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Branch Classification for the Spectrum of the Orr–Sommerfeld Equation for Plane Couette Flow

The spectrum of the Orr–Sommerfeld equation for Plane Couette Flow (solution continuum of the characteristic equation) undergoes only remarkably simple topological changes as the Reynolds number changes from 0 to ∞. The case of Reynolds number 0, which can be treated completely in full rigor, determines most of the structure for any finite Reynolds number, but not for infinite Reynolds number, the only complementary phenomenon being the so-called mode crossing.

1. Introduction

The purpose of this paper is to give a detailed rigorous branch classification of the eigenvalue spectrum of the Orr–Sommerfeld equation

$$\Big(\mathcal{R}^{-1}(\partial_y^2-\alpha^2)-i\alpha y-\lambda\Big)(\partial_y^2-\alpha^2)w(y)=0\,,\quad w(0)=0=w(1)\,,\ w'(0)=0=w'(1) \tag{1}$$

for plane Couette flow (pCf). This equation arises as the linearization of the Navier Stokes equation for a plane viscous incompressible shear flow between two parallel plates moving with respect to each other, as is detailed out, e.g., in [2],[4]. It has been shown by Romanov [3] that, for any Reynolds number $\mathcal{R}$ and any wave number α, all eigenvalues satisfy $\operatorname{Re}\lambda\le-\mu<0$, and also that asymptotic stability of the solution under the full Navier–Stokes equations ensues from this in the given case.

Given such a result, one still has to beware that the basin of attraction of the asymptotically stable solution may be very small, and it is in fact generally believed to shrink indefinitely as $\mathcal{R}\to\infty$ in the case of pCf. This explains superficially why instability is observed at large $\mathcal{R}$ in spite of Romanov's stability result. A more detailed account of the phenomenon is lacking. Trefethen and others [5] are working out scenarios where the non-normality of the linearized operator accounts for a large transient growth of initial perturbations, which will therefore be driven out of the basin of attraction by this effect. Much of that work is numerical, e.g. [6], or treats a model problem rather than the original equation [1], owing to the inherent difficulty. Any deeper understanding of the dynamics in a neighbourhood of the shear flow clearly requires not merely the half plane estimates for the spectrum, but a closer understanding of both the eigenvalues and the eigenfunctions coming with them (which amounts to a similar information as is contained in the pseudospectrum). This work gives rigorous results for the eigenvalues. The eigenfunctions will require further study, which is on the way. We give rough sketches of proofs only. Details will appear elsewhere.

2. The characteristic equation

By means of the transformation

$$2t=(\alpha\mathcal{R})^{1/3}\,,\quad \lambda=-\mathcal{R}^{-1}(4it^3+4t^2\delta)-\mathcal{R}^{-1}(8\mathcal{R}^{-1}t^3)^2\,,\quad \varepsilon=\mathcal{R}^{-1} \tag{2}$$

the eigenvalue condition of (1) can be written as $\Delta(\delta,t,\varepsilon)=0$ with the well-known Gram determinant

$$\Delta(\delta,t,\varepsilon):=\begin{vmatrix}\displaystyle\int_{-1}^{1}\mathrm{Ai}(\delta+itz)\sinh(4\varepsilon t^3z)\,dz & \displaystyle\int_{-1}^{1}\mathrm{Ai}(\delta+itz)\cosh(4\varepsilon t^3z)\,dz\\ \displaystyle\int_{-1}^{1}\mathrm{Bi}(\delta+itz)\sinh(4\varepsilon t^3z)\,dz & \displaystyle\int_{-1}^{1}\mathrm{Bi}(\delta+itz)\cosh(4\varepsilon t^3z)\,dz\end{vmatrix}\,. \tag{3}$$

It is sufficient to study this equation for $t>0$ and complex δ. Simplifications of (3) occur in various limiting cases. The case $\mathcal{R}\to0$, $\varepsilon\to\infty$, physically irrelevant as it may seem, leads to $\frac{\mathrm{Bi}}{\mathrm{Ai}}(\delta+it)=\frac{\mathrm{Bi}}{\mathrm{Ai}}(\delta-it)$ and can be handled comparatively easily and elegantly. It turns out that it not only gives the branch structure for sufficiently small $\mathcal{R}$, but actually up to arbitrarily large $\mathcal{R}$ with exception of only finitely many (depending on $\mathcal{R}$) branches. The limiting case $\mathcal{R}\to0$ is therefore a good starting point for the branch classification by homotopy arguments, and we shall give details below.

