Hwk for M443

We generally assume $z = x + iy$, $w = u + iv$ here, with x, y, u, v real and z, w complex.

Hwk 11: Using knowledge from real variable calculus, show that the function given by $f(x+iy) = e^x \operatorname{cis} y$ is holomorphic (analytic) in $\mathbb{C}$. This will be the complex exponential function.

Hwk 12: Using knowledge from real-variable calculus about ln and inverse trigs, show that the function given by $g(x + iy) = \ln r + i \arctan \frac{y}{x}$ is holomorphic (analytic) in the right half plane $\operatorname{Re} z > 0$.

Show that $f(g(z)) = z$ for all z in the right half plane $\operatorname{Re} z > 0$. Show also $g(f(z)) = z$ for a certain set of z, which you should specify (*).

Note that $g(f(z))$ is *not defined* for those z for which $f(z)$ falls out of the right half plane, and that $g(f(z)) = z$ may *fail to be true* for certain z, even if $g(f(z))$ is defined. (Give examples.)

This serves as one choice (among many) of a complex logarithm function. (Other domains and other choices of solutions to $e^w = z$ may be considered.)

(*) footnote: Remember that the arctan function is defined on all of $\mathbb{R}$, with values in the open interval $]-\frac{\pi}{2}, \frac{\pi}{2}[$. This should help specifying an appropriate domain for z.

Hwk 13: Find the two solutions $w_{1,2}$ of $z = w^2$, expressed in real and imaginary part: $u = u(x, y)$, $v = v(x, y)$. Show that these formulas are well-defined for $z \in \mathbb{C} \setminus]-\infty, 0[$.

Show that either function is holomorphic (analytic) in $\mathbb{C} \setminus]-\infty, 0]$. These serve as choices (among many more) for a complex square root function. (Other choices of domain may be considered.)

Note: The domains refer to the complex plane minus the negative real axis. So the *real* symbol $-\infty$, not the complex symbol ∞, is used in these domain specifications. The context of interval notation should have made this clear anyways, just pointing it out again to be safe.

Hwk 14: The real function

$$f(x) := x|x| = \begin{cases} x^2 & \text{for } x \geq 0 \\ -x^2 & \text{for } x < 0 \end{cases}$$

is known to be continuously differentiable, but not twice differentiable. In this problem you will attempt to construct a complex function displaying a similar feature (and learn that none of the attempts works). Subsequent theorems will imply that any such attempt (to construct a once, but not twice differentiable function) is doomed and cannot succeed. But trying it out anyways *is instructive nevertheless.*

Consider these four functions: $f_1(z) := x|z|$, $f_2(z) := z|x|$, $f_3(z) := z|z|$, $f_4(z) := \begin{cases} z^2 & \text{for } \operatorname{Re} z \geq 0 \\ -z^2 & \text{for } \operatorname{Re} z < 0 \end{cases}$
and determine where each of these fails to be differentiable.

From book: pg 43ff: #20, #21 (read #16 for definition of 'entire linear transformation'); #22; #24