# Necessary and Sufficient Conditions in the Calculus of Variations

Compare this with the corresponding leaflet from the beginning of the semester, which contains the same diagram as the reverse side of this leaflet.

We assume:

$$I[y] := \int_{t_0}^{t_1} L(t, y(t), \dot{y}(t))\, dt \text{ where } L \in C^3$$

and consider the following conditions and equations:

**Euler-Lagrange Equation:** $\dfrac{d}{dt} L_{\dot{y}_i}\Big(t, y_*(t), \dot{y}_*(t)\Big) = L_{y_i}\Big(t, y_*(t), \dot{y}_*(t)\Big)$

**Jacobi Equation:** $\displaystyle\sum_j \frac{d}{dt}\Big(L^*_{\dot{y}_i\dot{y}_j}(t)\dot{\varphi}_j + L^*_{\dot{y}_i y_j}(t)\varphi_j\Big) = \sum_j L^*_{y_i\dot{y}_j}(t)\dot{\varphi}_j + L^*_{y_i y_j}(t)\varphi_j$

**Weierstrass E-Function:** $\mathfrak{E}(t, y, \dot{y}, \dot{Y}) := L(t, y, \dot{Y}) - L(t, y, \dot{y}) - L_{\dot{y}}(t, y, \dot{y})(\dot{Y} - \dot{y})$

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**(Simple) Legendre Condition:** The matrix $L^*_{\dot{y}_i\dot{y}_j}(t)$ is positive semidefinite for all $t \in [t_0, t_1]$

**Strict Legendre Condition:** The matrix $L^*_{\dot{y}_i\dot{y}_j}(t)$ is positive definite for all $t \in [t_0, t_1]$

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**(Simple) Weierstrass Condition:** $\mathfrak{E}(t, y_*(t), \dot{y}_*(t), \dot{Y}) \geq 0$ for all $t \in [t_0, t_1]$ and all $\dot{Y}$.

(This implies the simple Legendre Condition. Conversely, a Legendre condition " globalized with respect to the velocity variables", namely "$L_{\dot{y}\dot{y}}(t, y_*(t), \dot{Y})$ is positive semidefinite for all $t \in [t_0, t_1]$ and all $\dot{Y}$" implies the Weierstrass Condition. )

**Strict Weierstrass Condition:** $\mathfrak{E}(t, y, \dot{y}, \dot{Y}) \geq 0$ for all $t \in [t_0, t_1]$, for all $(y, \dot{y})$ satisfying $|y - y_*(t)| < \varepsilon$, $|\dot{y} - \dot{y}_*(t)| < \varepsilon$ and for all $\dot{Y}$, AND the strict Legendre Condition.

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We will consider the Jacobi Conditions as undefined, whenever the strict Legendre condition is violated. (There is no use for a Jacobi condition in this case.) In case the strict Legender condition holds, we define:

**(Simple) Jacobi Condition:** Every nontrivial solution $\varphi$ to the Jacobi equation subject to $\varphi(t_0) = 0$ does NOT have another zero in the interval $t \in ]t_0, t_1[$.

**Strict Jacobi Condition:** Every nontrivial solution $\varphi$ to the Jacobi equation subject to $\varphi(t_0) = 0$ does NOT have another zero in the interval $t \in ]t_0, t_1]$.

