Examples Jordan Normal Form
UTK – M531 – Ordinary Differential Equations I
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(These examples were generated with the help of symbolic algebra software, and also the calculations were done using such software.)

$\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ denotes the diagonal matrix (size $n \times n$) with the diagonal entries as specified.

I may write column vectors like $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ as transposed rows $[1\,,\ 2\,,\ 3]^T$ for typographical reasons.

Example 1:

The matrix

$$A = \begin{bmatrix} \frac{44}{3} & -\frac{82}{3} & -20 \\ \frac{19}{3} & -\frac{35}{3} & -10 \\ 4 & -8 & -4 \end{bmatrix}$$

has characteristic polynomial

$$\det(A - \lambda I) = -\lambda^3 - \lambda^2 + 10\lambda - 8 = -(\lambda - 2)(\lambda - 1)(\lambda + 4)$$

where the factorization is based on eyeballing (guesswork) to find $\lambda = 1$ as one root, and then the quadratic formula.

Since all eigenvalues are distinct, we can diagonalize the matrix $A = SDS^{-1}$ with $D = \mathrm{diag}(1, 2, -4)$. (Any other order like eg. $\tilde{D} = \mathrm{diag}(1, -4, 2)$ would do just as well, with a different $\tilde{S} \neq S$ (similarly permuted. So we have chosen the numbering $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = -4$. Let's find corresponding eigenvectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$:

$\boldsymbol{v}_1$ is calculated as a solution to the linear system $(A - \lambda_1)\boldsymbol{v}_1 = \mathbf{0}$:

$$\begin{aligned} \tfrac{41}{3}v_1^{(1)} + \ \tfrac{82}{3}v_1^{(2)} \ + (-20)v_1^{(3)} &= 0 \\ \tfrac{19}{3}v_1^{(1)} + \ \tfrac{-38}{3}v_1^{(2)} \ + (-10)v_1^{(3)} &= 0 \\ 4v_1^{(1)} \ + (-8)v_1^{(2)} + \ (-5)v_1^{(3)} &= 0 \end{aligned}$$

(Upper indices denote components of the vector, lower indices identify vectors.) Gauss elimination from this system produces: $v_1^{(1)} = 2v_1^{(2)}$, $v_1^{(3)} = 0$. We may therefore choose $\boldsymbol{v}_1 = [2\,,\ 1\,,\ 0]^T$. Any nonzero multiple would have been just as legitimate a choice (and would have led to a somewhat different matrix S).

Likewise we obtain $\boldsymbol{v}_2 = [1\,,\ -1\,,\ 2]^T$ and $\boldsymbol{v}_3 = [4\,,\ 2\,,\ 1]^T$, where again nonzero multiples would have been just as legitimate choices. We have therefore found (our choice of) the matrix S:

$$S = \left[\begin{array}{c|c|c} 2 & 1 & 4 \\ 1 & -1 & 2 \\ 0 & 2 & 1 \end{array}\right]$$

(Vertical lines inserted to make eigenvectors more clearly visible). It is another bunch of linear equations to find its inverse (if you need it):

$$S^{-1} = \frac{1}{3}\begin{bmatrix} 5 & 7 & -6 \\ 1 & -2 & 0 \\ -2 & 4 & 3 \end{bmatrix}$$

You can now check explicitly that $AS = SD$, or $A = SDS^{-1}$.

Example 2:

This example must be higher dimensional, because its purpose is to display several bells and whistles at the same time. Took me quite a while to come up with an example with *somewhat* decent numbers. Let

$$A = \begin{bmatrix} 28 & 4 & -1 & 1 & -9 & -15 & -8 \\ 90 & 4 & -2 & 2 & -36 & -45 & -27 \\ 93 & 0 & -2 & 0 & -40 & -45 & -28 \\ 93 & 0 & 0 & -2 & -40 & -45 & -28 \\ -560 & -12 & 3 & -3 & 235 & 275 & 169 \\ -554 & 4 & -1 & 1 & 243 & 267 & 168 \\ 1790 & 24 & -6 & 6 & -759 & -875 & -540 \end{bmatrix}$$

