Homework
UTK – M532 – Ordinary Differential Equations II
Sprind 2005, Jochen Denzler, MWF 11:10–12:05, Ayres 104

36. Eigenvalues and Matrix Exponential
Show that, if $A\boldsymbol{x} = \lambda\boldsymbol{x}$, then $\exp(tA)\boldsymbol{x} = e^{t\lambda}\boldsymbol{x}$.

37. Wrong Reasoning on Matrix Exponentials and Variable Coefficient ODEs
J.H. Quick (a student invented by Donald E. Knuth along with the TeX typesetting system) reasons that he can solve the IVP $\dot{\boldsymbol{x}} = A(t)\boldsymbol{x}$, $\boldsymbol{x}(0) = \boldsymbol{x}_0$ in virtually the same way as the system with *constant* coefficients $\dot{\boldsymbol{x}} = A\boldsymbol{x}$, $\boldsymbol{x}(0) = \boldsymbol{x}_0$. Namely, he reasons:

Let $B(t) := \int_0^t A(\tau)\,d\tau$ an antiderivative of $A(t)$ such that $\dot{B}(t) = A(t)$ and $B(0) = 0$. We claim that $\boldsymbol{x}(t) = e^{B(t)}\boldsymbol{x}_0$ solves the IVP: The initial value is trivially verified, and for the ODE we calculate

warning: wrong!!

$$\dot{\boldsymbol{x}}(t) = \frac{d}{dt}e^{B(t)}\boldsymbol{x}_0 = \dot{B}(t)e^{B(t)}\boldsymbol{x}_0 = A(t)\boldsymbol{x}(t)\ .$$

J.H. Quick's reasoning was too quick and wrong.

(a) Try it with $A(t) = \begin{bmatrix} 1 & 1 \\ 0 & 2t \end{bmatrix}$, $\boldsymbol{x}_0 = \begin{bmatrix} a \\ b \end{bmatrix}$, calculate $e^{B(t)}\boldsymbol{x}_0$ explicitly and convince yourself that for general (a, b), it indeed does *not* solve the ODE $\dot{\boldsymbol{x}} = A(t)\boldsymbol{x}$.

(b) Explain the error in Quick's reasoning. Which step exactly is false, and why? — In contrast, for constant $A(t) \equiv A$, we have $B(t) = tA$, and Quick's reasoning does prevail. So his source of error must be either ineffective or absent in this special case. Why?

38. Total Derivative of the Matrix Exponential
Let exp denote the matrix exponential. Give a formula for its total derivative (as easy as possible, as complicated as necessary); i.e., give a formula for $D\exp(A)\,B$.

39. Bessel Equation and Wronskian The Bessel equation reads

$$t^2u''(t) + tu'(t) + (n^2 - t^2)u = 0.$$

Denote by $U_1(t)$ and $U_2(t)$ the solutions with the initial values $U_1(1) = 1$, $U_1'(1) = 0$ and $U_2(1) = 0$, $U_2'(1) = 1$ respectively.

Write the Bessel equation as a 2×2 system of 1st order. Write out the Wronskian of U_1 and U_2 in terms of U_1 and U_2 and their derivatives. Then, for reminiscence, reread hwk problem 9 from last semester, including the italics, and its solution.

Without calculating the solutions themselves, evaluate their Wronskian explicitly. (Explicitly means here: a formula so plain and simple that your M125 students wouldn't balk at it.)

We said that solutions of linear ODEs exist for *all* t, and therefore their Wronskian should also exist for all t. The formula you found should indicate otherwise. Resolve the apparent paradox. How does the explanation you found show up in the original (scalar 2nd order) Bessel equation, when all matrices are camouflaged?

40. Fundamental Matrix What is the fundamental matrix for those solutions to $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ whose initial vectors at $t = 0$ are the coordinate unit vectors $\boldsymbol{e}_i$?

