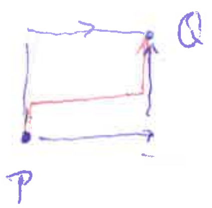
A domain is called simply connected if any closed path can be continuously contracted to a point all within the domain

Will skip a formal proof of this stmt, but here is at least an idea : show that $\int = \int$



<u>Review:</u> We claimed: If the cross derivatives of $\vec{F}$ coincide and the domain of $\vec{F}$ is simply connected then $\vec{F}$ is conservative (ie there is some f whose gradient $\vec{F}$ is: $\vec{\nabla} f = \vec{F}$)

→ we'll prove this later.

A few comments on this:

(1) We can write this somewhat differently:

Given a vector field $\vec{F}$, we define curl $\vec{F}$ (also $\vec{\nabla} \times \vec{F}$) to be

$$\begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{bmatrix}$$

(meaning of that thing to be discussed yet)

The matching of the cross derivatives of $\vec{F}$ is tantamount to $\vec{\nabla} \times \vec{F} = \vec{0}$

Clairaut's thm tells us : if $\vec{F} = \vec{\nabla} f$, then curl $\vec{F} = \vec{0}$

$$\left(\vec{\nabla} \times (\vec{\nabla} f) = \vec{0} \right)$$

and our statement above says : If $\vec{\nabla} \times \vec{F} = \vec{0}$ and domain of $\vec{F}$ is simply connected , then $\vec{F}$ is somebody's gradient.

Curl doesn't apply to two-dim vector field b/c $\times$ product only is available in $\mathbb{R}^3$.

However, if you deal with $\begin{bmatrix} F_1(x,y) \\ F_2(x,y) \end{bmatrix}$ you can formally write it as $\begin{bmatrix} F_1(x,y,z) \\ F_2(x,y,z) \\ 0 \end{bmatrix}$ but with F_i not really depending on z

then curl of this is $\begin{bmatrix} 0 \\ 0 \\ \dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} \end{bmatrix}$

(2) An important example illustrating the role of "simply connected" (every closed curve in a s/c domain can be contracted to a point without leaving the domain)

Consider the vector field $\begin{bmatrix} -y/(x^2+y^2) \\ +x/(x^2+y^2) \end{bmatrix}$

(enhanced with a 3^{d} component 0 , this v f is the magnetic field around a wire with current along the z-axis)

cross derivatives coincide :

$$\frac{\partial}{\partial y} \frac{-y}{x^2+y^2} = \frac{-1(x^2+y^2) + y\,2y}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\frac{\partial}{\partial x} \frac{+x}{x^2+y^2} = \frac{+1(x^2+y^2) - x \cdot 2x}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

Is this vf conservative? (Thm doesn't claim so b/c

domain $= \mathbb{R}^2 \setminus$ origin (punctured plane; origin is cut out)

is not simply connected.

Calculate the vector line integral over a circle C of radius a about origin

$$\vec{r}(t) = \begin{bmatrix} a\cos t \\ a\sin t \end{bmatrix} \qquad \vec{r}\,'(t) = \begin{bmatrix} -a\sin t \\ a\cos t \end{bmatrix} \qquad 0 \le t \le 2\pi$$

$$\oint_C \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \begin{bmatrix} -a\sin t/a^2 \\ +a\cos t/a^2 \end{bmatrix} \cdot \begin{bmatrix} -a\sin t \\ a\cos t \end{bmatrix} dt$$

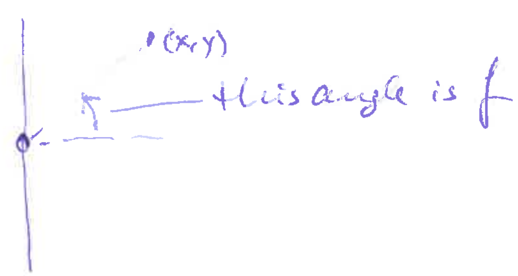
$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt = 2\pi \neq 0$$

vf is indeed <u>not</u> conservative b/c circulation is not 0.

Suppose we study this same $\vec{F}$ <u>only</u> in the right half plane $x>0$

(that domain is simply connected, so there should be a potential fct...)

$f = \arctan \frac{y}{x}$ is a potential fct. there (you check the calc)



If we do it in upper half plane

then $\operatorname{arccot} \frac{x}{y}$ is a potential function

(and the two coincide in the 1st quadrant)

(3) In physics, when $\vec{F}$ is a force field and

$\vec{F} = \vec{\nabla} f$, they like to call $-f =: V$ (potential energy)

$\vec{F} = -\vec{\nabla} V$

Suppose a particle moves in a conservative force field, and at time t, it is at location $\vec{r}(t)$.

Then its potential energy is $V(\vec{r}(t))$

Its kinetic energy is $\frac{1}{2} m \text{ speed}^2 = \frac{1}{2} m \|\vec{r}\,'(t)\|^2$

$= \frac{1}{2} m \, \vec{r}\,'(t) \cdot \vec{r}\,'(t)$

Total energy is $\frac{1}{2} m \, \vec{r}\,'(t) \cdot \vec{r}\,'(t) + V(\vec{r}(t))$

$$\frac{d}{dt}(\text{total energy}) = m \, \vec{r}\,'(t) \cdot \vec{r}\,''(t) + \vec{\nabla} V(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

$$= \vec{r}\,'(t) \cdot \Big(\underbrace{m \, \vec{r}\,''(t) - \vec{F}(\vec{r}(t))}_{= \vec{0}} \Big) = 0$$

by Newton's law

Hence total energy is constant in time.