Example: integrate x^2+y^2 over the ball (B_R) of radius R

(Application: this occurs in mechanics, calc'ing the moment of inertia of a solid ball)

ball: $0 \le \rho \le R$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$

$$x^2+y^2 = r^2 = \rho^2 \sin^2\phi$$

$$\iiint_{B_R} (x^2+y^2)\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} (\rho^2 \sin^2\phi)\,\overbrace{\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta}^{dV}$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin^3\phi \left[\frac{\rho^5}{5}\right]_0^R d\phi\, d\theta$$

$$= \frac{R^5}{5}\int_0^{2\pi} \int_0^{\pi} \sin\phi\,(1-\cos^2\phi)\,d\phi \quad d\theta$$

$u = -\cos\phi$

$du = \sin\phi\, d\phi$

$$= \frac{R^5}{5}\int_0^{2\pi} \underbrace{\int_{-1}^{1} (1-u^2)\,du}_{\frac{4}{3}} \quad d\theta \qquad = \frac{R^5}{5}\cdot\frac{4}{3}\cdot 2\pi$$

$$= \frac{2}{5}R^2 \cdot \underbrace{\frac{4\pi}{3}R^3}_{\text{vol (ball)}}$$

Change of Variables Formula for 2dim integrals → (Ch 15.6)

let u, v be <u>some</u> coordinates, eg

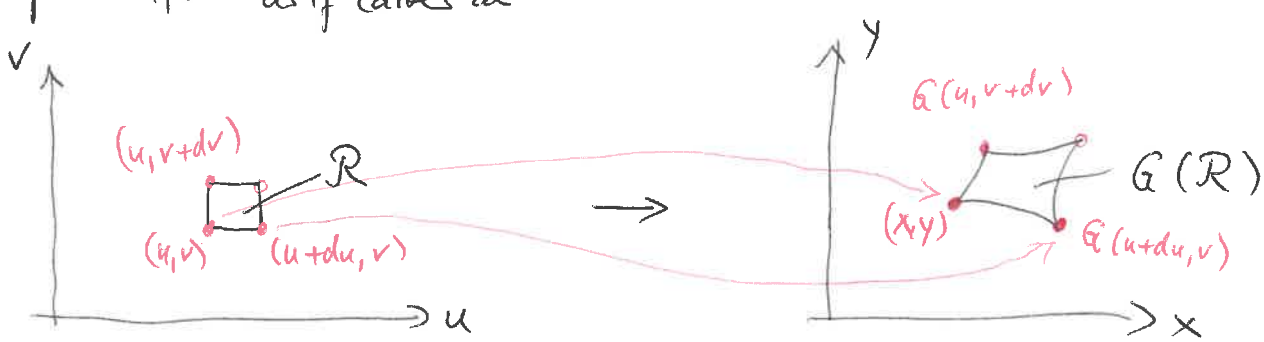
as convenient when studying bees in a honeycomb

and x, y cartesian coords (for which we know $dA = dx\,dy$)

<u>Question:</u> $dA =$ " ?? $du\,dv$ "

We are studying a map $(x, y) = G(u, v) = (x(u,v), y(u,v))$

plot u, v "as if cartesian"



(u,v) plane

Example if $(u,v) = (r, \phi)$

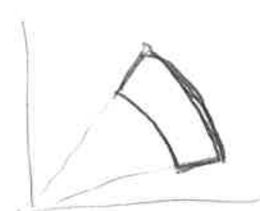


Figure above drew a "large" rectangle R to show that $G(R)$ may have curved sides; however for small rectangles, the deviation from straight sides may be imperceptible; in the limit of Riemann sums (where meshsize of partition → 0), the impact from the curvy-ness of the sides will go to 0.

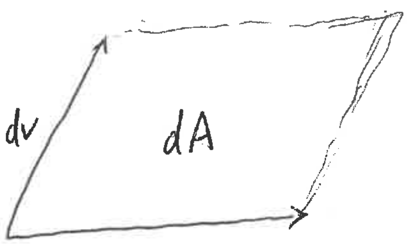
very small rectangle (but zoomed-in big)

$$G(u, v+dv) = \left(x(u,v) + \frac{\partial x}{\partial v}(u,v)\, dv,\ y(u,v) + \frac{\partial y}{\partial v}\cdot dv\right)$$

$(x,y) = G(u,v)$

$$G(u+du, v)$$
$$= \left(x(u+du,v),\ y(u+du,v)\right)$$
$$\approx \left(x(u,v) + \frac{\partial x}{\partial u}(u,v)\cdot du,\ y(u,v) + \frac{\partial y}{\partial u}(u,v)\cdot du\right)$$

$$\begin{bmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \end{bmatrix} \cdot dv$$



$$\begin{bmatrix} \frac{\partial x}{\partial u}\cdot du \\ \frac{\partial y}{\partial u}\cdot du \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \end{bmatrix} \cdot du$$

area of this parallelogram is $\left\| \begin{bmatrix} \partial x/\partial u \\ \partial y/\partial u \\ 0 \end{bmatrix} \times \begin{bmatrix} \partial x/\partial v \\ \partial y/\partial v \\ 0 \end{bmatrix} du\, dv \right\|$

$$= \underbrace{\left| \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right|}\, du\, dv$$

$$= \left| \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \right| du\, dv$$

call this thing Jacobi determinant

$$\frac{\partial(x,y)}{\partial(u,v)}$$