Homework Chapter 1
UTK – M251 – Matrix Algebra
Jochen Denzler

1. Which of the following are linear equations in the unknowns x_1, x_2, x_3, or x, y, z, as applicable:

(a)	$x_1 + x_2^2 + 3x_3 = 7$	(b)	$\frac{x}{2} + 2y - 3z = 0$
(c)	$\sqrt{2}x_1 - 3x_2 + x_3 = 5$	(d)	$2\sqrt{x_1} - x_2 + 8x_3 = 2$
(e)	$x + yz = 2$	(f)	$\frac{2}{x} + 2y - 3z = 0$

2. Find the augmented matrix for the following system of linear equations:

$$\begin{aligned} &5x + 3y - 2z + 11w = 9 \\ &3x - 4w = 11 \\ &x + y + z = \sqrt{5} \end{aligned}$$

3. For which values of the constant k does the system

$$\begin{aligned} x + 2y &= 7 \\ 3x + 6y &= k \end{aligned}$$

have (a) no solution (b) exactly one solution (c) infinitely many solutions?

4. Solve each of the following systems by Gauss–Jordan elimination:

$$(a)\ \begin{aligned} 3x + 2y - 2z &= 1 \\ -x + y + 2z &= 2 \\ 2x - 3y \qquad &= 1 \end{aligned} \qquad (b)\ \begin{aligned} 3x_1 - 2x_2 - x_3 - x_4 &= 0 \\ 2x_1 + x_2 \qquad + 3x_4 &= 1 \end{aligned} \qquad (c)\ \begin{aligned} 2x + y + 3z - w &= 4 \\ x - 3y + 2z + 4w &= 1 \\ 4x + 9y + 5z - 11w &= 7 \end{aligned}$$

5. Solve (a) in the previous problem by Gauss elimination and back substitution.

6. For which values of λ does the system

$$\begin{aligned} (3-\lambda)x + y &= 0 \\ 3x + (5-\lambda)y &= 0 \end{aligned}$$

have nontrivial solutions? (Note: Later in the semester, this type of problem will become known as an “eigenvalue problem” and will be studied in more generality.)

7. Give an example of a linear system of equations that has more equations than unknowns, but still has infinitely many solutions.

8. Give an example of a linear system of equations that has more unknowns than equations, but has no solutions.

9. Find numbers a, b and c such that the curve $y = ax^2 + bx + c$ passes through the points $(1, 3)$, $(2, 3)$ and $(4, 9)$. (Note that this problem leads to *linear* equations even though $f(x) = ax^2 + bx + c$ is a *non*linear function of x.)

10. Generally adopted convention: An $m \times n$ matrix B is a matrix with m rows and n columns; in other words, the first number gives the number of rows, the second number the number of columns. How to memorize that it isn't the other way round? — I haven't seen a trick for that purpose. The best I can offer is a poor translation of something that works much better in the German language, in which I learned it as a student:

Row fi**R**st, colum**N** seco**N**d. If you can invent a better memorization device, hand your suggestion in for bonus hwk points.

11. Suppose the following matrices have the sizes given:

matrix	A	B	C	D	E
size	3×7	7×2	3×7	3×2	2×3

Determine the sizes of the following matrices, or say "undefined", if this is the case. AB, BA, $A - C$, $A^T B$, $A^T C$, $D^T A$, $CB + D$, DE, $DD^T + E^T E$, $(A + C)BE$

12. Compute

$$\begin{bmatrix} 1 & 2 & 0 & -2 \\ 2 & 2 & 1 & -3 \\ 3 & -1 & 7 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 0 \\ -1 & 3 \\ 7 & -4 \end{bmatrix}$$

13. Compute AB and BA for

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & -4 \\ 3 & -1 \end{bmatrix}$$

14. Compute A^2 (a shorthand for AA) if $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

15. In the following matrix expression, $*$ denotes numbers that I decline to reveal to you. Compute as many entries of the result as is possible, based on this limited information, and fill in $*$ for the entries that cannot be determined.

$$\begin{bmatrix} * & * & * & 0 & * \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & * & 0 \\ -1 & * & 0 \\ 2 & * & 0 \\ 4 & 0 & 0 \\ -1 & * & 0 \end{bmatrix}$$

16. Write the matrix equation

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 7 & 7 & 7 & 7 \\ -\frac{1}{5} & 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 88 \\ 888 \end{bmatrix}$$

as a system of linear equations.

Also write the system of linear equations whose augmented matrix can be found on page 1 of the introductory notes as a matrix equation; you may invent your favorite names for the unknowns.

17. For the matrices

$$A = \begin{bmatrix} -2 & 1 \\ 0 & \frac{3}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & \frac{1}{2} & \frac{1}{3} \end{bmatrix}, \qquad C = \begin{bmatrix} 5 & -2 \\ -3 & 4 \end{bmatrix},$$

calculate the following expressions. If some expression is not defined, say so and explain why it is not defined.

$$(a)\ (A+B^T)C\,, \qquad (b)\ A^TA+C\,, \qquad (c)\ B+CA\,, \qquad (d)\ A^T-2CB$$

18. For the matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ calculate $\operatorname{tr} A^TA$ and $\operatorname{tr} AA^T$. (If you are efficient, you need to calculate only part of A^TA and of AA^T.)

