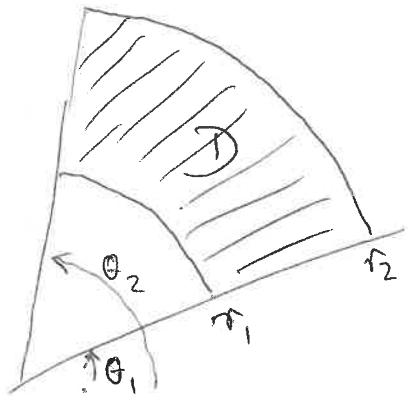
$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\det \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \det \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$= r\cos^2\theta - (-r\sin^2\theta)$$
$$= r(\cos^2\theta + \sin^2\theta)$$
$$= r$$

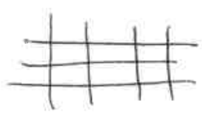
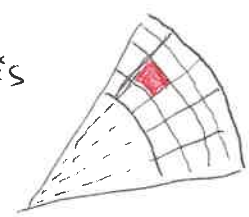
$$dA = dx\,dy = r\,dr\,d\theta$$

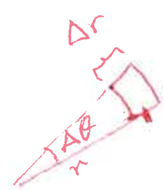
Now let's understand this formula (obtained from the yet-mysterious sneak preview formula) directly



a radially simple domain

$$\iint_D f(x,y)\,dA = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

If we were to tackle this ~~express~~ integral from scratch with Riemann sums it would make sense to divide D up not like this  , but rather like this 



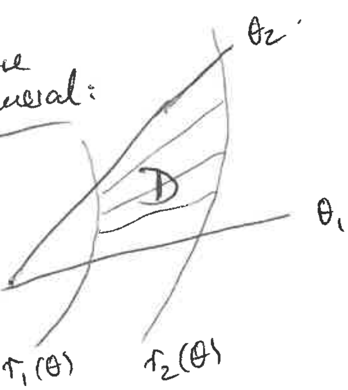
area of such a "polar rectangle"

$$\Delta A = \tfrac{1}{2}\left(r+\tfrac{\Delta r}{2}\right)^2\Delta\theta - \tfrac{1}{2}\left(r-\tfrac{\Delta r}{2}\right)^2\Delta\theta =$$

[recall area of sector  $= \frac{1}{2}R^2\,\Delta\theta$]

$\Rightarrow r\,\Delta r\,\Delta\theta$

more general:



$$\iint_D f(x,y)\,dA = \int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d$$