Homework Chapter 1
UTK – M251 – Matrix Algebra
Fall 2003, Jochen Denzler, MWF 1:25–2:15, Ayres 318

1. Which of the following are linear equations in the unknowns x_1, x_2, x_3, or x, y, z, as applicable:

(a)	$x_1 + x_2^2 + 3x_3 = 7$	(b)	$\frac{x}{2} + 2y - 3z = 0$
(c)	$\sqrt{2}x_1 - 3x_2 + x_3 = 5$	(d)	$2\sqrt{x_1} - x_2 + 8x_3 = 2$
(e)	$x + yz = 2$	(f)	$\frac{2}{x} + 2y - 3z = 0$

2. Find the augmented matrix for the following system of linear equations:

$$\begin{aligned} &5x + 3y - 2z + 11w = 9 \\ &3x - 4w = 11 \\ &x + y + z = \sqrt{5} \end{aligned}$$

3. For which values of the constant k does the system

$$\begin{aligned} x + 2y &= 7 \\ 3x + 6y &= k \end{aligned}$$

have (a) no solution (b) exactly one solution (c) infinitely many solutions?

4. Solve each of the following systems by Gauss–Jordan elimination:

$$(a)\ \begin{aligned} 3x + 2y - 2z &= 1 \\ -x + y + 2z &= 2 \\ 2x - 3y &= 1 \end{aligned} \qquad (b)\ \begin{aligned} 3x_1 - 2x_2 - x_3 - x_4 &= 0 \\ 2x_1 + x_2 + 3x_4 &= 1 \end{aligned} \qquad (c)\ \begin{aligned} 2x + y + 3z - w &= 4 \\ x - 3y + 2z + 4w &= 1 \\ 4x + 9y + 5z - 11w &= 7 \end{aligned}$$

5. Solve (a) in the previous problem by Gauss elimination and back substitution.

6. For which values of λ does the system

$$\begin{aligned} (3 - \lambda)x + y &= 0 \\ 3x + (5 - \lambda)y &= 0 \end{aligned}$$

have nontrivial solutions? (Note: Later in the semester, this type of problem will become known as an "eigenvalue problem" and will be studied in more generality.)

7. Give an example of a linear system of equations that has more equations than unknowns, but still has infinitely many solutions.

8. Give an example of a linear system of equations that has more unknowns than equations, but has no solutions.

9. Find numbers a, b and c such that the curve $y = ax^2 + bx + c$ passes through the points $(1, 3)$, $(2, 3)$ and $(4, 9)$. (Note that this problem leads to *linear* equations even though $f(x) = ax^2 + bx + c$ is a *non*linear function of x.)

10. Generally adopted convention: An $m \times n$ matrix B is a matrix with m rows and n columns; in other words, the first number gives the number of rows, the second number the number of columns. How to memorize that it isn't the other way round? — I haven't seen a trick for that purpose. The best I can offer is a poor translation of something that works much better in the German language, in which I learned it as a student:

Row fi**R**st, colum**N** seco**N**d. If you can invent a better memorization device, hand your suggestion in for bonus hwk points.

11. Suppose the following matrices have the sizes given:

matrix	A	B	C	D	E
size	3×7	7×2	3×7	3×2	2×3

Determine the sizes of the following matrices, or say "undefined", if this is the case. AB, BA, $A - C$, $A^T B$, $A^T C$, $D^T A$, $CB + D$, DE, $DD^T + E^T E$, $(A + C)BE$

12. Compute

$$\begin{bmatrix} 1 & 2 & 0 & -2 \\ 2 & 2 & 1 & -3 \\ 3 & -1 & 7 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 0 \\ -1 & 3 \\ 7 & -4 \end{bmatrix}$$

13. Compute AB and BA for

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & -4 \\ 3 & -1 \end{bmatrix}$$

14. Compute A^2 (a shorthand for AA) if $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

15. In the following matrix expression, $*$ denotes numbers that I decline to reveal to you. Compute as many entries of the result as is possible, based on this limited information, and fill in $*$ for the entries that cannot be determined.

$$\begin{bmatrix} * & * & * & 0 & * \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & * & 0 \\ -1 & * & 0 \\ 2 & * & 0 \\ 4 & 0 & 0 \\ -1 & * & 0 \end{bmatrix}$$

16. Write the matrix equation

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 7 & 7 & 7 & 7 \\ -\frac{1}{5} & 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 88 \\ 888 \end{bmatrix}$$

as a system of linear equations.

Also write the system of linear equations whose augmented matrix can be found on page 1 of the introductory notes as a matrix equation; you may invent your favorite names for the unknowns.

17. For the matrices

$$A = \begin{bmatrix} -2 & 1 \\ 0 & \frac{3}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & \frac{1}{2} & \frac{1}{3} \end{bmatrix}, \qquad C = \begin{bmatrix} 5 & -2 \\ -3 & 4 \end{bmatrix},$$

calculate the following expressions. If some expression is not defined, say so and explain why it is not defined.

$$(a)\ (A + B^T)C\,, \qquad (b)\ A^T A + C\,, \qquad (c)\ B + CA\,, \qquad (d)\ A^T - 2CB$$

18. For the matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ calculate $\operatorname{tr} A^T A$ and $\operatorname{tr} AA^T$. (If you are efficient, you need to calculate only part of $A^T A$ and of AA^T.)

