Homework Chapter 2
UTK – M251 – Matrix Algebra
Spring 2006, Jochen Denzler, MWF 1:25–2:15, Ayres 102

1. How many inversions does the permutation $(3, 1, 5, 2, 7, 4, 6)$ have? (In other words, we are taking about the permutation that gives the term $a_{13}a_{21}a_{35}a_{42}a_{57}a_{64}a_{76}$ in the determinant of a 7×7 matrix) – Is it an even or an odd permutation?

2. Read the glossary entry on even and odd permutations. Find a sequence of swaps of pairs of numbers that obtains $(3, 1, 5, 2, 7, 4, 6)$ from $(1, 2, 3, 4, 5, 6, 7)$. How many swaps did you take?

3. How many inversions does the permutation $(n, n-1, n-2, \ldots, 3, 2, 1)$ have? List the number of inversions for the cases $n = 2, 3, 4, \ldots, 9$ in a table.

4. Look at ex. 8 on p. 86 and make sure that you understand the special rule memorization device for 3×3 determinants. Use it to calculate $\begin{vmatrix} 2 & -1 & 3 \\ 4 & 1 & -2 \\ 1 & 1 & 5 \end{vmatrix}$. *It's also a good idea to try to do this same technique without actually copying the two columns on paper.*

5. Complete the following sentence: If two columns (or two rows) in a square matrix are equal, then the determinant of this matrix ....

6. Calculate the following determinant by the use of row transformations:

$$\begin{vmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{vmatrix}$$

7. Have a close look at your work in the previous problem. You should discern a pattern. I give you a 100×100 matrix with the same pattern, namely 2's on the diagonal and -1's next to them, 0's everywhere else. What is the determinant of this matrix?

8. Calculate

$$\det \begin{bmatrix} 3 & 5 & 1 & 2 \\ 1 & 10 & 1 & 2 \\ 0 & -5 & 1 & 2 \\ 4 & 15 & 2 & 1 \end{bmatrix}$$

9. Find the following determinant, using a smart sequence of row or column transformations to get many zeros easily:

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 1 \\ 3 & 3 & 3 & 3 & 3 & 2 & 1 \\ 4 & 4 & 4 & 4 & 3 & 2 & 1 \\ 5 & 5 & 5 & 4 & 3 & 2 & 1 \\ 6 & 6 & 5 & 4 & 3 & 2 & 1 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{vmatrix}$$

Hint: There are at least two choices: create zeros in the first column below a leading one in the first row, then cofactor expansion of the first column. Or: bottom-up row operations: for $i = n, n-1, \ldots 2$, subtract the $(i-1)^{\text{st}}$ row from the i^{th} row.

10. Let

$$V := \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

Before you start calculating, answer the question: What value does V have, if $x = y$? — Now calculate V for general x, y, z. Here comes the punchline: I want you to write the result not as a mess with 6 terms, but transform it into a product $(* - *)(* - *)(* - *)$, where you have to figure out what the $*$ stand for. And the very first question in this problem will serve as a hint how to accomplish this task.

11. In the following matrix the symbols e and o stand for even and odd integers respectively. I don't reveal for which exactly. Different e's may stand for different even numbers, and similarly for o's. I claim that from this information you can already know that the determinant

$$\begin{vmatrix} o & e & e & e & e \\ e & o & e & e & e \\ e & e & o & e & e \\ e & e & e & o & e \\ e & e & e & e & o \end{vmatrix}$$

is not zero. Explain *how* you can come to this conclusion.

12. For which numbers a does the linear system

$$\begin{aligned} ax + 2y - 3z &= 5 \\ 2x + 3y - 2z &= 7 \\ 3x + 2y - 5z &= 9 \end{aligned}$$

have exactly one solution? Use Cramer's rule to find y in this case. (Finding x and z is not required.)

13. For A as given below, find $\operatorname{adj}(A)$.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & a & -3 \\ 0 & -2 & a \end{bmatrix}$$

14. Let A be a 5×5 matrix with determinant -2. Simplify $\operatorname{adj}(\operatorname{adj}(A))$ to an expression that can be typed on the computer keyboard with 5 or less keystrokes. Make sure that you give a clear calculation to justify your result. *Hint: If you find this difficult, you should collect formulas about the adjoint of a matrix and the determinant of matrices from the glossary, and you need to look which are useful and try to put them together.* Quote the formulas you are using.

15. If all entries of a square matrix A are integers and if furthermore $\det A = 1$, can you conclude that all entries of A^{-1} are integers? If the answer is yes, explain why; if the answer is no, explain why not (counterexample).