~~Ex~~ Giving away a secret here:

$$\vec{F} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{is somebody's gradient}$$

$$\vec{G} = \begin{bmatrix} -z \\ 3x \\ y \end{bmatrix} \quad \text{is nobody's gradient}$$

For suppose G were $\vec{\nabla} g$, $\quad g_x = -z$
$g_y = 3x$
$g_z = y$

If there were indeed such a g, then $g_{xy} = (-z)_y = 0$
and $g_{yx} = (3x)_x = 3$
but these would have to be equal by Clairaut's thm.

Can we find f st. $F = \vec{\nabla} f$

$f_x = x \Rightarrow f = \frac{x^2}{2} + \text{some expr}(y, z)$

$f_y = y \longrightarrow f = \frac{x^2}{2} + \frac{y^2}{2} + \text{some expr}(\cancel{x}, z)$

$f_z = z \longrightarrow f = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} + C$

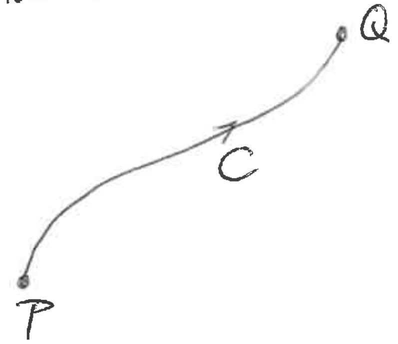
Sneak preview: If a vf $\vec{F}$ passes all the tests by which we ~~find~~ weed out from Clairaut's thm such vf that cannot be any body's gradient, does that mean $\vec{F}$ is indeed somebody's gradient.

Answer: "yes, but..." (The "but" has to do with geometry, not with algebra)

<u>Claim:</u> If $\vec{F} = \vec{\nabla} f$, then

$$\int_C \vec{F} \cdot d\vec{s} = f(\text{endpoint of } C) - f(\text{start point of } C)$$

How come?



$C: \ \vec{r}(t)$ where $a \le t \le b$

$\vec{r}(a) = \vec{P} \qquad \vec{r}(b) = \vec{Q}$

$$\int_C \vec{F} \cdot d\vec{s} = \int_a^b \vec{\nabla} f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt \underset{\substack{\uparrow \\ \text{Chain rule for paths}}}{=} \int_a^b \frac{d}{dt} f(\vec{r}(t))\, dt \underset{\substack{\uparrow \\ \text{FTC}}}{=} \Big[f(\vec{r}(t)) \Big]_a^b =$$

$$= f(\vec{r}(b)) - f(\vec{r}(a)) = f(Q) - f(P)$$

<u>A few names:</u> (1) a vector field $\vec{F}$ is called <u>conservative</u> if it is somebody's gradient. (There is some f s.t. $\vec{F} = \nabla f$)

(2) if C is a closed curve (end pt = start pt), then we like to write $\int_C \vec{F} \cdot d\vec{s}$ as $\oint_C \vec{F} \cdot ds$

and call it the circulation of $\vec{F}$ around C.

Easy consequence: If $\vec{F}$ is conservative and C is closed, then $\oint_C \vec{F} \cdot d\vec{s} = 0$