We proved 2.3.4 (ODE for the Wronskian) first by avoiding the formula for the derivative of a determinant, using $e^{\operatorname{tr} A} = \det \exp A$ in a sneaky way; after proving $e^{\operatorname{tr} A} = \det \exp A$ via the Jordan normal form. Then we gave a direct proof of 2.3.4 using the formula for $D\det$, but not the JNF. So I now want you to prove conversely that $e^{\operatorname{tr} A} = \det \exp A$ follows from 2.3.4. (Prove $e^{\operatorname{tr} tA} = \det \exp tA$ for all t, using 2.3.4)

41. On the Derivative of a Determinant

Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & -1 \\ 3 & -1 & -2 \\ 0 & 2 & -1 \end{bmatrix}$.

Calculate $\det(A + tB)$ explicitly. Check $\frac{d}{dt}|_{t=0} \det(A + tB) = D\det(A)\,B$ by using the formula for $D\det$.

42. More on Matrix Exponentials

Calculate $\frac{d}{dt}|_{t=0} e^{tA}e^{tB}e^{-tA}e^{-tB}$ and $\frac{d^2}{dt^2}|_{t=0} e^{tA}e^{tB}e^{-tA}e^{-tB}$.

In similar spirit, show: If $e^{t(A+B)} = e^{tA}e^{tB}$ for all $t \in \,]-\varepsilon, \varepsilon[$, then $AB = BA$.

43. Coordinate free Ddet

For a square matrix A, let A^{adj} be its adjoint in the sense 'transpose of the cofactor matrix'. I.e., $(A^{\text{adj}})_{ji} = (-1)^{i+j}$ times the determinant obtained by deleting the i^{th} row and j^{th} column from A. (To be distinguished from a completely unrelated different meaning of the word 'adjoint'). Remember from linear algebra that $A^{-1} = A^{\text{adj}}/\det A$.

Show that $D\det(A)B = \operatorname{tr}(A^{\text{adj}}B)$.

Hint: Show it first for $A = I$. Then calculate $\frac{d}{dt}|_{t=0} \det(A + tB)$ for invertible A, using $A + tB = A(I + tA^{-1}B)$. How do you finally generalize this to non-invertible A?

44. Phase Plane Example

Consider the ODE

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Find the invariant lines corresponding to the eigenvalue problem for the matrix A. Let u, v be coordinates in the plane corresponding to eigenvectors of the matrix A. Transform the ODE into a decoupled system $\dot{u} = ?u$, $\dot{v} = ?v$. Use this and the transformation to plot the precise orbits of the original system. Explicit calculation possible: eliminate t and solve the separable ODE involving dv/du.

45. Phase Plane Example Continued; Linear as a Special Case of Nonlinear

With the same A as before, consider the (pretendedly nonlinear) system

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}$$

with $\boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0}$, $D\boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0}$.

In the special case, where $\boldsymbol{f}$ happens to vanish identically, find local 1-dim manifolds W^S_{loc} and W^U_{loc} where a, b in Thm. 2.3.9 are chosen such that S and U each contain one eigenvalue. Are they unique in this example?

46. Another Phase Plane Example

Plot the phase portrait of the *linearization at the origin of* the system

$$\dot{u} = u + 3u^2 + 5v^2 + 3u^3 + 5uv^2 + u^4 - v^4$$
$$\dot{v} = -2v - 2uv + 2u^2v + 2u^3v + 4v^3 + 2uv^3$$