19. Find a square matrix A such that $\operatorname{tr}(A^2) \neq (\operatorname{tr} A)^2$. (Unless it's your really unlucky day, your first best random choice of a 2×2 matrix should do the trick.)

20. Suppose that a square matrix A satisfies $A^2 - 3A + 2I = 0$. Show that A is invertible, and that $A^{-1} = \frac{3}{2}I - \frac{1}{2}A$.

21. Let

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}$$

Calculate $(A+B)^2$ and $A^2 + 2AB + B^2$, and explain how it comes that they are not the same. – Give the *correct* binomial formula for matrices:

$$(A+B)^2 = A^2 + B^2 + \underline{\qquad\qquad}\ ?$$

22. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 2 & 1 \\ -2 & 0 & 1 \end{bmatrix}$$

23. Find the inverses of the following matrices:

$$B = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & & & & \\ & \frac{4}{5} & & & \\ & & 2 & & \\ & & & \pi & \\ & & & & -3 \end{bmatrix}$$

(Note: Vacant entries are understood to be 0. This convention is popular when dealing with large matrices that have only a comparatively small number of non-zero entries, so-called sparse matrices.)

24. Find an (the) elementary matrix E for which $A = EB$ when

$$A = \begin{bmatrix} 8 & 3 & 3 \\ 4 & 2 & 1 \\ -5 & 2 & -3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 8 & 3 & 3 \\ 4 & 2 & 1 \\ 3 & 6 & -1 \end{bmatrix}$$

25. Solve the linear system

$$\begin{aligned} -x - 2y - 3z &= 2 \\ w + x + 4y + 4z &= 5 \\ w + 3x + 7y + 9z &= -1 \\ -w - 2x - 4y - 6z &= 3 \end{aligned}$$

by actually calculating the inverse of the coefficient matrix and then evaluationg the matrix product. (Remember from the lecture that this method is usually not encouraged, and not efficient for a single system of linear equations. Just once so that you see it work in practice.)

26. Which of the following matrices are invertible? (Decide by eyeballing; you are not required to find the inverse)

$$A = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & -2 & 1 & 9 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 7 & 5 \end{bmatrix}$$

27. I have done the following calculation for you already. You job is to determine, by eyeballing alone, if the two matrices on the left hand side commute or not; and to explain your reasoning in one or two sentences. If you find it difficult to come up with a compelling argument, you may need to study the glossary (again).

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 11 & 5 & -2 \\ 5 & 0 & 15 \\ -2 & 15 & -11 \end{bmatrix} = \begin{bmatrix} 15 & 50 & -5 \\ 15 & 25 & -30 \\ 38 & 15 & 9 \end{bmatrix}$$

28. In the following calculations, the $*$ represent entries which I do not reveal to you. Carry out the evaluations to the extent possible and put $*$ for entries that cannot be determined with the given information. Note that vacant entries are understood to be zero.

$$\begin{bmatrix} 1 & * & * \\ & 2 & * \\ & & 3 \end{bmatrix} \cdot \begin{bmatrix} 6 & * & * \\ & 5 & * \\ & & 4 \end{bmatrix} = ?\,, \qquad \begin{bmatrix} 1 & * & * \\ & 2 & * \\ & & 3 \end{bmatrix}^{-1} = ?$$

29. Find a diagonal matrix A that satisfies

$$A^{-4} = \begin{bmatrix} 16 & & \\ & 1 & \\ & & 1/9 \end{bmatrix}$$

(I didn't mention this in class, but hope you find the notation obvious:
$A^{-4} := A^{-1} \cdot A^{-1} \cdot A^{-1} \cdot A^{-1}$.)

30. A and B are given square matrices of the same size (they may or may not be symmetric). Which of the following expressions will always produce a symmetric matrix, whatever matrices A and B I may have in mind?

$$(a) \quad AB + BA\,, \qquad (b) \quad A^T A\,, \qquad (c) \quad AB - BA\,, \qquad (d) \quad ABA^T B^T$$

31. Answer the same question as before, but this time with A and B given *symmetric* matrices of the same size.

32. Find lower and upper triangular matrices L and U such that $A = LU$, where A is the matrix

$$A = \begin{bmatrix} 2 & 8 & -2 \\ -1 & -1 & -14 \\ 2 & 11 & -19 \end{bmatrix}$$

33. Here is a list of problems about concepts and clear reasoning. They are taken from the book, and I give you some explanations such as to make sure you understand the questions right. (Note that understanding this type of questions right is not trivial, but is at least half of the goal I want you to achieve here.)

Are the following statements always true or sometimes false?