The characteristic polynomial is

$$\det(A - \lambda I) = -\lambda^7 - 10\lambda^6 - 41\lambda^5 - 90\lambda^4 - 120\lambda^3 - 112\lambda^2 - 80\lambda - 32 = -(\lambda+2)^5(\lambda^2+1)$$

where the factorization is a conspiratively designed lucky 'coincidence'. So $\lambda_1 = -2$ is an eigenvalue with algebraic multiplicity 5 and $\lambda_{6,7} = \pm i$ are single eigenvalues. To find the geometric multiplicity of λ_1, we must actually solve the linear system $(A + 2I)\boldsymbol{v} = \mathbf{0}$. Let me do this 'by hand', with 'unsystematic' row transformations (i.e., smart pivoting) to preserve nice numbers:

$$\begin{array}{rrrrrrr} 30 & 4 & -1 & 1 & -9 & -15 & -8 \\ 90 & 6 & -2 & 2 & -36 & -45 & -27 \\ 93 & 0 & 0 & 0 & -40 & -45 & -28 \\ 93 & 0 & 0 & 0 & -40 & -45 & -28 \\ -560 & -12 & 3 & -3 & 237 & 275 & 169 \\ -554 & 4 & -1 & 1 & 243 & 269 & 168 \\ 1790 & 24 & -6 & 6 & -759 & -875 & -538 \end{array} \quad \xrightarrow{\text{step1}} \quad \begin{array}{rrrrrrr} 30 & 4 & -1 & 1 & -9 & -15 & -8 \\ -3 & 6 & -2 & 2 & 4 & 0 & 1 \\ 93 & 0 & 0 & 0 & -40 & -45 & -28 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -560 & -12 & 3 & -3 & 237 & 275 & 169 \\ 6 & 16 & -4 & 4 & 6 & -6 & -1 \\ 110 & -12 & 3 & -3 & -48 & -50 & -31 \end{array}$$

<ins>step 1</ins>: subtract 3rd row from 2nd and 4th; subtract 5th row from 6th and add three times to 7th.

<ins>step 2</ins>: add 5×last row to 5th, then subtract 3rd row from last, then use 2nd row to produce leading 0's in 1st, 3rd, 6th row

<ins>step 3</ins>: move 4th row to bottom, 2nd row to top, and use it to create more zeros in the 1st column

$$\begin{array}{rrrrrrr} 0 & 64 & -21 & 21 & 31 & -15 & 2 \\ -3 & 6 & -2 & 2 & 4 & 0 & 1 \\ 0 & 186 & -62 & 62 & 84 & -45 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & -72 & 18 & -18 & -3 & 25 & 14 \\ 0 & 28 & -8 & 8 & 14 & -6 & 1 \\ 17 & -12 & 3 & -3 & -8 & -5 & -3 \end{array} \quad \xrightarrow{\text{step3}} \quad \begin{array}{rrrrrrr} -3 & 6 & -2 & 2 & 4 & 0 & 1 \\ 0 & 64 & -21 & 21 & 31 & -15 & 2 \\ 0 & 186 & -62 & 62 & 84 & -45 & 3 \\ 0 & -92 & \frac{74}{3} & \frac{-74}{3} & \frac{-49}{3} & 25 & \frac{32}{3} \\ 0 & 28 & -8 & 8 & 14 & -6 & 1 \\ 0 & 22 & \frac{-25}{3} & \frac{25}{3} & \frac{44}{3} & -5 & \frac{8}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

step 4: subtract 2nd row 3 times from 3rd, add it to 4th, subtract 5th row from 6th

step 5: add 5th row to 4th, subtract it 5/2 times from 2nd; subtract 3rd row from 6th

$$\begin{array}{rrrrrrr} -3 & 6 & -2 & 2 & 4 & 0 & 1 \\ 0 & 64 & -21 & 21 & 31 & -15 & 2 \\ 0 & -6 & 1 & -1 & -9 & 0 & -3 \\ 0 & -28 & \frac{11}{3} & \frac{-11}{3} & \frac{44}{3} & 10 & \frac{38}{3} \\ 0 & 28 & -8 & 8 & 14 & -6 & 1 \\ 0 & -6 & \frac{-1}{3} & \frac{1}{3} & \frac{2}{3} & 1 & \frac{5}{3} \end{array} \quad \overset{\text{step5}}{\longrightarrow} \quad \begin{array}{rrrrrrr} -3 & 6 & -2 & 2 & 4 & 0 & 1 \\ 0 & -6 & -1 & 1 & -4 & 0 & \frac{-1}{2} \\ 0 & -6 & 1 & -1 & -9 & 0 & -3 \\ 0 & 0 & \frac{-13}{3} & \frac{13}{3} & \frac{86}{3} & 4 & \frac{41}{3} \\ 0 & 28 & -8 & 8 & 14 & -6 & 1 \\ 0 & 0 & \frac{-4}{3} & \frac{4}{3} & \frac{29}{3} & 1 & \frac{14}{3} \end{array}$$

step 6: multiply 4th and 6th row by 3; use 3rd row to produce 0's in column2 of 2nd and 5th row

step 7: exchange 3rd and 2nd row; use new 3rd (old 2nd) row to produce 0's in 3rd column