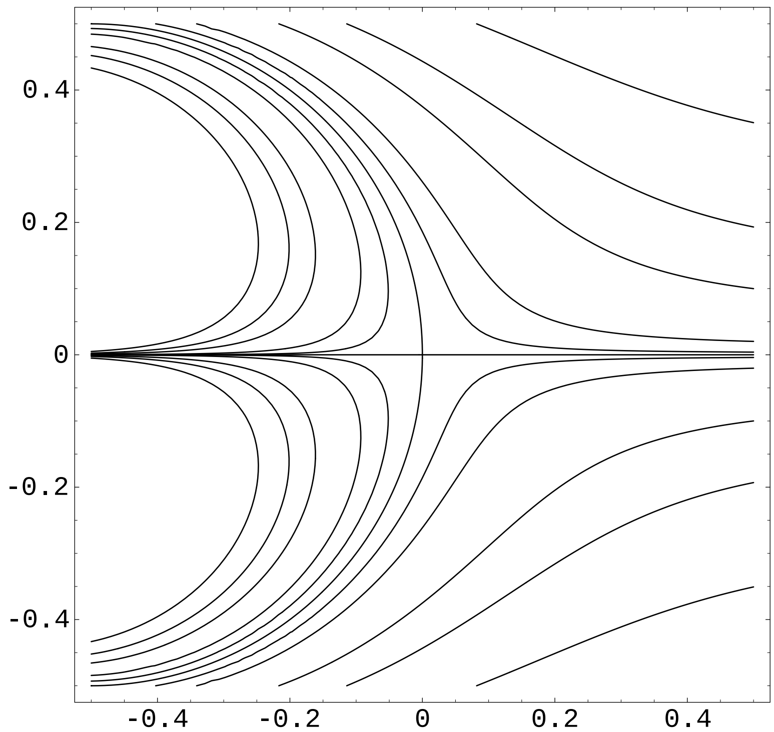
To this end, denote the variables in the linearized system by (x, y), and describe the solution orbits analytically in the form $h(x, y) = \text{constant}$.

Next show that, by 'miracle', the curves level curves of

$$\frac{v(u + u^2 + v^2)^2}{[(1 + u)^2 + v^2]^3}$$

are invariant under the flow of the nonlinear system. You are not actually required to carry out the calculation in algebraic detail (unless you have a computer algebra system available). Merely write what you would want to show, up to routine calculus and algebra manipulations.

Note: The 'miracle' has been carefully and surreptitiously crafted: I have transformed the linearized system by means of the complex transformation $w = z/(1 - z)$ with $z = x + iy$, $w = u + iv$, just to obtain a nonlinear 2×2 system that I myself can solve easily. This way I can now draw a phase portrait for you without resorting to numerical ODE solvers.



Identify the local stable and unstable manifolds and compare with the corresponding manifolds for the linearized system. Put directional arrows in the phase portrait.

47. Complex conjugate eigenvalues

For the system

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

show that the orbits are ellipses. Find the directions of their main axes and the ratio of their semiaxes. To this end, use the complex diagonalization to find a real matrix T such that $A = TRT^{-1}$ with $R = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$.

48. Sketch the phase portrait for

$$\begin{aligned} \dot{x} &= x^2 \\ \dot{y} &= -y \end{aligned}$$

along with the phase portrait of its linearization at the origin. Notice that the orbits can be obtained explicitly. Splitting the set of eigenvalues in two singleton sets S and U, what are the local invariant manifolds matching Thm 2.3.9?

Also observe the phenomenal failure of a Hartman-Grobman type of result, due to the failure in its hypothesis.

49. **Trotter Product Formula:** We have seen that for matrices, e^{A+B} is, in general, neither $e^A e^B$ nor $e^B e^A$. The true formula is a 'homogenized' compromise:

$$\lim_{n\to\infty} (e^{A/n} e^{B/n})^n = e^{A+B}$$

You'll prove it here. (The proof in Chicone's book is not correct, as is also noted in his errata web page.)

Rewrite the formula $X^n - Y^n = (X - Y)(X^{n-1} + X^{n-2}Y + \ldots + XY^{n-2} + Y^{n-1}$ from commutative algebra in a form that does not depend on the commutative law, so that the modified form applies to matrices as well.

With $X = e^{A/n}e^{B/n}$ and $Y = e^{(A+B)/n}$, estimate $\|(e^{A/n}e^{B/n})^n - e^{A+B}\|$, using that (why?) $\|e^{A/n}e^{B/n} - e^{(A+B)/n}\| \le c/n^2$.

