$$\int_{t(\alpha)}^{t(\beta)} f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt \overset{?}{=} \int_{\alpha}^{\beta} f(\vec{r}(t(\tau)))\,\|\underbrace{\dot{\vec{r}}(\tau)}\|\,d\tau$$

$$\frac{d\vec{r}}{d\tau} = \frac{d\vec{r}}{dt}\,\frac{dt}{d\tau}$$

$t(\tau)$ = new variable t

$$\int_{t(\alpha)}^{t(\beta)} f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\cdot\underbrace{\underbrace{\left|\frac{dt}{d\tau}\right|}_{=\frac{dt}{d\tau}\ \text{b/c}\ \frac{dt}{d\tau}>0}\,d\tau}_{=dt}$$

Applications: (1) $\int_C 1\,ds = \int_a^b \|\vec{r}\,'(t)\|\,dt$

= length of curve

(2) δ mass density along a curve

$\int_C \delta(\vec{r}(t))\,ds$ total mass of the wire shaped along the curve.

(3,4) center of mass, or moment of inertia of a wire w/ density δ

$x_{cm} = \int \delta\,ds / \int 1\,ds \qquad I_z = \int (x^2+y^2)\,\delta(\dots)\,ds$

(5) potential of a point charge ρ_{total} (located at $\vec{x}_0$) is at $\vec{x}$

$$\frac{\rho_{total}}{\|\vec{x}-\vec{x}_0\|}$$

$\rho(\vec{r})$ $\qquad$ $\rho_{total} = \int \rho(\vec{r})\,ds$

with a charge density along a wire

the potential will be $\displaystyle\int_C \frac{\rho(\vec{r}(t))}{\|\vec{x}-\vec{r}(t)\|}\,ds$

$$= \int_C \frac{\rho(\vec{r}(t))\ \|\vec{r}\,'(t)\|\,dt}{\|\vec{x}-\vec{r}(t)\|}$$

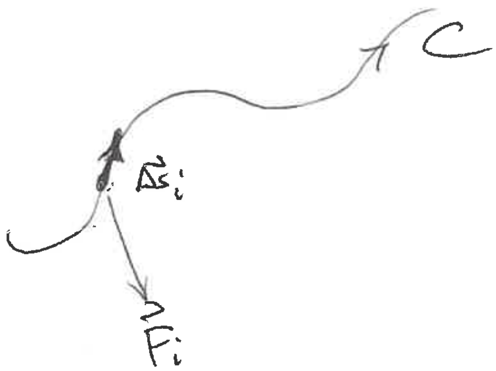
Vector Line Integrals

Application: Work against a force field while transporting an object affected by this force

"Baby-physics case" with constant force F and distance traveled Δs. If the two are in same direction: $W = F \cdot \Delta s$

If directions don't necessarily match $W = \vec{F} \bullet \vec{\Delta s}$

"True life physics" where all these quantities vary by location



$$\Delta W_i = \vec{F}_i \cdot \vec{\Delta s}_i$$

$$W = \sum \Delta W_i \qquad \text{Riemann sum}$$

lim of these Riemann sums is an integral

$C: \vec{r} = \vec{r}(t)$,
$a \le t \le b$

$$W = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

This is an ("the prime") example of a vector line integral,

$$\int_C \vec{F}(\vec{r}) \cdot d\vec{r} := \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

if C is param'd as above

Again, given the (geometric) curve C,

this vector line integral does NOT depend on which parametrization $\vec{r} = \vec{r}(t)$ you choose for C.

(as long as you maintain orientation)

~~So think of C~~

So, actually, C represents not a curve but an oriented curve

(i.e. we discern in which direction we traverse the curve)

Properties of vector line integral see boxed Thm 3 in Ch 16.2

unsurprising

Examples:

$$\vec{F}(x,y,z) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Motivated by the ~~question~~ claim from physics that work should only depend on start & end point not on path itself.

$$\vec{G}(x,y,z) = \begin{bmatrix} -z \\ 3x \\ y \end{bmatrix}$$

two curves: $C_1 : \vec{r}(t) = \begin{bmatrix} \cos t \\ \sin t \\ t \end{bmatrix} \qquad 0 \le t \le 2\pi$

$C_2 : \vec{r}(t) = \begin{bmatrix} 1 \\ 0 \\ t \end{bmatrix} \qquad 0 \le t \le 2\pi$

$$\int_{C_1} \vec{F} \cdot d\vec{r} \overset{?}{=} \int_{C_2} \vec{F} \cdot dr$$

$$\int_{C_1} \vec{G} \cdot d\vec{r} \overset{?}{=} \int_{C_2} \vec{G} \cdot d\vec{r}$$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left(\cos t \, (-\sin t) + \sin t \cdot \cos t + t \cdot 1 \right) dt = \frac{(2\pi)^2}{2}$$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (1 \cdot 0 + 0 \cdot 0 + t \cdot 1) \, dt = \frac{(2\pi)^2}{2}$$

here the two are the same

$$\int_{C_1} \vec{G} \cdot d\vec{r} = \int_0^{2\pi} \Big(\overbrace{-t}^{-z} \cdot \overbrace{(-\sin t)}^{x'} + \overbrace{3\cos t}^{3x} \cdot \overbrace{\cos t}^{y'} + \overbrace{\sin t}^{y} \cdot \overbrace{1}^{z'} \Big) dt$$

$$= \Big[-t\cos t + \sin t \Big]_0^{2\pi} + 3\Big[\frac{t}{2} + \frac{1}{4}\sin 2t \Big]_0^{2\pi} + \Big[-\cos t \Big]_0^{2\pi}$$

$$= \pi$$

$$\int_{C_2} \vec{G} \cdot d\vec{r} = \int_0^{2\pi} (-t \cdot 0 + 3 \cdot 0 + 0 \cdot 1) \, dt = 0$$

here the two integrals differ!