3. Series expansion

There is a remarkable series representation of (3) that seems to be completely new:

Theorem 1. *Let $x := 4\delta t^2$ and $A := 64\varepsilon^2 t^6$. Then we have the locally, in $(\mathbb{C}\setminus\{0\})\times\mathbb{R}\times[0,\infty[\,\ni(\delta,t,\varepsilon)$, uniformly absolutely convergent series expansion*

$$-\pi\frac{\Delta(\delta,t,\varepsilon)}{16i\varepsilon t^4} = \sum_{j=0}^{\infty}\frac{\delta^{-3j}}{(A+x)^{4j+2}}\Big[P_j^{00}(x,A) + P_j^{\mathrm{cc}}(x,A)\cos\sqrt{x}\,\cosh\sqrt{A} + \\ + P_j^{\mathrm{cs}}(x,A)\cos\sqrt{x}\,\frac{\sinh\sqrt{A}}{\sqrt{A}} + P_j^{\mathrm{sc}}(x,A)\frac{\sin\sqrt{x}}{\sqrt{x}}\,\cosh\sqrt{A} + P_j^{\mathrm{ss}}(x,A)\frac{\sin\sqrt{x}}{\sqrt{x}}\,\frac{\sinh\sqrt{A}}{\sqrt{A}}\Big] \tag{4}$$

with polynomials P_j^{**} consisting of monomials of which the joint degrees with respect to (x,A) are within the range given by the table:

deg of mon's in	P_j^{00}	P_j^{cc}	P_j^{cs}	P_j^{sc}	P_j^{ss}
...from	$4j$	$4j$	$4j+1$	$4j+1$	$4j+1$
...to	$5j$	$5j+\lfloor j/2\rfloor$	$5j+\lfloor (j+1)/2\rfloor$	$5j+\lfloor (j+1)/2\rfloor$	$5j+\lfloor j/2\rfloor+1$
deg wrt $A\le$	$2j$	$4j$	$4j+1$	$4j$	$4j+1$

The apparent pole at $x=-A$ is compensated, in each order j, by a zero of order at least $4j+2$ of the term in brackets.

The theorem is proved by means of a full Taylor expansion with respect to t of all terms under the integrals in (3). A lot of combinatorics is involved to recombine the terms into series (4). Explicit (though rather unwieldy) combinatorial formulas for the polynomials are available.

In the case $t < O(|\delta|^{1/4})$, the $j=0$ term of (4) in Theorem 1 is actually dominant and permits a full classification of branches in that domain:

Theorem 2. *Given $\varepsilon\ge 0$ and C sufficiently small, independent of ε, the zero set*

$$\Big\{(\delta,t)\in\mathbb{C}\times\mathbb{R}^+ \mid \varepsilon^{-1}\Delta(\delta,t,\varepsilon)=0\,,\ t<C|\delta|^{1/4}\Big\} \tag{5}$$

(*with the obvious interpretation of the case $\varepsilon=0$ as a limit $\varepsilon\to 0$*) *consists of* real *branches $(\delta_n(t),t)$, $n\ge 2$, only. These branches are totally ordered: $\delta_n(t)<\delta_{n+1}(t)$, the limit $\omega_n := \lim_{t\to 0} 2\sqrt{\delta_n(t)}\,t$ exists and is independent of $\varepsilon\in[0,\infty[$. It holds*

$$(n-1)\pi < 2\sqrt{\delta_n(t)}\,t < \begin{cases} (n+\frac{1}{2})\pi & \textit{for } n \textit{ even} \\ n\pi & \textit{for } n \textit{ odd}\end{cases}, \qquad \begin{matrix} \omega_n = n\pi & \textit{for } n \textit{ even} \\ \omega_n/2 = \tan\omega_n/2 & \textit{for } n \textit{ odd}\end{matrix}\ . \tag{6}$$

For $\varepsilon=\infty$ (where in (5) the corresponding limiting equation $\frac{\mathrm{Bi}}{\mathrm{Ai}}(\delta+it)=\frac{\mathrm{Bi}}{\mathrm{Ai}}(\delta-it)$ appears), it holds $\lim_{t\to 0}2\sqrt{\delta_n(t)}\,t=(n-1)\pi$. For arbitrary C, the same still applies to those branches indexed by $n\ge n_0(C)$.

These real solutions are the basis of the branch classification. Based on either the implicit function theorem (IFT) or, where $\partial_\delta\Delta=0$, on the Weierstraß preparation theorem (WPT), any solution to $\Delta(\delta,t,\varepsilon)=0$ can be followed for ever decreasing $t>0$, until it finally enters the domain covered by Theorem 2: In points where $\partial_\delta\Delta=0$, the IFT fails, but WPT shows that a branch point does actually occur, and one can follow the solution continuum for further decreasing t (nonuniquely, though, with a choice of branches). Therefore *all* solution branches are connected to the real branches discussed in Theorem 2 through (possibly several) bifurcations.

The proof of Theorem 2 is a Rouché counting argument, combined with an existence proof for real solutions by the intermediate value theorem, where the real solutions thus found already exhaust the total number of solutions determined by Rouché.