50. **Variation of Parameters – sophomore vs graduate:** On the extra sheet, find variation of parameters explained without matrices for the consumption of a sophomore/junior ODE course in the case of a 2nd order scalar ODE. Identify every numbered equation in this explanation with an analog for the matrix formalism.

51. **Gronwall's inequality:** Show that if $f \in C^0([0,T] \to \mathbb{R})$ is nonnegative and satisfies

$$f(t) \le \alpha(t) + \int_0^t \beta(s) f(s)\, ds$$

with $\alpha(\cdot)$ continuous and nondecreasing and $\beta(\cdot)$ nonnegative and continuous, then

$$f(t) \le \alpha(t) \exp\left(\int_0^t \beta(s)\, ds\right)$$

Hint: Assuming K:= $\max\{f(t) : t \in [0,T]\}$*, show inductively, for all* $k \in \mathbb{N}$*, that*

$$\begin{aligned} f(t) \le \alpha(t)\Bigg(1 &+ \int_0^t \beta(s_1)\, ds_1 + \int_0^t \int_0^{s_1} \beta(s_1)\beta(s_2)\, ds_2 ds_1 + \ldots + \\ &+ \int_0^t \int_0^{s_1} \cdots \int_0^{s_{k-1}} \beta(s_1)\beta(s_2)\cdots\beta(s_k)\, ds_k \cdots ds_2 ds_1\Bigg) + \\ &+ K \int_0^t \int_0^{s_1} \cdots \int_0^{s_k} \beta(s_1)\beta(s_2)\cdots\beta(s_{k+1})\, ds_{k+1} \cdots ds_2 ds_1 \end{aligned}$$

Note that the k-fold integral equals $(\int_0^t \beta(s)\, ds)^k / k!$ *(why?).*

BTW: An altogether different proof can be found in Chicone, or in Walter.

52. A Perturbation by a Short Pulse:
Let $\boldsymbol{x}_n(\cdot)$ solve the IVP

$$\dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x}(t) + \delta_n(t-\tau)\boldsymbol{b}(t)\,,\ \ \boldsymbol{x}(0) = \mathbf{0}$$

Here δ_n is some nonnegative continuous function that vanishes outside the interval $[-1/n, 1/n]$.

Consider this differential equation on the interval $t \in [0,T]$ and assume $\tau \in [1/n, T - 1/n]$. We choose α, β such that $\|A(t)\| \le \alpha$ and $|\boldsymbol{b}(t)| \le \beta$ on $[0,T]$; and we choose as norm for functions $\boldsymbol{x}$ the following:

$$\|\boldsymbol{x}(\cdot)\| := \sup_{t\in[0,T]} e^{-2\alpha t}|\boldsymbol{x}(t)|$$

(cf. pblm. 20 last semester). Show that the Picard-Lindelöf argument can be carried out in the ball $\|\boldsymbol{x}(\cdot)\| \le 2\beta$, independent of n. Then show (using the sketch from the lecture) that $\boldsymbol{x}_n(\tau + 1/n) \to \boldsymbol{b}(\tau)$ as $n \to \infty$.

53. As an alternative to the Picard-Lindelöf part of the previous problem, obtain an upper estimate for $\|\boldsymbol{x}_n(\cdot)\|$ from the Gronwall inequality.

54. Complex Derivatives: In $\mathbb{R}$, we can define the function $f(x) := x|x| = \begin{cases} x^2 & \text{if } x \ge 0 \\ -x^2 & \text{if } x < 0 \end{cases}$ and thus construct an element $f \in C^1 \setminus C^2$. Similar attempts in $\mathbb{C}$ are doomed.

