A few comments about induction

We are speaking of the *axiom* of induction. By calling it an axiom, we are saying it is foundational to the theory. It is *not* a theorem to be proved; rather its truth is assumed. These unproved and unprovable assumptions called axioms actually specify the objects with which we are dealing; in this particular case of induction, the axiom of induction is a crucial part of explaining what natural numbers are: it specifies one crucial property of the natural numbers. You may find it difficult to grasp why we should need such specification; after all, the numbers one, two three, etc., belong to your elementary school, if not preschool, experience. Think of it this way:

If you want to learn something about water, don't ask a fish.

In order to help you appreciate the meaning of the axiom of induction, I'll give you a situation where it does *not* hold; so to say, I'll lift the fish out of the water (but it won't be harmful).

The French language has a gender distinction in the grammar, that also affects numbers: if you say "one woman" the word "one" translates as "une", if you say "one man", the same "one" translates as "un". The distinction does not show up for "two". "three", etc., but recurs at "twenty-one". You could say that the natural numbers of the French language are structured differently from the natural numbers of mathematics:

$$\begin{array}{cccccc} 1m \searrow & & \nearrow & 21m \searrow & \\ & 2 \to 3 \to 4 \to \dots 19 \to 20 & & & 22 \to \dots \\ 1f \nearrow & & \searrow & 21f \nearrow & \end{array}$$

A similar, more sophisticated three-way distinction applies to Russian:

$$\begin{array}{cccccccc} 1m \searrow & & & & \nearrow 21m \searrow & & & \\ & 2mn \searrow & & & & 22mn \searrow & & \\ 1n \nearrow & & 3 \to 4 \to \dots 19 \to 20 & \to & 21n \nearrow & & 23 \to 24 \to \dots \\ 1f \longrightarrow & 2f \nearrow & & & \searrow 21f \longrightarrow & 22f \nearrow & \end{array}$$

Now you may ask: Why is he telling me that stuff; it has nothing to do with mathematics!? — My response is: That's exactly why! The distinctions mentioned are indeed irrelevant to mathematics, and the axiom of induction is part of our way of saying so:

This axiom says: If some property holds for *the* first number ("$P(1)$ is true") and if its validity for some number implies the validity for *its* successor ("$P(k)$ implies $P(k+1)$ for every k"), then it holds for all numbers. Note the definite article in "*the* first number". Even if we abandoned it in favor of an indefinite "a beginning number to account for the language examples mentioned above, the axiom would still preclude these models from being the mathematical natural numbers, because the successor mechanism would leave out the competing beginning numbers. Another property of the mathematical numbers is hidden in the phrase "*its* successor $n+1$". It tacitly assumes that there is just one successor (which we call $n+1$), unlike the forks after 20 in the above language models. These details could be formalized more in a true axiomatic foundation of the set of natural numbers, but such an endeaver goes beyond the scope of this course. Instead, we have only grabbed the one piece of this axiomatic foundation that you need in "everyday" mathematical life, namely induction.

Let me give you another illustration (fish out of the water) how the axiom of induction makes more precise what we mean by the set of natural numbers: The English language has this nice number "a gazillion", which is perceived as a number *beyond* all other imaginable numbers. Not the largest number of course, because, clearly, what's more than a gazillion? A gazillion and one, right? Let me invent the symbol ω for the "number one gazillion". The intuitive idea behind the English use of this word may be close to the following:

$$1 \to 2 \to 3 \to 4 \to 5 \to \dots \to 1000000 \to 1000001 \to \dots \qquad \Big| \; \omega \to \omega + 1 \to \omega + 2 \to \dots$$

If you try to follow the steps of the axiom of induction, you will reach all natural numbers, but you won't get as far as "one gazillion", let alone beyond it. The axiom of induction therefore stipulates that this "gazillion" ω is *not* part of our (mathematical, and science) number system.[1]

[1] with due apologies to those colleagues in set theory that do study theories containing ω and its brethren

The role of the axiom of induction would be clearer if we viewed it as part of a larger set of axioms that would together give a precise description of the set $\mathbf{N}$. In this framework, we would even *define* addition and multiplication inductively. Such an endeavour however goes far beyond the scope and rationale of M300. Among the whole bunch of axioms, we only grab the one that you encounter in proofs in everyday mathematical life, namely induction. It is however also for this limitation of purpose that I choose to skip Thm. 4.1 of the course notes and advise you to ignore its proof.

However, I want to show you how a definition by induction works: We have defined

$$\sum_{k=1}^{n} a_k := a_1 + a_2 + \ldots + a_n$$

The conspicuous part of the definition is the '$\cdots$'. In reality, a definition by induction is hiding behind these dots:

$$\sum_{k=1}^{1} a_k := a_1 \ , \quad \sum_{k=1}^{n+1} a_k := \left(\sum_{k=1}^{n} a_k \right) + a_{n+1} \quad \text{for each} \quad n \in \mathbf{N}$$