*** 1:** *[3 points]*
Determine positive integers a and b such that the following statement becomes true for every $x \in \mathbb{Z}$:

$x \equiv 2 \pmod 5$ and $x \equiv 1 \pmod 3$ if and only if $x \equiv a \pmod b$

$x \equiv 7 \quad \text{mod } 15$

2: *[6 points]*

* (a) Evaluate the Euler function $\varphi(30)$.

*** (b) Write down the general formula for $\varphi(n)$, given a prime factor decomposition $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ and use it to find all those n for which $\varphi(n) = 4$. *(For partial credit, find some such n)*

$\varphi(30) = \varphi(2 \cdot 3 \cdot 5) = 1 \cdot 2 \cdot 4 = 8 \qquad (2 \text{pts})$

$\varphi(p_1^{\alpha_1} \cdots p_k^{\alpha_k}) = (p_1 - 1) p_1^{\alpha_1 - 1} \cdot \ldots \cdot (p_k - 1) p_k^{\alpha_k - 1} \qquad (2 \text{pts})$

For $\varphi(n) = 4$ all prime divisors of n must satisfy $(p-1) \mid 4$
so $p \in \{2, 3, 5\}$ Moreover $3, 5$ could not occur in higher powers.
Complete inspection of cases:
$n \in \{2^3, 2^2 \cdot 3, 2 \cdot 5, 5\} = \{5, 8, 10, 12\} \qquad (4 \times \tfrac{1}{2} \text{pt})$

Note: if the formula isn't present:
1 pt for 1st sol'n
$\tfrac{1}{2}$ pt for each further sol'n

*** 3:** *[3 points]*
Complete the sentence correctly:

The ring $\mathbb{Z}_n$ is a field if and only if n is a prime number.

*****4:** *[6 points]*

3 pts each

(a) Find a single-digit number x such that $143^{572} \equiv x \pmod 7$

(b) Find a positive integer n such that $47^n \equiv 1 \pmod{113}$.

Explain your reasoning.

(a) $143^{572} \equiv 3^{572}$ since $143 \equiv 3 \pmod 7$

$3^6 \equiv 1 \text{ mod } 7 \quad [\, a^{\varphi(n)} \equiv 1 \text{ mod } n \text{ if } \gcd(a, n) = 1 \,]$

$572 = 95 \cdot 6 + 2$. So $3^{572} \equiv (3^6)^{95} \cdot 3^2 \equiv 9 \equiv \underline{2} \text{ mod } 7$

(b) Since ~~p~~ 113 is prime $a^{p-1} \equiv 1 \text{ mod } p$ for all a with $(a, p) = 1$

So $n = 112$ is a solution. (It wasn't asked whether n is the smallest solution.