Hwk for M443

Hwk 15: Show that the series $\sum_{n=0}^{\infty} \frac{1}{(z-n)^2}$ converges uniformly on every compact subset of $\mathbb{C} \setminus \mathbb{N}_0$ (where $\mathbb{N}_0$ denotes the set of nonnegative integers).

Comment: in $\mathbb{C}$, or $\mathbb{R}^n$, 'compact' is equivalent to 'closed and bounded'; either by the theorem of Bolzano Weierstrass for those who have studied advanced calculus and have seen an independent definition of compact; or by definition adopted for this class, for those who have not encountered the word compact independently before.

Hwk 16: Show that the series $\sum_{n=0}^{\infty} \frac{1}{z^2-n^2}$ converges uniformly on every compact subset of $\mathbb{C} \setminus \mathbb{Z}$.

Note: We will study these series in more detail later.

Hwk 17: (This is a pure real variable problem, inserted for comparison purposes:)
Show that the series $\sum_{n=0}^{\infty} \frac{\cos nx}{n^2}$ converges uniformly on $\mathbb{R}$ (hence uniformly on every compact set of $\mathbb{R}$), but that the formally twice differentiated series $-\sum_{n=0}^{\infty} \cos nx$ does not converge for any x.

Comment: This is in contrast to the theorem that series of holomorphic functions in a domain in $\mathbb{C}$, that converge uniformly on every compact set, can be differentiated term by term, resulting in a series that is again uniformly convergent on every compact set. We see here that there is no real-variable analog for the stated complex variables theorem. In concluding so, we do not need to decide whether the 'inheritance' of the uniform convergence property on compact sets gets lost in the first or in the second differentiation step.

$\mathbb{R}$ is not open in $\mathbb{C}$, hence not a domain. We will soon define trig functions for complex arguments and identify the cos function as holomorphic in $\mathbb{C}$. So the issue is with the domain hypothesis, not with the holomorphic hypothesis.

Hwk 18: Comment: Obviously, if a series is uniformly convergent in a domain, then it is uniformly convergent on every compact subset of that domain (on *any* subset, for that matter). So 'uniformly convergent in G' is the *stronger* property than 'uniformly convergent on every compact subset of G'. But it is *not* the 'better' or 'more useful' property, in the following sense: While we have seen that 'uniformly convergent on compact subsets' is inherited from a series of holomorphic functions to the series of derivatives, the stronger property 'uniformly convergent on the whole domain' is not inherited. This is the purpose of the present problem.

Show that $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ is uniformly convergent in the disc $|z| < 1$ (*), the twice formally derived series is not uniformly convergent in this disc. Again we don't care to decide whether the uniform convergence gets lost in the first or in the second differentiation step. It actually gets lost in the first differentiation here, but that's more subtle to prove than what I want for this homework.

At (*), note that you can actually show a little bit more than stated here, with the same effort.