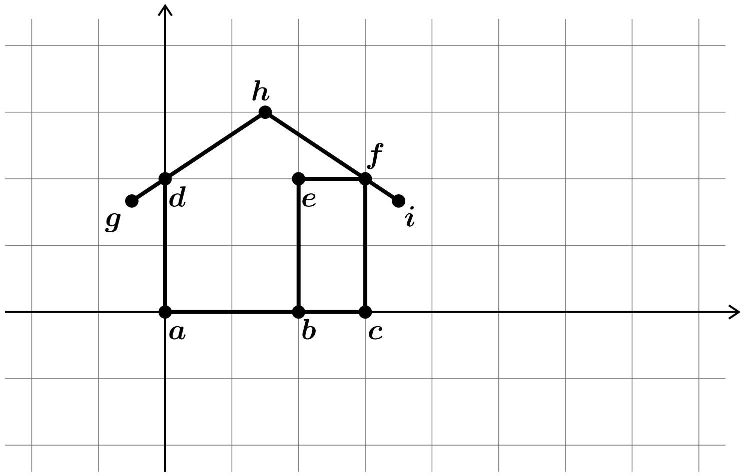
Homework on Matrix Transformations (Chapter 4.9-11)
UTK – M251 – Matrix Algebra

1–6 In the following figure, showing the $\boldsymbol{x}$ plane, the points $\boldsymbol{a}$,..., $\boldsymbol{i}$ are as pictured, and their coordinates are given as well:



$\boldsymbol{a} = [0, 0]^T$
$\boldsymbol{b} = [2, 0]^T$
$\boldsymbol{c} = [3, 0]^T$
$\boldsymbol{d} = [0, 2]^T$
$\boldsymbol{e} = [2, 2]^T$
$\boldsymbol{f} = [3, 2]^T$
$\boldsymbol{g} = [-\frac{1}{2}, \frac{5}{3}]^T$
$\boldsymbol{h} = [\frac{3}{2}, 3]^T$
$\boldsymbol{i} = [\frac{7}{2}, \frac{5}{3}]^T$

You now have a list of matrices A, B, C, ... that generate linear mappings $\boldsymbol{x} \mapsto \boldsymbol{y} = A\boldsymbol{x}$, $\boldsymbol{x} \mapsto \boldsymbol{y} = B\boldsymbol{x}$,...

For each mapping, draw the image of the above figure in the $\boldsymbol{y}$ plane and determine the effect of the matrix: rotation, dilation, reflection, or give an informal description of what the matrix does, if none of these applies.

Each matrix counts as a separate problem and is worth 2 points

Also write the determinant in each figure.

I'll make available for download a number of blank sheets with coordinate grids.

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad D = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 3 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

7. For a *unit* (column) vector $\boldsymbol{u}$, consider the matrix $R = I - 2\boldsymbol{u}\boldsymbol{u}^T$. Show that $R^2 = I$, show (generally, not for a specific example only) that $\boldsymbol{u}$ is an eigenvector with eigenvalue -1 for R.

Also write down the matrix R in the specific example where $\boldsymbol{u} := [\frac{1}{3}, \frac{2}{3}, \frac{-2}{3}]^T$. Calculate $\det R$ in this specific example.

8. Find the eigenvalues λ_1 and λ_2 of the matrix $A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$. For each of these two eigenvalues, give a corresponding eigenvector.

9. Show that the matrix

$$Q = \frac{1}{25}\begin{bmatrix} 16 & -15 & 12 \\ 12 & 20 & 9 \\ -15 & 0 & 20 \end{bmatrix}$$

satisfies $Q^T Q = I$ and $\det Q = 1$. It should therefore represent a rotation in 3-space. Find the axis of rotation. (Read the relevant glossary entries from pages 19 and 21 if you don't know how to do this.)

10. Find the matrix of a reflection in the plane, where the line of reflection is given by $x_2 = \frac{1}{2}x_1$.

11. Find the missing entries, if the following matrix is orthogonal (i.e., satisfies $Q^T Q = I$):

$$Q = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/\sqrt{17} & * & -2/\sqrt{17} \\ * & * & -7/(3\sqrt{17}) \end{bmatrix}$$

The following exercises review previous material:

12. Is the following always true? If yes, give an argument (proof), if no, give a counterexample.

"If A is invertible and $AB = 0$, then $B = 0$"

Same question for the statement

"If A is symmetric and $AB = 0$, then $B = 0$"

Same question for the statement

"If A is orthogonal and $AB = 0$, then $B = 0$"

13. Is the following always true? If yes, give an argument (proof), if no, give a counterexample.

"If A is a square matrix and A^2 is symmetric, then A is symmetric."

14. Is the following always true? If yes, give an argument (proof), if no, give a counterexample.

"If A is a square matrix and AA^T is singular, then A is singular"

Same question for the statement

"If A is a square matrix and AA^T is invertible, then A is invertible"

15. You know (at least I hope you know) that for square matrices A, B of the same size, $\det(AB) = \det(BA)$, because both are equal to $(\det A)(\det B)$. Now for nonsquare matrices A and B of sizes $m \times n$ and $n \times m$ respectively, is it still true that $\det(AB) = \det(BA)$? If yes, give a proof; if no, give a counterexample.

16. Let me remind you about the trace: we had discussed the fact that for matrices A and B of sizes $m \times n$ and $n \times m$ respectively, it is true that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. Now below, I take three matrices U, V, and W, all square of the same size. One of the following two statements is true for any such choices of U, V, W, the other sometimes (usually) false (I.e., false for *some* choices of U, V, W). You have to find which of the two statements is true (and explain why) and which is false (and give a counterexample for it).

Statement (a): $\mathrm{tr}(UVW) = \mathrm{tr}(VUW)$
Statement (b): $\mathrm{tr}(UVW) = \mathrm{tr}(VWU)$

17. Suppose you have vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, with $\boldsymbol{w} \neq \boldsymbol{0}$. Can you conclude from $\boldsymbol{u} \cdot \boldsymbol{w} = \boldsymbol{v} \cdot \boldsymbol{w}$ that $\boldsymbol{u} = \boldsymbol{v}$? If yes, give an argument, if no, give a counterexample.

18. For column vectors ($n \times 1$ matrices) $\boldsymbol{u}$ and $\boldsymbol{v}$ and an $n \times n$ matrix A, is it always true that $\boldsymbol{u} \cdot (A\boldsymbol{v}) = (A^T\boldsymbol{u}) \cdot \boldsymbol{v}$, or can it sometimes fail to be true? If "always true", give an argument, if "sometimes false", give a counterexample.

19. If a square matrix P satisfies $P^2 = P$, what can you conclude about $\det P$?

20. If a square matrix P satisfies $P^2 = P$ and you have a nonzero vector $\boldsymbol{v}$ and a number λ such that $P\boldsymbol{v} = \lambda\boldsymbol{v}$, what can you conclude about the number λ?