If you claim that the statement is 'always true', you must give a proof, which means a sequence of arguments like a simple calculation, using rules like those we have learned for matrices (e.g., $(AB)^T = B^T A^T$ or $A(B + C) = AB + AC$. But normally you must add some text for clarification. An *example* where the statement is true does *not* qualify, because you would be claiming 'always true' based on only a simple example. That would be like saying 'it rains everyday' just because it is raining today.

If you claim a statement is 'sometimes false', you must give a counterexample. This means one example (satisfying all the if's of the statement) for which the claim (the 'then' part) is false. In math language 'once' already qualifies for 'sometimes'.

(a) From p.23, #31c: Claim: "If the r.r.e.f. of a matrix has a row of zeros, then the system must have infinitely many solutions" (T/F?)

(b) From p.23, #32a: Claim: "A linear system of three equations in five unknowns must be consistent (i.e., must have (a) solution(s)." (T/F?)

(c) From p.23, #32b: Claim: "A linear system of five equations in three unknowns cannot be consistent (i.e., cannot have a solution." (T/F?)

(d) From p.23, #32c: Claim: "If a linear system of n equations in n unknowns has n leading 1's in the r.r.e.f. of its augmented matrix, then the system has exactly one solution." (T/F?) *From now on, words like 'always' and 'must' may be omitted. When we make a general claim, then we are implicitly claiming that it is always true, not just occasionally. So if you choose 'T', this means you choose 'always true', whereas if you choose 'F', this means you are choosing 'sometimes false'.*

(e) Claim: "If a linear system of n equations in n unknowns has n leading 1's in the r.r.e.f. of its coefficient matrix, then the system has exactly one solution." (T/F?) *If you don't see the difference between this and the previous one, your reading isn't sufficiently careful for math purposes. Did you know that genuine math often requires more diligent reading than languages do!?*

(f) From p.23, #32d: Claim: "If a linear system of n equations in n unknowns has two equations that are multiples of one another, then the system is inconsistent."

34. More in the same style:

(a) From p.49, #35: Let A and B $n \times n$ matrices. (This means, you are told that A and B are square matrices of the same size. But there is no other information about A and B that is available to you.) Are the following statements always true or sometimes false? Give a proof of a counterexample, as appropriate.

(i) $(AB)^2 = A^2B^2$ (ii) $(A-B)^2 = (B-A)^2$ (iii) $(AB^{-1})(BA^{-1}) = I_n$
(In example (iii), you may assume that A and B are invertible. This is implicitly stated by writing A^{-1} and B^{-1} in the question.
(iv) $AB \neq BA$
(Remember: They ask "Is it always true that AB is different from BA?)
(v) $AB = BA$
(Remember: They ask "Is it always true that AB is the same matrix as BA?)

(b) From p.58, #19–21: Give a proof or a counterexample. True or false?
(i) Every square matrix can be expressed as a product of elementary matrices.
(ii) If A is invertible and an elementary row operation is applied to it, then the resulting matrix must also be invertible.
(iii) If A is invertible and $AB = 0$ (with B any matrix such that the product makes sense), then B must be 0.
(iv) If A is a singular (i.e., not invertible) matrix, then the system $A\mathbf{x} = \mathbf{0}$ must have infinitely many solutions.
(v)If A is a singular matrix, then the r.r.e.f. of A must have a row of zeros ('a row' means: 'at least one row')
(vi) Is there a 2×2 matrix A such that $A\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b & d \\ a & c \end{bmatrix}$ for all choices of a, b, c, d?

35. More in the same style:

(a) From p.66 (modified): What condition does it take on a square matrix A to guarantee that the system $A\mathbf{x} = \mathbf{x}$ has exactly one solution? — Answer format: $A\mathbf{x} = \mathbf{x}$ has exactly one solution if and only if the matrix ________ is ________________.
(Brief explanation wanted)

(b) From p.66: Is it possible that $AB = I$ without B being the inverse of A? Explain.

(c) (From p.73): How many distinct (that is, different) entries can a symmetric $n \times n$ matrix have? If you find this difficult, you'd first answer the question for a 2×2 matrix, then 3×3 and so on. It is ok if your answer for general n looks different from the answer in the back of the book. We can later discuss why your answer and the one in the book amount to the same, and in the meanwhile you just build confidence by checking that they do amount to the same for a few samples of n.

(d) (From p.73): If A is a square matrix and D is a diagonal matrix and $AD = I$, what can you conclude about A? Explain.

36. Are the following always true or sometimes false? (Give a counterexample or an argument): (Modified from p. 74) Clarification: claiming a matrix to be singular, or invertible, or symmetric includes the claim that this matrix is square, because these notions are defined for square matrices only.

(a) If AA^T is invertible and A is square, then A is (necessarily) invertible.
(b) If AA^T is invertible, then A is (necessarily) invertible.
(c) If AA^T is singular and A is square, then A is (necessarily) singular.
(d) If AA^T is singular, then A is (necessarily) singular.
(e) If A^2 is symmetric, then A is (necessarily) symmetric.
(f) If A is symmetric, then A^2 is (necessarily) also symmetric.