$$\begin{array}{rrrrrrr} -3 & 6 & -2 & 2 & 4 & 0 & 1 \\ 0 & 0 & -2 & 2 & 5 & 0 & \frac{5}{2} \\ 0 & -6 & 1 & -1 & -9 & 0 & -3 \\ 0 & 0 & -13 & 13 & 86 & 12 & 41 \\ 0 & 0 & -\frac{10}{3} & \frac{10}{3} & -28 & -6 & -13 \\ 0 & 0 & -4 & 4 & 29 & 3 & 14 \end{array} \quad \overset{\text{step7}}{\longrightarrow} \quad \begin{array}{rrrrrrr} -3 & 6 & -2 & 2 & 4 & 0 & 1 \\ 0 & -6 & 1 & -1 & -9 & 0 & -3 \\ 0 & 0 & -2 & 2 & 5 & 0 & \frac{5}{2} \\ 0 & 0 & 0 & 0 & \frac{107}{2} & 12 & \frac{99}{4} \\ 0 & 0 & 0 & 0 & \frac{-109}{3} & -6 & \frac{-103}{6} \\ 0 & 0 & 0 & 0 & 19 & 3 & 9 \end{array}$$

step 8: multiply row5 by 3/2 and add row 4 to it: it becomes $(0,0,0,0,-1,3,-1)$; use it to produce 0's in column 5.

step 9: the 4th row has become a multiple of the 6th and can be dropped; normalize the 6th row.

$$\begin{array}{rrrrrrr} -3 & 6 & -2 & 2 & 4 & 0 & 1 \\ 0 & -6 & 1 & -1 & -9 & 0 & -3 \\ 0 & 0 & -2 & 2 & 5 & 0 & \frac{5}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{345}{2} & \frac{-115}{4} \\ 0 & 0 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 60 & -10 \end{array} \quad \overset{\text{step9}}{\longrightarrow} \quad \begin{array}{rrrrrrr} -3 & 6 & -2 & 2 & 4 & 0 & 1 \\ 0 & -6 & 1 & -1 & -9 & 0 & -3 \\ 0 & 0 & -2 & 2 & 5 & 0 & \frac{5}{2} \\ 0 & 0 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 6 & -1 \end{array}$$

We therefore find two eigenvectors: $v^{(7)}$ can be chosen arbitrarily, but the 5th eqn requires $6v^{(6)} - v^{(7)} = 0$. The 4th eqn then determines $v^{(5)}$, whereas $v^{(4)}$ is again arbitrary. The other components are then determined again. The eigenspace is therefore 2-dimensional (geometric multiplicity 2), and we choose two linearly independent eigenvectors $\boldsymbol{v}$ and $\boldsymbol{w}$ by letting $v^{(7)} = 2, v^{(4)} = 0$, and $w^{(7)} = 0, w^{(4)} = 1$. Other choices would be equally legitimate and would simply produce a different matrix S.

We thus find the eigenvectors $\boldsymbol{v} = [\frac{1}{3}\,,\ \frac{1}{2}\,,\ 0\,,\ 0\,,\ -1\,,\ \frac{1}{3}\,,\ 2]^T$ and $\boldsymbol{w} = [0\,,\ 0\,,\ 1\,,\ 1\,,\ 0\,,\ 0\,,\ 0]^T$.

We try to find further vectors by solving $(A+2I)\boldsymbol{v}' = \boldsymbol{v}$ and $(A+2I)\boldsymbol{w}' = \boldsymbol{w}$ (if possible). And then, we try to solve $(A+2I)\boldsymbol{v}'' = \boldsymbol{v}'$ and $(A+2I)\boldsymbol{w}'' = \boldsymbol{w}'$ (if possible). We wouldn't know yet if any particular among these equations has a solution, but the JNF theorem guarantees that altogether we find three solutions, i.e., among the tentative vectors $\{\boldsymbol{v}', \boldsymbol{v}'', \boldsymbol{v}''', \ldots, \boldsymbol{w}', \boldsymbol{w}'', \boldsymbol{w}''', \ldots\}$ three will actually exist (such as to bring the total to 5 generalized eigenvectors, according to the algebraic multiplicity 5).

I skip the remaining linear systems (the row transformations could be reused) and merely give the results:

The system $(A+2I)\boldsymbol{v}' = \boldsymbol{v}$ has the solutions $\boldsymbol{v}' = [0\,,\ \frac{1}{12}\,,\ 0\,,\ 0\,,\ 0\,,\ 0\,,\ 0]^T + c_1\boldsymbol{v} + c_2\boldsymbol{w}$. The most convenient choice is of course $c_1 = c_2 = 0$, and we have $\boldsymbol{v}' = [0\,,\ \frac{1}{12}\,,\ 0\,,\ 0\,,\ 0\,,\ 0\,,\ 0]^T$.