Observe this for the following three attempts, assuming the notation $z = x + iy$:
(a) $f(z) := z|z|$
(b) $g(z) := \begin{cases} z^2 & \text{if } x \ge 0 \\ -z^2 & \text{if } x < 0 \end{cases}$
(c) $h(z) := \begin{cases} x^2 & \text{if } x \ge 0 \\ -x^2 & \text{if } x < 0 \end{cases}$

Where are these functions continuous, where are they differentiable as maps $\mathbb{R}^2 \to \mathbb{R}^2$, and where are they differentiable as maps $\mathbb{C} \to \mathbb{C}$? Remember that the latter condition means that the Jacobi matrix ($\in \mathbb{R}^{2\times 2}$) represents a linear mapping that can also be represented by multiplication with a complex number, i.e., it is of the form ___???___.

55. Practical Estimate of a Convergence Radius: We have seen that the solution to the IVP $\dot{x} = t + x^2$, $x(0) = 0$ is given by the power series

$$x(t) = t^2 \sum_{k=0}^{\infty} b_k (t^3)^k \quad \text{where} \quad b_0 = \frac{1}{2}\,,\ b_k = \frac{1}{3k+2}\sum_{l=0}^{k-1} b_l b_{k-1-l}$$

and
(a) We have shown inductively that $|b_k| \le Cr^k$ with $C = \frac{1}{2}$ and $r = \frac{1}{6}$.

(b) Find conversely constants C, r such that an inductive proof succeeds for the claim $b_k \ge Cr^k$.

(c) Also find other constants C, r such that $|b_k| \le Cr^k$ for $k \ge 1$, but not necessarily for $k = 0$.

(d) Similarly as in (c), one can show (and I save you the work) that $b_k \ge Cr^k$ for $k \ge 1$ with $C = \frac{2}{5}$, $r = \frac{1}{8}$.

What conclusions can you draw about the existence interval from each of these four calculations, when you choose C, r optimally in each case. (They are already chosen optimally in (a),(d).)

56. A Bit of Differential Operator Algebra: Show inductively, for sufficiently smooth functions f of a single variable x, that

$$x\left(\frac{d}{dx}\right)^n f(x) = \left(\frac{d}{dx}\right)^n (xf(x)) - n\left(\frac{d}{dx}\right)^{n-1} f(x)$$

and

$$x^n\left(\frac{d}{dx}\right)^n f(x) = \left(x\frac{d}{dx}\right)\left(x\frac{d}{dx}-1\right)\cdots\left(x\frac{d}{dx}-(n-1)\right) f(x)$$

57. Euler Equation: Show that the Euler ODE

$$a_n x^n y^{(n)}(x) + \ldots + a_1 x y'(x) + a_0 y(x) = 0$$

where we assume $x > 0$ for the moment, can be transformed into a constant coefficient ODE by means of the substitution $x = e^t$, $u(t) = y(e^t) = y(x)$.

Using this information, what is the general solution of the following Euler ODEs?

$$\begin{array}{ll} (a) & 2x^2y'' - xy' - 2y = 0 \\ (b) & 4x^2y'' + y = 0 \\ (c) & x^2y'' + xy' - 2y = 0 \\ (d) & x^2y'' + xy' + 4y = 0 \end{array}$$

Make a general statement about what kind of functions can occur as solutions of an Euler equation (based on your sophomore (or 531/2) knowledge of constant coefficient equations).

Do there exist solutions for every / for certain IVP's $y(0) = y_0$, $y'(0) = y_1$? Are they unique? Decide for each of the four examples which IVPs have how many solutions.

58. Regular Singular Points?
In the following examples, determine for each $x_0 \in \mathbb{C} \cup \{\infty\}$ whether x_0 is an ordinary, a regular singular, or an irregular singular point.