19. Find a square matrix A such that $\operatorname{tr}(A^2) \neq (\operatorname{tr} A)^2$. (Unless it's your really unlucky day, your first best random choice of a 2×2 matrix should do the trick.)

20. Suppose that a square matrix A satisfies $A^2 - 3A + 2I = 0$. Show that A is invertible, and that $A^{-1} = \frac{3}{2}I - \frac{1}{2}A$.

21. Let

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}$$

Calculate $(A + B)^2$ and $A^2 + 2AB + B^2$, and explain how it comes that they are not the same. – Give the *correct* binomial formula for matrices:

$$(A + B)^2 = A^2 + B^2 + \underline{\qquad\qquad}\ ?$$

22. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 2 & 1 \\ -2 & 0 & 1 \end{bmatrix}$$

23. Find the inverses of the following matrices:

$$B = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & & & & \\ & \frac{4}{5} & & & \\ & & 2 & & \\ & & & \pi & \\ & & & & -3 \end{bmatrix}$$

(Note: Vacant entries are understood to be 0. This convention is popular when dealing with large matrices that have only a comparatively small number of non-zero entries, so-called sparse matrices.)

24. Find an (the) elementary matrix E for which $A = EB$ when

$$A = \begin{bmatrix} 8 & 3 & 3 \\ 4 & 2 & 1 \\ -5 & 2 & -3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 8 & 3 & 3 \\ 4 & 2 & 1 \\ 3 & 6 & -1 \end{bmatrix}$$

25. Solve the linear system

$$\begin{aligned} -x - 2y - 3z &= 2 \\ w + x + 4y + 4z &= 5 \\ w + 3x + 7y + 9z &= -1 \\ -w - 2x - 4y - 6z &= 3 \end{aligned}$$

by actually calculating the inverse of the coefficient matrix and then evaluationg the matrix product. (Remember from the lecture that this method is usually not encouraged, and not efficient for a single system of linear equations. Just once so that you see it work in practice.)

26. Which of the following matrices are invertible? (Decide by eyeballing; you are not required to find the inverse)

$$A = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & -2 & 1 & 9 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 7 & 5 \end{bmatrix}$$

27. I have done the following calculation for you already. You job is to determine, by eyeballing alone, if the two matrices on the left hand side commute or not; and to explain your reasoning in one or two sentences. If you find it difficult to come up with a compelling argument, you may need to study the glossary (again).

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 11 & 5 & -2 \\ 5 & 0 & 15 \\ -2 & 15 & -11 \end{bmatrix} = \begin{bmatrix} 15 & 50 & -5 \\ 15 & 25 & -30 \\ 38 & 15 & 9 \end{bmatrix}$$

28. In the following calculations, the $*$ represent entries which I do not reveal to you. Carry out the evaluations to the extent possible and put $*$ for entries that cannot be determined with the given information. Note that vacant entries are understood to be zero.

$$\begin{bmatrix} 1 & * & * \\ & 2 & * \\ & & 3 \end{bmatrix} \cdot \begin{bmatrix} 6 & * & * \\ & 5 & * \\ & & 4 \end{bmatrix} = ?, \qquad \begin{bmatrix} 1 & * & * \\ & 2 & * \\ & & 3 \end{bmatrix}^{-1} = ?$$

29. Find a diagonal matrix A that satisfies

$$A^{-4} = \begin{bmatrix} 16 & & \\ & 1 & \\ & & 1/9 \end{bmatrix}$$

(I didn't mention this in class, but hope you find the notation obvious:
$A^{-4} := A^{-1} \cdot A^{-1} \cdot A^{-1} \cdot A^{-1}$.)

30. A and B are given square matrices of the same size (they may or may not be symmetric). Which of the following expressions will always produce a symmetric matrix, whatever matrices A and B I may have in mind?

$$(a) \quad AB + BA, \qquad (b) \quad A^T A, \qquad (c) \quad AB - BA, \qquad (d) \quad ABA^T B^T$$

31. Answer the same question as before, but this time with A and B given *symmetric* matrices of the same size.

32. Find lower and upper triangular matrices L and U such that $A = LU$, where A is the matrix

$$A = \begin{bmatrix} 2 & 8 & -2 \\ -1 & -1 & -14 \\ 2 & 11 & -19 \end{bmatrix}$$

33. As a preparation for the exam, and review of chapter 1, study the following problems from the book; you will find solutions in the book, but in each case you should be able to give a reason why a statement is true or false, and this info may not be in the solution part:
p. 23,#31c,32 — p. 49,#35 — p. 58,#19,20,21 — p. 66,#28,29 — p. 73,#26 (you may do $n = 3$, $n = 4$, and $n = 10$ if you find this easier than working with general n) — p. 73,74,#28,30

Here is the deal: I'll have a quiz with 3 questions on Monday, each counting like one hwk problem. You may bring in a list of well worked out questions, concerning any of the problems listed above. These questions should display a bona-fide attempt to solve a problem and the question will be a question whose answer you would think helpful to continue where you got stuck. You may hand them in together with the quiz. Up to two questions will substitute for the two weakest quiz questions; so if you have two bona-fide attempts, with a good question, you'll enter the quiz with 4 out of 6 points pocketed safely. We will later continue to work on difficulties you may have with these problems; the simpler questions in this style are fair game for exam 1; the more subtle ones (that require more thought and time) may become any kind of material (hwk, quiz, exam) for grade at a later time, when I feel confident that you are / ought to be prepared for it.