The converse is also true : on a connected domain

If a vector field $\vec{F}$ has the property that

$\int_C \vec{F} \cdot d\vec{s}$ depends only on start & end point of C, but not on the route C takes in between, then $\vec{F}$ is conservative,

ie. there is a function f s.t. $\vec{F} = \vec{\nabla} f$.

(may call f an antiderivative of $\vec{F}$)

<u>How come?</u>

We obtain f from $\vec{F} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$ by a vector line integral

Choose <u>some</u> point P in the domain to start with

then for any Q in the domain, define

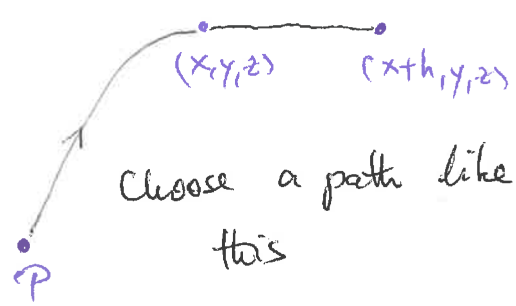
$$f(Q) := \int_{\text{path from } P \text{ to } Q} \vec{F} \cdot d\vec{s}$$

(whatever path chosen, the result will be the same b/c we assume that $\vec{F}$ has the property of path independence)

Claim
$$f_x = F_1$$
$$f_y = F_2$$
$$f_z = F_3 .$$

To see that this is the case, calc'

$$\frac{f(x+h,y,z) - f(x,y,z)}{h} :$$



Choose a path like this

$$f(x+h, y, z) - f(x, y, z) = \int_L \vec{F} \cdot d\vec{s}$$

where L is a straight segment from (x, y, z) to $(x+h, y, z)$

$$L : \vec{r}(t) = [x+t, y, z]^T \qquad 0 \le t \le h$$

$$\vec{r}\,'(t) = [1, 0, 0]^T$$

$$\cancel{\int_0^h f(x+t, y, z)\,dt}$$

$$= \int_0^h \begin{bmatrix} F_1(x+t, y, z) \\ F_2(x+t, y, z) \\ F_3(x+t, y, z) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} dt$$

$$= \int_0^h F_1(x+t, y, z)\,dt$$

$$\frac{f(x+h, y, z) - f(x, y, z)}{h} = \frac{1}{h} \int_0^h F_1(x+t, y, z)\,dt$$

as $h \to 0$, the average, which is the rhs integral, converges to $F_1(x, y, z)$

$$f_x(x, y, z) = F_1(x, y, z)$$

Similar calc's for $f_y = F_2$, $f_z = F_3$ (with appropriately different c)

<u>Note:</u> we can also write the stmt

(1) $\left\{ \int_C \vec{F} \cdot d\vec{s} \right.$ depends only on start & end of C, not path in between

in the following way:

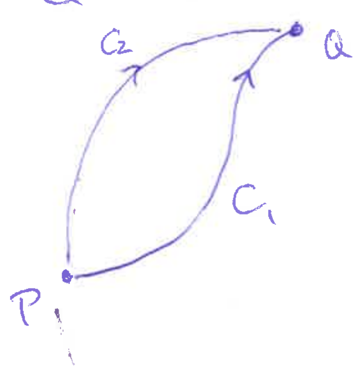
(2) $\left\{ \oint_C \vec{F} \cdot d\vec{s} \right.$ over any closed curve C is 0.

If C is closed

and (1) is true, then we can replace C with a constant path (that stays @ P all the time) and $\oint_C$ doesn't change

but then $\oint_C \vec{F} \cdot d\vec{s} = \underbrace{\int_a^b \vec{F}(P) \cdot \underbrace{\vec{r}\,'(t)}_{=0} dt}_{=0}$

if (2) is true and we compare two paths C_1, C_2 from P to Q



the Consider $C = C_1 - C_2$

go from P via C_1 to Q and then continue back to P via C_2 in reverse orientation

C is a closed path, and

$$\oint_C \vec{F} \cdot d\vec{s} = 0$$
$$\| $$
$$\int_{C_1} \vec{F} \cdot d\vec{s} - \int_{C_2} \vec{F} \cdot d\vec{s}$$

(reversing the order in a vector line integral gives the negative)

Therefore $\int_{C_1} \vec{F} \cdot d\vec{s} = \int_{C_2} \vec{F} \cdot d\vec{s}$

We know the following fact from weeks ago: given a vf $\vec{F}$:

If there is an f st. $\vec{F} = \nabla f$, then

the "cross-derivatives" must coincide by Clairaut's thm

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \qquad (\text{ b/c } (f_x)_y = (f_y)_x)$$

$$\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$$

$$\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$$

<u>Thus:</u> If I check the cross deriv's for $\vec{F}$ and they don't match,

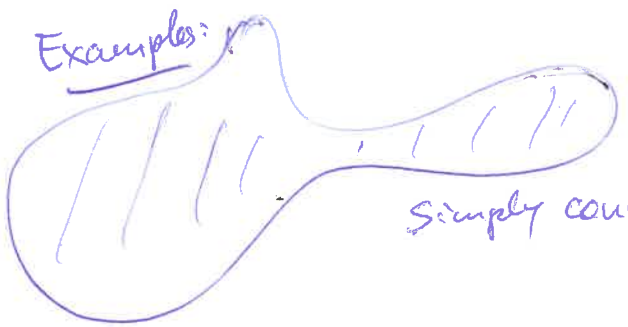
then $\vec{F}$ cannot be conservative (cannot be a gradient)

<u>Q:</u> If the cross derivatives <u>do</u> match, will $\vec{F}$ be a gradient indeed?

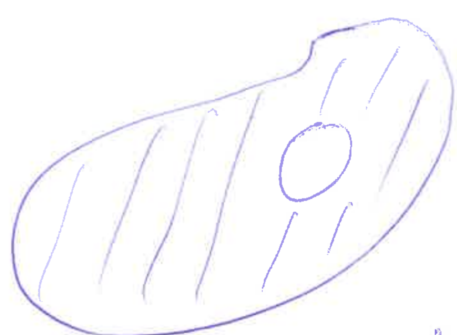
<u>A:</u> "yes, but..."

The answer is yes if the domain of $\vec{F}$ is simply connected.

<u>Examples:</u>



simply connected



<u>not</u> simply connected

A domain is called simply connected if any closed path can be continuously contracted to a point all within the domain

Will skip a formal proof of this stmt, but here is at least an idea