The system $(A+2I)\boldsymbol{w}' = \boldsymbol{w}$ has the solutions $\boldsymbol{w}' = [2\,,\ \frac{3}{2}\,,\ 0\,,\ 0\,,\ -1\,,\ 5\,,\ 0]^T + c_3\boldsymbol{v} + c_4\boldsymbol{w}$, and again we choose $\boldsymbol{w}' = [2\,,\ \frac{3}{2}\,,\ 0\,,\ 0\,,\ -1\,,\ 5\,,\ 0]^T$.

The system $(A+2I)\boldsymbol{v}'' = \boldsymbol{v}'$ has the solutions $\boldsymbol{v}'' = [\frac{1}{4}\,,\ \frac{1}{3}\,,\ \frac{-1}{6}\,,\ 0\,,\ \frac{-3}{4}\,,\ \frac{1}{4}\,,\ \frac{3}{2}]^T + c_5\boldsymbol{v} + c_6\boldsymbol{w}$. We choose $c_5 = c_6 = 0$.

The system $(A+2I)\boldsymbol{w}'' = \boldsymbol{w}'$ has NO solutions.

We have already 5 vectors, and indeed the system $(A+2I)\boldsymbol{v}''' = \boldsymbol{v}''$ (which we might yet consider trying) has no solutions. At this moment we know that the two Jordan blocks for $\lambda = -2$ have sizes 3 and 2 respectively, corresponding to the sets of vectors $\{\boldsymbol{v}_1 := \boldsymbol{v}, \boldsymbol{v}_2 := \boldsymbol{v}', \boldsymbol{v}_3 := \boldsymbol{v}''\}$ and $\{\boldsymbol{v}_4 := \boldsymbol{w}, \boldsymbol{v}_5 := \boldsymbol{w}'\}$. (Now that we know the sizes of the Jordan blocks, we can number them sensibly.)

We still need eigenvectors $\boldsymbol{v}_6$ and $\boldsymbol{v}_7$ for the eigenvalues i and $-i$ respectively. Then we put them all as columns in our matrix S. We get $AS = SJ$ with :

$$S = \begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{4} & 0 & 2 & -1 & -1 \\ \frac{1}{2} & \frac{1}{12} & \frac{1}{3} & 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & \frac{-1}{6} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & \frac{-3}{4} & 0 & -1 & -1+i & -1-i \\ \frac{1}{3} & 0 & \frac{1}{4} & 0 & 5 & \frac{-9+8i}{5} & \frac{-9-8i}{5} \\ 2 & 0 & \frac{3}{2} & 0 & 0 & 1-4i & 1+4i \end{bmatrix} \qquad J = \left[\begin{array}{ccc|cc|c|c} -2 & 1 & 0 & & & & \\ 0 & -2 & 1 & & & & \\ 0 & 0 & -2 & 0 & & & \\ \hline & & & -2 & 1 & & \\ & & & 0 & -2 & & \\ \hline & & & & & i & \\ \hline & & & & & & -i \end{array}\right]$$

S is invertible as predicted by the general theory. I won't write down S^{-1}, but note that $\det S = i/540$, therefore $A = SJS^{-1}$.

$\ker(A+2I)$ is spanned by columns 1 and 4 of S, $\ker(A+2I)^2$ is spanned by columns 1,2,4,5 of S. The complete eigenspace for eigenvalue -2 is $\ker(A+2I)^3 = \ker(A+2I)^j$ for any $j \geq 3$, spanned by columns 1–5 of S. — Have a look at the (3,4) entry of J, which is 0. It is this 0 that distinguishes between the case of a 3-Jordan block and a 2-Jordan block, as opposed to a single 5-Jordan block.

The span of columns 6 and 7 of S can also be spanned by real vectors (which are then no longer eigenvectors), namely the real and imaginary parts of these two vectors,
$[-1\,,\ 0\,,\ 0\,,\ 0\,,\ -1\,,\ -\frac{9}{5}\,,\ 1]^T$ and $[0\,,\ 0\,,\ 0\,,\ 0\,,\ 1\,,\ \frac{8}{5}\,,\ -4]^T$.

Exponentials

The exponential of a Jordan block of size n can be calculated explicitly:

$$\exp t \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix} = \exp\left(t\lambda I + t \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \right) = e^{t\lambda} \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n-1}}{(n-1)!} \\ & 1 & t & \ddots & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2!} \\ & & & \ddots & t \\ & & & & t \end{bmatrix}$$

Therefore the JNF theorem implies that $\exp(tA) \to 0$ as $t \to \infty$ *if and only if* all eigenvalues of A have $\operatorname{Re}\lambda < 0$.

Moreover, $\exp(tA)$ remains bounded as $t \to \infty$ *if and only if* all eigenvalues of A have $\operatorname{Re}\lambda \leq 0$ *and* all Jordan blocks for eigenvalues with $\operatorname{Re}\lambda = 0$ have size 1.