(a)	Bessel equation	$x^2y'' + xy' + (x^2 - \nu^2)y = 0$
(b)	hypergeometric equation	$x(1-x)y'' + [\gamma - (\alpha+\beta+1)x]y' - \alpha\beta y = 0$
(c)	Airy equation	$y'' - xy = 0$
(d)	Euler equations	$a_n x^n y^{(n)}(x) + \ldots + a_1 x y'(x) + a_0 y(x) = 0$
(e)	constant coefficient equations	$a_n y^{(n)}(x) + \ldots + a_1 y'(x) + a_0 y(x) = 0$

In (a),(b), $\alpha, \beta, \gamma, \nu$ stand for given real (or complex) numbers. To simplify your work, you may decide the status of ∞ for (d),(e) in the special case of 2nd order only.

59. Airy Functions: Let y be any nontrivial solution to the Airy equation $y''(x) - xy(x) = 0$ for $x \in \mathbb{R}$. Show:
(a) y can have at most one zero in $x \in [0, +\infty[$.
(b) Given $-r^2 < 0$, y has infinitely many zeros in $]-\infty, -r^2]$, and the distances between any two of them is less than π/r.
(c) Given any two zeros in $[-r^2, 0]$, their distance is larger than π/r.

Hints:
(a) a maximum principle (which here means: Rolle's theorem & the 2nd derivative test for local maxima/minima)
(b),(c): Sturm comparison principle: here, this means comparison with solutions of $z'' + r^2 z = 0$ on an interval where neither y nor z changes sign, but one of them vanishes at the boundary; use a mixed Wronskian as in Pblm 9 last semester.

60. **Using the Complex Logarithm:** Taking all possible definitions of the complex logarithm together, what is the list of possible values for i^i? (Remember that such powers should be understood as $x^y := e^{y \ln x}$.) Which definition of i^i results, if we take a branch cut at the negative real axis and require $\ln 1 = 0$?

Explain why the defintion $x^y := e^{y \ln x}$, in combination with the availability of many choices for a function called ln does *not* introduce ambiguity in the meaning of (e.g.) x^2.

61. **The Gamma Function:** *Per se it doesn't arise in ODE's, but it is a useful auxiliary function when studying Airy, Bessel, and hypergeometric functions (solutions of 2nd order ODEs carrying the same name) by means of power series methods.*

Definition: $\Gamma(z) := \int_0^\infty t^{z-1} e^{-t}\, dt$ for $\operatorname{Re} z > 0$. (With $t^{z-1} := e^{(z-1)\ln t}$.)

(a) Show that the integral indeed converges for $\operatorname{Re} z > 0$.
(b) Show that $\Gamma(z+1) = z\Gamma(z)$, and $\Gamma(1) = 1$.

Note: (b) implies that $\Gamma(n) = (n-1)!$ for positive integers. The same formula is also used to extend the definition of $\Gamma(z)$ to all complex z other than $0, -1, -2, -3, \ldots$

(c) Show $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. To this end, retrieve the proof that $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$ from (e.g.) MV calculus; it goes by transforming $\int\int_{\mathbb{R}^2} e^{-x^2-y^2}\, dx\, dy$ into polar coordinates. Then relate $\Gamma(\frac{1}{2})$ to $\int_{-\infty}^{\infty} e^{-x^2}\, dx$ by a Calc'2 type substitution.

That should be good enough for us; more wisdom on Γ can be obtained by complex variable techniques. Here are a few such formulas FYI:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}\,, \qquad \Gamma(2z) = \pi^{-1/2} 2^{2z-1} \Gamma(z)\Gamma(z+\tfrac{1}{2})$$

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{k=1}^{\infty} \left[\left(1 + \frac{z}{k}\right) e^{-z/k}\right] \quad \textit{with} \quad \gamma := -\Gamma'(1) = \lim_{n\to\infty} \left(\sum_{j=1}^{n} \frac{1}{j} - \ln n\right)$$

62. **The Bessel Function J_ν:** Show that

$$\frac{1}{\pi} \int_0^\pi \cos(z \cos\theta)\, d\theta = J_0(z)$$

(Use the Taylor expansion for the outer cos and integrate termwise. Obtain a formula for the integrals of powers of cosines that arise, using repeated integration by parts.)

