#47 (NEW VERSION):

Let's list inverses in $\mathbb{Z}_{13}$ for convenience: ($[\cdot]_{13}$ omitted from notation)

X	1	2	3	4	5	6	7	8	9	10	11	12
X^{-1}	1	7	9	10	8	11	2	5	3	4	6	12

I get these by trial & error, plus wisdom, eg:

$2\cdot 6 \equiv -1 \mod 13$, hence $2^{-1} = -6 = 7$ in $\mathbb{Z}_{13}$ (and $7^{-1} = 2$)

$3\cdot 4 \equiv -1 \mod 13$, hence $3^{-1} = -4 = 9$ in $\mathbb{Z}_{13}$ (and $9^{-1} = 3$)

Or I could start factoring $\underset{2\cdot 7}{\underset{\shortparallel}{13+1}}$, $\underset{3\cdot 9}{\underset{\shortparallel}{2\cdot 13+1}}$, $\underset{5\cdot 8}{\underset{\shortparallel}{\underset{4\cdot 10}{\underset{\shortparallel}{3\cdot 13+1}}}}$, ...

and fill in the table as I produce these factors

$$a = a_0 = X^3 - 7X^2 + 3X + 3$$

$$b = a_1 = X^3 - 6X^2 + X + 7$$

$$a_0 = 1\cdot a_1 + \underbrace{-X^2 + 2X - 4}_{a_2}$$

$$a_1 = (-X+4)\, a_2 + \underbrace{2X+10}_{a_3}$$

$$a_2 = (6X-3)\, a_3 + 0$$

$\boxed{\text{So, } 2X+10 \text{ is a gcd}}$

long divisions: in $\mathbb{Z}_{13}$

$$\begin{array}{r|l} & -X+4 \\ \hline -X^2+2X-4 & X^3 - 6X^2 + X + 7 \\ & X^3 - 2X^2 + 4X \\ \hline & -4X^2 - 3X + 7 \\ & -4X^2 + 8X - 3 \\ \hline & 2X + 10 \end{array}$$

of course, one could write (eg) $2X-3$ just as well

$$\begin{array}{r|l} & 6X-3 \\ \hline 2X+10 & -X^2 + 2X - 4 \\ & 12X^2 + 8X \\ \hline & -6X - 4 \\ & -6X - 30 \\ \hline & 0 \end{array}$$

$(-4-(-30) = 26 = 0$ in $\mathbb{Z}_{13})$

If you want a monic polynomial:

$$2^{-1}(2X+10) = X + 2^{-1}\cdot 10 = X + 70 = \underline{\underline{X+5}}$$