4. Small or finite Reynolds numbers

The behaviour outside the domain $t<O(|\delta|^{1/4})$ is determined by the following theorem:

Theorem 3. *For sufficiently large ε (small Reynolds number), the zero set $\mathcal{Z}_\varepsilon^+ := \{(\delta,t)\mid t>0\,,\ \Delta(\delta,t,\varepsilon)=0\}$ is homeomorphic to $\mathcal{Z}_\infty^+ := \{(\delta,t)\mid t>0\,,\ \frac{\mathrm{Bi}}{\mathrm{Ai}}(\delta+it)=\frac{\mathrm{Bi}}{\mathrm{Ai}}(\delta-it)\}$, namely it consists of*

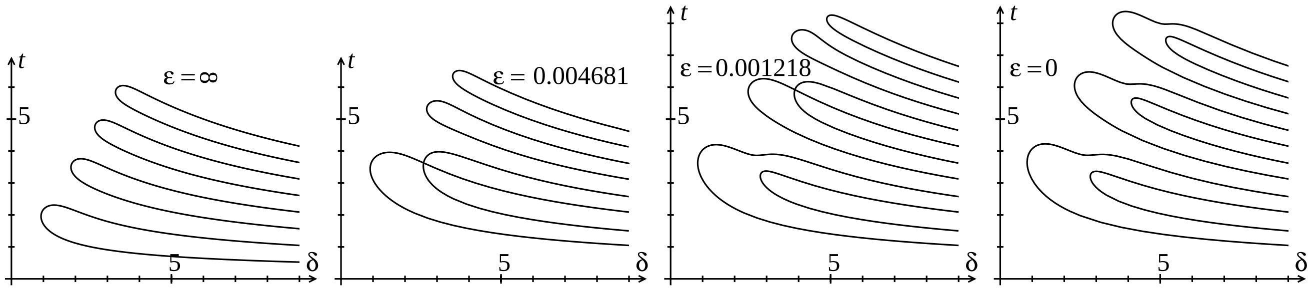
(1) *A family of real branches $\tilde{R}_j$, $j \geq 1$, connecting the $(\delta_{2j}(t), t)$ and $(\delta_{2j+1}(t), t)$ of Theorem 2 into a single curve on which the coordinate t has exactly one critical point, a nondegenerate maximum. (Call this point $\boldsymbol{r}_j$.)*

(2) *A corresponding branch $\tilde{C}_j$ of complex solutions $(\delta_j(t), t)$ and $(\bar{\delta}_j(t), t)$ branching off at $\boldsymbol{r}_j$ for increasing t and globally parametrized by t. As $t \to \infty$, $(\delta_j(t) - it)e^{-2\pi i/3}$ converges to $a_{-j} < 0$, the j^{th} zero of the Airy function* Ai *(independent of ε).*

For arbitrary $\varepsilon > 0$ (not necessarily large), the same applies to those branches indexed by $j \geq j_0(\varepsilon)$.

Given this result, it follows that changes in the topology of $\mathcal{Z}_\varepsilon^+$ can only accumulate as $\varepsilon \to 0$ ($\mathcal{R} \to \infty$) with $t \to \infty$ at the same time. A quantitative analysis shows that they actually are confined to a critical domain where $|t - \sqrt{3}\delta| < O(t^{-1/2} \ln t)$ as $t \to \infty$. By means of a separate calculation for the case $\varepsilon = 0$ and comparison to $\varepsilon > 0$ one sees that bifurcations actually do occur for all branches (see Theorem 4 below).

The figure shows the first few real branches for different values of ε. The asymptotic behaviour as $t \to 0$ is clearly visible, both for $\varepsilon < \infty$ as stated in (6) and for $\varepsilon = \infty$, where the asymptotic behaviour is shifted. The change of topology which affects the first real branches, one at a time, will be discussed in the following section.



5. The inviscid limit

Here the following branch classification holds. (Remember that the δ_n of Theorem 2 are numbered from $n = 2$.)

Theorem 4. *The solution continuum $\mathcal{Z}_0^+$ consists of*

(1a) *real branches $\tilde{R}_{2j}$, j sufficiently large, connecting the $(\delta_{4j-1}(t), t)$ and $(\delta_{4j}(t), t)$ of Theorem 2 into a single curve on which the coordinate t has exactly one critical point, a nondegenerate maximum. (Call this point $\boldsymbol{k}_{2j}$.)*

(1b) *real branches $\tilde{R}_{2j-1}$, j sufficiently large, connecting the $(\delta_{4j-2}(t), t)$ and $(\delta_{4j+1}(t), t)$ of Theorem 2 into a single curve on which the coordinate t has exactly three critical points, a nondegenerate minimum $\boldsymbol{k}_{2j-1}$ and two nondegenerate maxima $\boldsymbol{r}_{2j-1}$ and $\boldsymbol{r}_{2j}$.*