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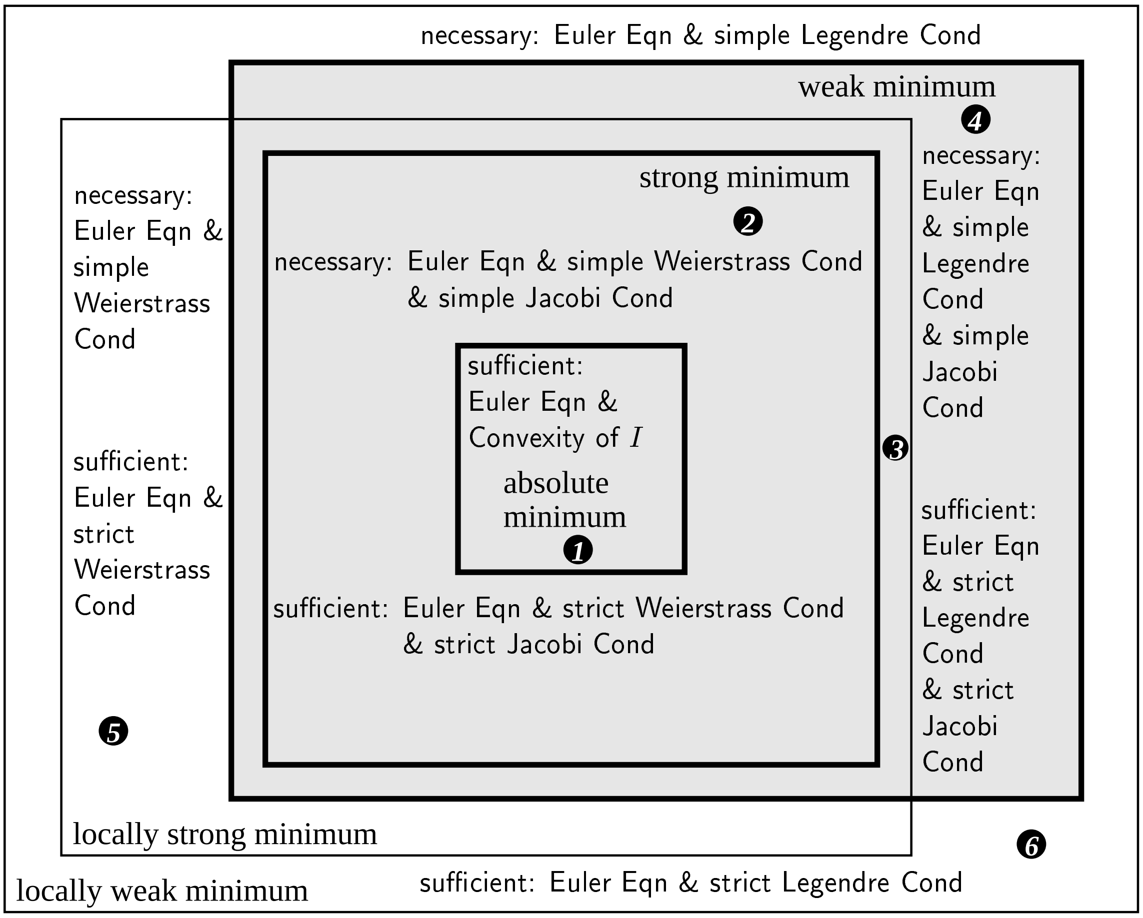
**Remarks:** It is more common to call the 'simple' vs 'strict' NN conditions 'weak' vs 'strong' NN conditions. I am using this slightly different wording simply because this distinction has nothing to do with the distinction of weak vs strong minimals. One might consider another variant of the strict Weierstrass condition, namely "$\mathfrak{E} > 0$ unless $\dot{y} = \dot{Y}$", rather than $\mathfrak{E} \geq 0$. This one however characterises the uniqueness of strong minimals, but does not concern the distinction between necessary and sufficient condition. – Also note that in the literature, one author may swap the order of the 3rd and 4th entry of $\mathfrak{E}$ as compared to another author.

The diagram on the reverse side gives the necessary and sufficient conditions. (I write shortly 'Euler Eqn' instead of 'Euler Lagrange Eqn')




Given a solution to the Euler equation (such a solution is called an extremal), we can therefore distinguish the various minimality properties by matching adjectives and conditions:

| | |
|---|---|
| for necessary conditions: simple NN condition | for sufficient conditions: strict NN condition |
| for locally `xx` minimum no more | for globally `xx` minimum Jacobi Condition |
| for weak minimum Legendre Condition | for strong minimum Weierstrass Condition |

Moreover, by constructing *fields* of extremals ($n > 1$: Mayer fields), which is possible in some neighborhood of the extremal, whenever the Jacobi condition is verified, we can automatically deduce a domain, within which a relative minimum is indeed (weakly or strongly) minimal. In case it is possible to construct a Mayer field that covers all of $[t_0, t_1] \times \mathbb{R}^n$, and if the strict Legendre condition is verified in all of $[t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^n$, then each segment of the Mayer field is an absolute minimum with respect to its own boundary conditions.

There remains a very small margin between the necessary and sufficient conditions, which is why it is not at all obvious how to construct an example of the type $\int L(t, x, \dot{x})$ to illustrate the case ❸ . This is why I gave a more general type of functional for this case in the introduction.

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