#54: We have to show: if F is a field with infinitely many elements, then $p(x_i) = q(x_i)$ for all $x_i \in F$ implies $p = q$

Let $g := p - q$. We then have, by hypothesis,

$$g(x_i) = 0 \text{ for all } x_i \in F$$

But then $X - x_i$ must be a factor of g. This is impossible if F has more elements than $\deg g$

#55: By trying all possibilities (or by using the prime factor decomposition in $\mathbb{Z}$): the roots of X^2 in $\mathbb{Z}_{25}$ are: $0, 5, 10, 15, 20$

$X \cdot X = (X-5)(X+5) = (X-10)(X+10) = (X-15)(X+15) = (X-20)(X+20)$ are three (!) different factorizations of X^2 in $\mathbb{Z}_{25}$

Three only? - Yes. $X - 5 = X + 20$. less delusively written:

$$X \cdot X = (X+5)(X+20) = (X+10)(X+15)$$

#56: The elements of $\mathbb{Z}_2[X]/I$ are:

$[0]_I =: 0$
$[1]_I =: 1$
$[X]_I =: j$
$[X+1]_I = j+1$
$[X^2]_I = j^2$
$[X^2+1]_I = j^2+1$
$[X^2+X]_I = j^2+j$
$[X^2+X+1]_I = j^2+j+1$

Note: Since $[X^3+X+1]_I = [0]_I$ we have $j^3+j+1 = 0$

I'll omit the trivial rows/columns 0 and 1 from the table.

	j	$j+1$	j^2	j^2+1	j^2+j	j^2+j+1
j	j^2	j^2+j	$j+1$	1	j^2+j+1	j^2+1
$j+1$	j^2+j	j^2+1	j^2+j+1	j^2	1	j
j^2	$j+1$	j^2+j+1	j^2+j	j	j^2+1	1
j^2+1	1	j^2	j	j^2+j+1	$j+1$	j^2+j
j^2+j	j^2+j+1	1	j^2+1	$j+1$	j	j^2
j^2+j+1	j^2+1	j	1	j^2+j	j^2	$j+1$

From the table we can read of now: $(j+1)^{-1} = j^2+j$