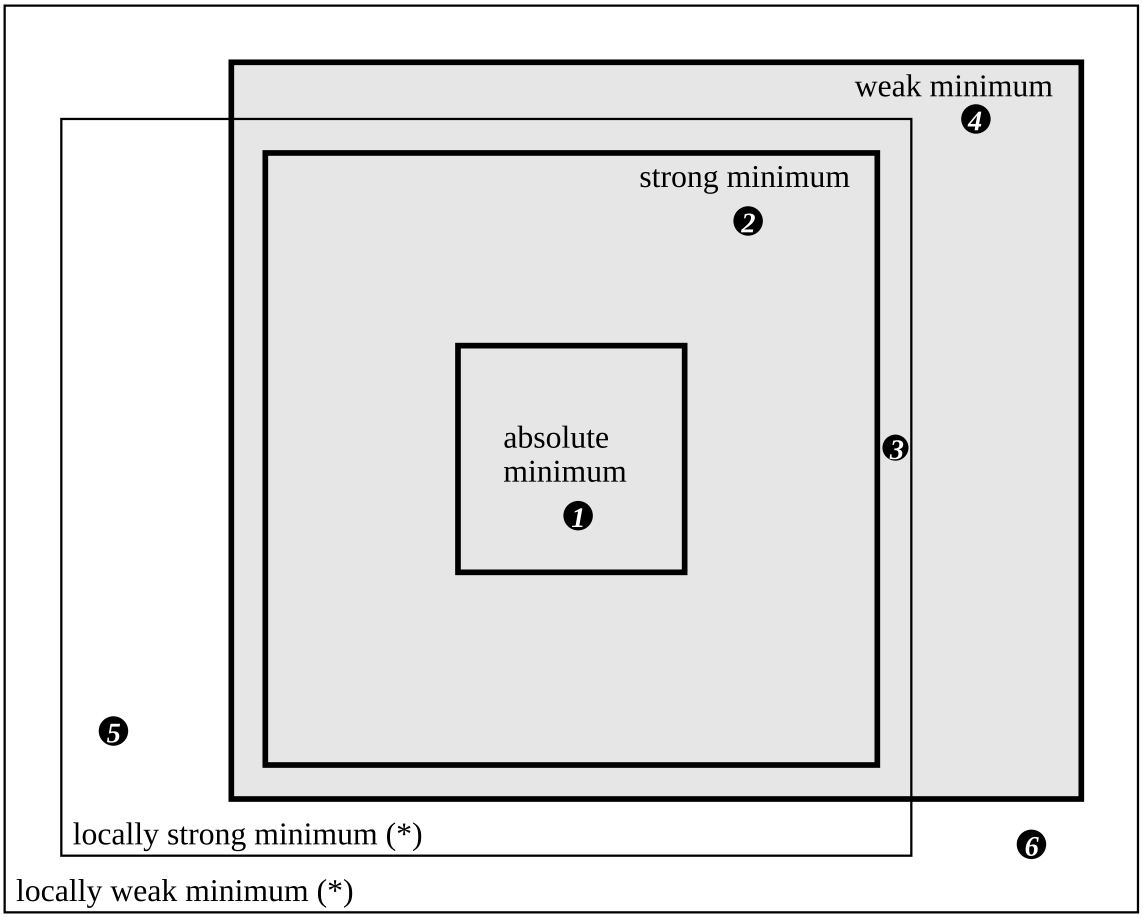
## Comparing Various Notions of Minimality in Calculus of Variations



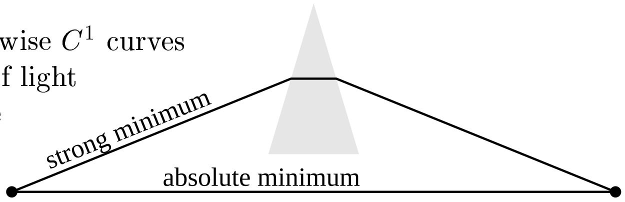

(*) Here we denote as *locally* strong or *locally* weak minima those extremals for which the strong or weak minimality property is only satisfied on sufficiently short segments. In other words every sufficiently short subsegment will be a strong or weak minimal. (I have made up these definitions of "locally strong" or "locally weak" ad hoc. They are not part of generally used mathematical language.)

