This is a careful study of the eigenvalue problem $-\Delta u = \lambda u$ in the rectangle $\Omega = ]0, \frac{\pi}{3}[ \times ]0, \frac{\pi}{2}[$ with $u = 0$ on $\partial\Omega$.

Separation of variables retrieves the eigenfunctions $u_{k,l} = \sin 3kx \sin 2ly$, corresponding to the eigenvalues $\lambda = 9k^2 + 4l^2$. It can be shown by methods that remind us more of abstract algebra than of calculus (namely consideration of symmetry) that 'every' solution to the problem in the rectangle can be found by separation of variables. (We'll see a bit below why I put the word 'every' in quotation marks.) Of course, for odd-shaped domains, separation of variables will likely not retreive a single solution. So the concern whether our method retrieves all solutions is certainly legitimate. In this example however we can be sure that we have found 'all' solutions, bacause we can practically verify the claim that 'every reasonable function' can be written as a(n infinite) superposition of the eigenfunctions (generalized Fourier series). This claim would be guaranteed to fail, if our method had failed to retrieve even a single soution. Since in our examples the eigenfunctions we have retrieved by separation are products of sines, we know that we can write every reasonable function $u(x, y)$ as a double Fourier series

$$u(x, y) = \sum_{k=1}^{\infty} a_k(y) \sin 3kx$$

where $a_k(y) = \frac{2}{\pi/3} \int_0^{\pi/3} u(x, y) \sin 3kx \, dx$; and then $a_k(y)$ can likewise be so written:

$$a_k(y) = \sum_{l=1}^{\infty} a_{kl} \sin 2ly$$

where $a_{kl} = \frac{2}{\pi/2} \int_0^{\pi/2} a_k(y) \sin 2ly \, dy$. Putting this together, we can write

$$u(x, y) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{kl} u_{k,l}(x, y)$$

which indeed writes an arbitrary function $u$ as a superposition of the eigenfunctions $u_{k,l}$ we have retrieved; and this bears witness to the fact that no eigenfunction has escaped our method of separation of variables.

In the table on the right we have listed all those eigenvalues $\lambda$ that are $\leq 500$, depending on $(k, l)$. All abstract theorems will of course list the eigenvalues indexed with one index, by size. We have done this for the first 32 eigenvalues in the figure below. You see that it can happen occasionally that eigenvalues appear repeatedly: in our example, 180 appears twice. This means that you have correspondingly many *linearly independent* eigenfunctions, in our example $u_{4,2}(x, y) = \sin 12x \sin 6y$ and $u_{2,6}(x, y) = \sin 6x \sin 12y$.

| $l \setminus k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 13 | 40 | 85 | 148 | 229 | 328 | 445 |
| 2 | 25 | 52 | 97 | 160 | 241 | 340 | 457 |
| 3 | 45 | 72 | 117 | 180 | 261 | 360 | 477 |
| 4 | 73 | 100 | 145 | 208 | 289 | 388 | |
| 5 | 109 | 136 | 181 | 244 | 325 | 424 | |
| 6 | 153 | 180 | 225 | 288 | 369 | 468 | |
| 7 | 205 | 232 | 277 | 340 | 421 | | |
| 8 | 265 | 292 | 337 | 400 | 481 | | |
| 9 | 333 | 360 | 405 | 468 | | | |
| 10 | 409 | 436 | 481 | | | | |
| 11 | 493 | | | | | | |

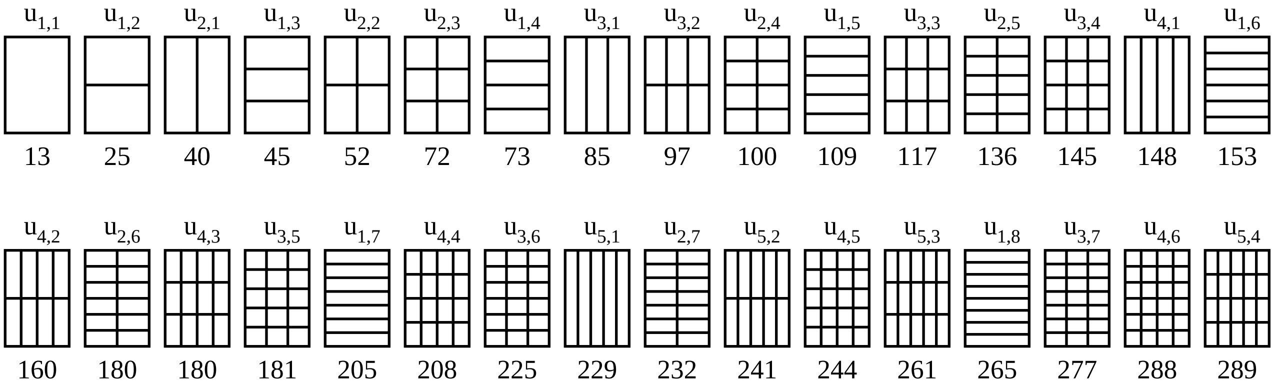
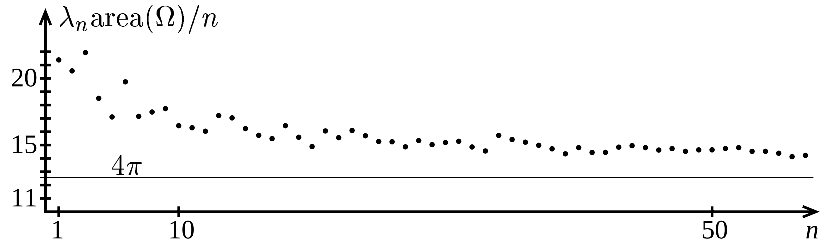



Figure 1: *This figure lists the eigenvalues and eigenfunctions for the equation $-\Delta u = \lambda u$ for the rectangle of sides $\pi/3$ and $\pi/2$ with Dirichlet boundary conditions. The lines show just where the eigenfunctions vanish (so-called nodal lines). In the regions between the lines (called nodal domains), the eigenfunction is alternatingly positive and negative.*

The easiest interpretation of such eigenfunctions is that of a vibrating membrane $\Omega$. The eigenfunctions give the shape of the vibration ($u > 0$: membrane up, $u < 0$ membrane down), and the eigenvalue gives the square of the frequency with which the membrane vibrates. The actual vibration would be given as $v(t, x, y) = \cos(\sqrt{\lambda}\, t)u(x, y)$. We'll see this soon when we study wave equations.

The figure illustrates also the fact that the number of nodal domains of the $n^{\text{th}}$ eigenfunction is at most $n$. We can also use the table to illustrate the claim that $\lambda_n^{2/2}\text{area}(\Omega)/n \to 4\pi$ as $n \to \infty$.




The speed of convergence is not spectacular, but note also the visual effect of shifting the origin of the coordinate system.

Finally let us have a look at the case $\lambda = 180$, where we have two linearly independent eigenfunctions $u_{4,2}(x, y) = \sin 12x \sin 6y$ and $u_{2,6}(x, y) = \sin 6x \sin 12y$. We could take other eigenfunctions in this case, for instance $\frac{1}{2}(u_{2,6} + u_{4,2})$. This eigenfunction would *not* have been retrieved by separation of variables alone, but it is retrieved by separation and superposition. It is this qualification that needs to be made above, where I said (with quotation marks, warning of some technicality being omitted) that our method retrieves 'every' solution.

For the function $\frac{1}{2}(u_{2,6} + u_{4,2})$, nodal lines take more effort to draw, and they would not be straight. The fact that the nodal lines in the figure are all straight is very exceptional, and due to the very special domain, a rectangle.