**5: [3 points]
Let E be the ring of even integers (with the usual addition and multiplication).

Prove or disprove: "$\theta : x \mapsto 2x, \mathbb{Z} \to E$ is a ring homomorphism."

θ does satisfy $\theta(x+y) = \theta(x) + \theta(y)$,
but it fails $\theta(xy) = \theta(x)\theta(y)$, so it is not a hom.

$\theta(xy) = 2xy$; $\theta(x)\theta(y) = 2x \cdot 2y = 4xy$

**6: [6 points]
Find a GCD of $a = X^3 - 3X^2 - 6X + 8$ and $b = X^3 + 2X^2 - X - 2$ in the ring $\mathbb{Q}[X]$.

$$\begin{array}{r|l} & 1 \\ X^3+2X^2-X-2 & X^3-3X^2-6X+8 \\ & X^3+2X^2-X-2 \\ \hline & -5X^2-5X+10 \end{array}$$

$$\begin{array}{r|l} & -\tfrac{1}{5}X - \tfrac{1}{5} \\ -5X^2-5X+10 & X^3+2X^2-X-2 \\ & X^3+X^2-2X \\ \hline & X^2+X-2 \\ & X^2+X-2 \\ \hline & 0 \end{array}$$

A gcd(a,b) is
$-5X^2-5X+10$

If we want the monic gcd:
it's X^2+X-2

**7: [5 points]
Use modular arithmetic to show that the following determinant is not 0:

$$\begin{vmatrix} \frac{71}{7} & 12 & -63 & 0 \\ 43 & 32 & 36 & -27 \\ 14 & -53 & 33 & 25 \\ 44 & 64 & 14 & -52 \end{vmatrix} = \frac{1}{7} \cdot \underbrace{\begin{vmatrix} 71 & 7\cdot 12 & -7\cdot 63 & 0 \\ 43 & 32 & 36 & -27 \\ 14 & -53 & 33 & 25 \\ 44 & 64 & 14 & -52 \end{vmatrix}}_{=:D}$$

and it suffices to show that this new determinant is $\neq 0$

For partial credit (lose $1\frac{1}{2}$ points) you may do the problem with 71 instead of $\frac{71}{7}$.

$$D \equiv \begin{vmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 1 \\ 2 & 1 & 2 & 2 \end{vmatrix} \pmod 3$$

$D \equiv -2\cdot 2 \cdot 1 \cdot 2 \equiv 1 \mod 3$

hence $D \neq 0$

Some calculated mod 4 or mod 7, which works, but not quite as neatly.

**8: [4 points]
The ring $\mathbb{Z}[\sqrt{2}]$ consists of all numbers $a + b\sqrt{2}$, with $a, b \in \mathbb{Z}$, and the usual addition and multiplication. The ring $\mathbb{Z}[i]$ is defined similarly (with $i^2 = -1$). Give an argument that there cannot exist an isomorphism $\theta : \mathbb{Z}[\sqrt{2}] \to \mathbb{Z}[i]$.

The eqn $x^2+1=0$ has two sol's in $\mathbb{Z}[i]$, but none in $\mathbb{Z}[\sqrt{2}] \subset \mathbb{R}$ so the two rings cannot be isomorphic

BUT As long as you don't have integers, you cannot calculate mod anything.