Assume R is a field. Let's try to find an inverse for $(a,b) \neq (0,0)$

$$(a,b)(x,y) = (ax-by,\ ay+bx) = (1,0)$$

We need: $ax - by = 1 \implies a\cdot a x - a\cdot b\cdot y = a$

$bx + ay = 0 \implies b\cdot b x + b a y = 0$

add 'em $(aa+bb)x = a$

Similarly - - - - - $(b\cdot b + a\cdot a)y = -b$

Therefore: If $a\cdot a + b\cdot b = 0$, but $(a,b) \neq 0$, we cannot have an inverse (since $0 \cdot$ anything $= 0$)

If $(a\cdot a + b\cdot b) \neq 0$, the only candidate for an inverse is

$\left((aa+bb)^{-1}a,\ -(aa+bb)^{-1}b\right)$. It is indeed an inverse, as can be checked by plugging it back in.

~~The~~ Conclusion: If $a\cdot a + bb = 0$ can happen only (in the field R) for $a=b=0$, then $R[i]$ is a field.

Therefore: $\mathbb{R}[i]$, $\mathbb{Q}[i]$ are fields, but $\mathbb{Z}_2[i]$ is not

$\mathbb{Z}_2 = \{E, O\}$

Note: One can simplify the condition

"$aa+bb=0$ only if $a=b=0$" : (1)

~~Reword it: "aa+bb=0 and b~~

to "$x\cdot x + 1 \neq 0$ for all x" (2)

Why? Suppose (1) holds. Then (2) holds, by choosing ~~x~~ $a := x$, $b := 1$

Suppose ~~(2) holds, and~~ (1) fails. Then there exists a, b with $a^2+b^2=0$, but $b \neq 0$ (or $a \neq 0$, which is analogous)