#9: We show :

(1) If $ab=0$ and a has a mult. inverse, then $b=0$.

(2) If $ab=0$ and b has a mult. inverse, then $a=0$.

Proof (1) If $ab=0$ and a^{-1} exists, then

$$\underbrace{a^{-1}(ab)} = \underbrace{a^{-1}0}$$

$= 0$ as we showed as a consequence of the ring axioms.

$= (a^{-1}a)\,b$ by (A$\cdot$)

$= 1\ b$ ~~by the~~ since a^{-1} is inverse of a

$= b$ by (N$\cdot$)

So we have showed $b=0$ as promised.

Question: It said "in any ring", it didn't say "in any ring with identity". So how could we use (N$\cdot$)?

Answer: When we say "if a has a mult. inverse", we require (N$\cdot$), since the concept of mult. inverse makes no sense without mult. identity.

Conversely, if R does not have a mult. identity, then clearly neither a nor b can have a mult. inverse, b/c the concept doesn't make sense. So in this case, the claim is trivially true.

(2) is proved analogously:

If $ab=0$ and b^{-1} exists then

$$\underbrace{(ab)b^{-1}} = 0\cdot b^{-1} = 0$$

$$= a(bb^{-1}) = a\cdot 1 = a$$