Homework Chapter 6
UTK – M251 – Matrix Algebra
Fall 2003, Jochen Denzler, MWF 1:25–2:15, Ayres 318

1. Which of the following are inner products on $\mathbb{R}^3$? For those that are not, determine which of the properties does/do not hold.
(a) $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = 2u_1v_1 + 3u_2v_2$
(b) $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = 2u_1v_1 + 3u_2v_2 + 3u_3v_3 - 2u_2v_3 - 2u_3v_2$
(c) $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = 2u_1v_1 + 3u_2v_2 - u_3v_3$
(d) $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2)^2$

2. If A is an invertible $n \times n$ matrix, then the formula $\langle \boldsymbol{u}, \boldsymbol{v}\rangle := (A\boldsymbol{u}) \cdot (A\boldsymbol{v})$ gives an inner product, which is called the *inner product generated by A*. Calculate the inner product generated by $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ of the vectors $\boldsymbol{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

3. In this problem, let $\|\boldsymbol{w}\|$ denote *not* the euclidean norm, but the norm obtained from the inner product generated by the matrix A from the previous problem.

What condition do x and y, the components of the vector $\boldsymbol{w}$ have to satisfy such that $\|\boldsymbol{w}\| = 3$ holds? Draw the curve $\|\boldsymbol{w}\| = 3$ in the plane in a good, to-scale figure. If $\|\boldsymbol{w}\|$ were the euclidean norm, then the curve would be a circle of radius 3. What kind of curve did you come up with in the present case? Suggested plot window: $x \in [-8, 8]$, $y \in [-5, 5]$.

4. On the space $\mathbb{R}^{3\times3}$, we take the inner product $\langle A, B\rangle := \operatorname{tr} A^T B$. Based on this inner product, what is the norm of the matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$?

Before calculating too much, remember that in a certain matrix product you are about to compute, not all of the entries need to be computed.

5. To show that $\operatorname{tr} AB$ is *not* an inner product, even for square matrices, find an example of a 2×2 matrix A for which $\operatorname{tr}(A^2) < 0$.

6. In the vector space $C^0[0, 2\pi]$ of all continuous functions defined on the interval $[0, 2\pi]$, we use the inner product $\langle f, g\rangle := \int_0^{2\pi} f(x)g(x)\,dx$. Write out explicitly what the Cauchy-Schwarz inequality says in the case $f(x) = x$ and $g(x) = \sin x$.

Also check this particular instance of the Cauchy-Schwarz inequality by explicit evaluation of all integrals. — I do expect Math, Phys and Engr majors to be able to do these integrals without help, but since this is not a calculus class, and for the benefit of those who have not used calculus for a while, I give some derivative-antiderivative formulas for your convenience:

$$\frac{d}{dx}(\sin x - x\cos x) = x \sin x\ , \qquad \frac{d}{dx}\,\frac{1}{2}(x - \sin x\cos x) = \sin^2 x$$

7. Continuing the previous problem, what is the "angle" between the functions f and g, in the sense determined by the inner product given there. — Note: If you have even the slightest gut feeling that this "angle" has anything to do with any angle you might see when you graph the function f and g, be assured emphatically that this is *not* the case.

8. For $m = 0, 1, 2, 3, \ldots$, we define the functions $c_m \in C^0[0, 2\pi]$ by $c_m(x) := \cos mx$. For $m = 1, 2, 3, \ldots$, we define the functions $s_m \in C^0[0, 2\pi]$ by $s_m(x) := \sin mx$. We share this workload among the class. Everybody select at random one of the three (equally easy/difficult) parts as homework.

(a) Show that $\langle c_m, c_n \rangle = 0$ if $m \neq n$.
(b) Show that $\langle s_m, s_n \rangle = 0$ if $m \neq n$.
(c) Show that $\langle c_m, s_n \rangle = 0$.

You may use the following trig' formulas (which I would not expect you to have memorized):

$$\begin{aligned}(\cos u)(\cos v) &= \tfrac{1}{2}(\cos(u+v) + \cos(u-v))\\ (\sin u)(\sin v) &= \tfrac{1}{2}(\cos(u-v) - \cos(u+v))\\ (\sin u)(\cos v) &= \tfrac{1}{2}(\sin(u+v) + \sin(u-v))\end{aligned}$$

9. Refer to the previous problem, and have a look back at hwk 10 from Ch. 5. Show that the set $S = \{c_0, c_1, c_2, \ldots, c_{100}, s_1, s_2, \ldots, s_{100}\}$ in $C^0[0, 2\pi]$ is linearly independent. — What is the dimension of $C^0[0, 2\pi]$? (Possible answers are either a nonnegative integer, or “infinite-dimensional”.)

10. In $\mathbb{R}^4$, with the euclidean inner product (dot product), let U be the subspace spanned by the vectors $[1,\ 1,\ 1,\ 1]^T$ and $[-1,\ 0,\ 1,\ 2]^T$. Find a basis for the orthogonal complement $U^\perp$ of U.

11. Same question, but this time, take the inner product $\langle \boldsymbol{u}, \boldsymbol{v} \rangle := u_1v_1 + 2u_2v_2 + 3u_3v_3 + u_4v_4$ instead of the dot product.

(I'd like to stress that the $U^\perp$ in this problem refers to a different space than the $U^\perp$ in the previous problem, because the meaning of the symbol $^\perp$ depends on the inner product chosen. This is why I give two so similar problems)

12. In $\mathbb{R}^3$ with the dot product, use the Gram–Schmidt process to obtain an orthonormal basis from the basis $S = \{[2,\ 2,\ 1]^T, [1,\ 2,\ 2]^T, [2,\ -1,\ 2]^T\}$.

13. In the vector space P_3 of polynomials of degree 3 or less, choose the inner product $\langle p, q \rangle := \int_{-1}^{1} p(x)q(x)\,dx$. Use the Gram–Schmidt process to obtain an orthogonal basis $\{p_0, p_1, p_2, p_3\}$ from the standard basis $S = \{1, x, x^2, x^3\}$. Normalize the vectors (i.e., polynomials) such that $\|p_0\|^2 = 2$, $\|p_1\|^2 = \frac{2}{3}$, $\|p_2\|^2 = \frac{2}{5}$, $\|p_3\|^2 = \frac{2}{7}$.

(Note: This is a convenient normalization that avoids square roots in the coefficients. — The polynomials you obtain are called Legendre polynomials, and the reason why they deserve to be given a name is exactly that they form an orthogonal basis. — You are not required to memorize their name, though. — Here is one piece of information about the polynomials p_j that I provide for the sole purpose that you can check your calculation in each step against possible calculational erors: $p_j(1) = 1$ for all j.)

14.

15.

16.

17.