Such extremals that are merely "locally weak minima" or "locally strong minima" are NOT relative minima in any functional analytic sense, but are saddle points. Genuine (relative) minima are found in the gray areas of the diagram.

## Examples

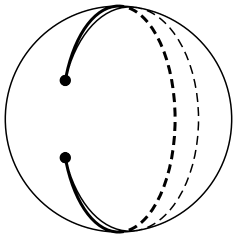
❶ The length functional, defined for all $C^1$ curves connecting two given points in the plane. The straight segment yieldss the absolute minimum of the length.

❷ Travel time for light, defined for all piecewise $C^1$ curves connecting two given points: The speed of light in glass is less than in air; this causes the refraction in the prism. The polygonal path is a strong minimum, but not the absolute minimum.




❹ The functional $I[y] := \int_{-1}^{1}(\dot y^2 - \dot y^4)\,dt$, defined for $y \in C^1[-1,1]$ with $y(-1) = y(1) = 0$: $y^* \equiv 0$ is a weak minimum, since for $|\dot y| < \varepsilon$ it holds: $\dot y^2 - \dot y^4 \geq (1-\varepsilon^2)\dot y^2$. But even short segments of $y^* \equiv 0$ aren't strong minimals; indeed, choose $\tilde y(t) := \varepsilon \sin^2 n(t-t_0)$ on $[t_0, t_0 + \pi/n]$ and $\tilde y = 0$ elsewhere: $I[\tilde y] = \frac{1}{8}\pi\varepsilon^2 n(4 - 3\varepsilon^2 n^2)$, which is negative for large $n$.

❺ The length functional, defined for all $C^1$ curves connecting two given points *on the sphere*: the great circle that connects the points 'on the back of the sphere' in the figure is not even a weak minimum. The smaller circle is shorter, but still $\varepsilon$-close in $C_1$. Nevertheless, sufficiently short segments of the great circle (short enough such as not to contain antipodes) are the absolutely shortest connections between their endpoints.



❻ Choose the functional $\int_0^\pi (\dot y^2 - \dot y^4 - 2y^2)\,dt$ on $C_0^1[0,\pi]$. Then $y \equiv 0$ is a critical point, but is not even weakly minimal. For $\tilde y(t) = \varepsilon \sin t$, it holds $I[\tilde y] = -\frac{1}{8}\pi\varepsilon^2(4 + 3\varepsilon^2)$. On short segments however, it is a weak minimum, but not a strong one; the reason is the same as in case ❹. To prove weak minimality, segments $[t_0, t_1] \subset [0,\pi]$ need to be so short that it always holds $\int_{t_0}^{t_1} \dot y^2\,dt > 2\int_{t_0}^{t_1} y^2\,dt$, and this happens if $(t_1 - t_0)^2 < \pi^2/2$.

❸ This case is a curiosity for which I could only find a somewhat artificial example: A weak minminal such that the functional can be made smaller by means of strong oscillations, but only if these strong oscillations occur on long segments. I am giving an example with vector-valued $y = (y_1, y_2)$, in which the functional itself is also more complicated than a simple integral expression:

$$I[y_1,y_2] := \int_0^\pi \left(\dot y_1^2 - \frac{1}{2}y_1^2\right)\,dt + \left(\int_0^\pi (\dot y_1^2 - 2y_1^2)\,dt\right)\left(\int_0^\pi \dot y_2^2\,dt\right)$$

on $C_0^1([0,\pi] \to \mathbb{R}^2)$. We conclude $I[y_1,y_2] \geq \left(\frac{1}{2} - \int \dot y_2^2\right)\left(\int \dot y_1^2\right)$, because $C_0^1[0,\pi]$ functions satisfy $\int \dot y^2 \geq \int y^2$; hence $(y_1,y_2) \equiv (0,0)$ is weakly minimal.

It is not strongly minimal, because for $\tilde y_1(t) = \varepsilon \sin t$, $\tilde y_2 = \varepsilon \sin nt$ it holds $4I[\tilde y_1, \tilde y_2] = \varepsilon^2\pi - n^2\varepsilon^4 n^2$, which is negative for large $n$.

However, if we consider short segments $[t_0, t_1]$ only, then the term $\int_{t_0}^{t_1}(\dot y_1^2 - 2y_1^2)\,dt$ will become $\geq 0$, and oscillations in $\int \dot y_2^2$ cannot do harm to the minimum property.