Show (by massaging the power series) that

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

63. An irregular singular point:
Show that the ODE $x^2y''-y'+y=0$ has a *formal* power series solution $y=\sum_{n=0}^{\infty}a_nx^n$.

By FORMAL *power series solution, I mean that if you plug it into the lhs and use the algebraic manipulations for power series, you indeed get 0.*

Show that the power series found has zero radius of convergence. *Our algebraically successful search for a solution has therefore* NOT *actually retrieved a solution.*

A more promising approach to study this ODE would be to note that ∞ is a regular singular point, to obtain therefore a convergent power series expansion in powers of $1/x$, which does of course not display information near $x=0$. Next try to cook up this power series into a definite integral formula (along the philosophy used in the previous hwk for J_0); Gamma functions are well-nigh omnipresent in this step. Then work with this integral formula, presumably introducing complex variable integrals. None of this is routine, but it can be successful with a combination of luck and wisdom. If good luck abounds (as it does in this example), part of the wisdom may be replaced by reference to a special functions handbook like Abramowitz/Stegun.

64. The Bessel Function Y_0: We have found the solution

$$J_0(x):=\sum_{n=0}^{\infty}\frac{(-x^2/4)^n}{n!^2}$$

to the Bessel equation $x^2y''+xy'+x^2y=0$. Find another solution $\tilde{Y}_0$ in the form

$$\tilde{Y}_0(x):=\sum_{k=0}^{\infty}b_kx^{2k}+c\ln x\,J_0(x)$$

with $b_0=0$ and $b_1=1$. Explain which changes of basis in the solution space allow to make the assumptions $b_0=0$ and $b_1=1$ with no loss of generality.

Make sure to give as explicit as possible a formula for the b_k, not merely a recursion formula. (I skipped this step in the example of the lecture.)

Note: For reasons of no concern here (convenient behavior as $x\to+\infty$), the function usually denoted as Y_0 is given by the linear combination

$$Y_0(x):=\frac{1}{2\pi}\tilde{Y}_0(x)+\frac{2}{\pi}(\gamma-\ln 2)J_0(x)$$

with γ the same quantity as in problem 61.

65. The Hypergeometric Function ${}_2F_1(a,b;c;z)$:
Assuming $a,b\in\mathbb{C}$, $c\in\mathbb{C}\setminus\mathbb{Z}$, find two linearly independent solutions (as power series centered at 0) of the hypergeometric ODE

$$z(1-z)y''(z)+[c-(a+b_1)z]y'(z)-aby(z)=0$$

Making appropriate assumptions on a,b,c, find two linearly independent solutions to the same ODE, but this time as power series centered at 1.

66. An Integral Representation for the Hypergeometric Function: Using the binomial series $(1+z)^\alpha = \sum \binom{\alpha}{n} z^n$, where $\binom{\alpha}{n} = \frac{1}{n!}\alpha(\alpha-1)\cdots(\alpha-n+1) = \frac{1}{n!}\Gamma(\alpha+1)/\Gamma(\alpha-n+1)$ (which you should know), and (without proof, this is likely new to you) the integral formula

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

show that

$$\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt = {}_2F_1(a,b;c;z)$$

provided the integral converges and $|z|<1$. Which hypotheses on a,b,c are needed to ascertain convergence of the integral?

Note: In a further step, which requires good understanding of complex variables and the Riemann surface of $\ln$, *one could concoct an integral with the very same integrand, but instead of* $t \in [0,1]$ *the integral now runs along a closed path in the complex plane that goes around the singularities* $t=0$ *and* $t=1$ *in a special way. One can obtain a very similar formula* without *size constraints on* a,b,c *that can also be used to define* ${}_2F_1(a,b;c;z)$ *beyond the disc of convergence of the original series.*

The special path needed has the following description: go around 0 counterclockwise, then around 1 counterclockwise, then around 0 clockwise, then around 1 clockwise.