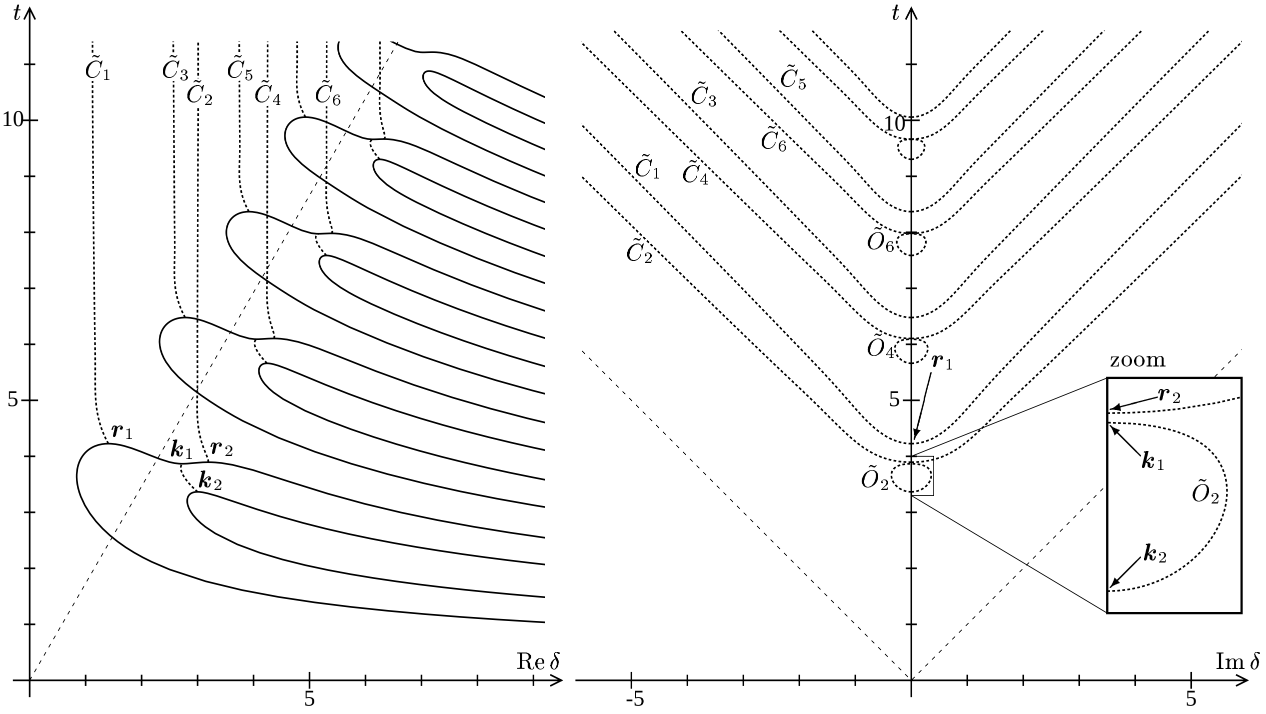
(2) *complex branches $\tilde{C}_{2j-1}$ and $\tilde{C}_{2j}$ bifurcating from $\tilde{R}_{2j-1}$ and $\tilde{R}_{2j}$ at $\boldsymbol{r}_{2j-1}$ and $\boldsymbol{r}_{2j}$ respectively. These consist of pairs of complex conjugate solutions parametrized globally by t as in Theorem 3. This time, however, $(\delta_j(t) - it)e^{-2\pi i/3}$ converges to zeros of $\int^\infty \mathrm{Ai}(x)\,dx$ instead, as $t \to \infty$.*

(3) *topological circles $\tilde{O}_{2j}$ connecting $\boldsymbol{k}_{2j-1}$ with $\boldsymbol{k}_{2j}$, and also consisting of pairs of complex conjugate solutions parametrized by t.*

Numerical evidence suggests that "j large enough" means actually "$j \geq 1$", thus exhausting all branches. Without resort to numerics, all asymptotic *statements hold for $j \geq 1$; it is only the way how the ends are connected inside a compact set that is affected by the hypothesis "j large enough".*

The proof of this theorem relies on asymptotic expansions of Airy functions. It is very likely possible, by enhancing the qualitative asymptotic estimates entering in the proof with quantitative error bounds, to abandon the hypothesis "j large" in favor of "$j \geq 1$". We have however not carried this through. Numerical evidence also suggests that the mode crossings that $\tilde{R}_{2j-1}$ and $\tilde{R}_{2j}$ undergo as $\varepsilon \to 0$ are the only bifurcations to occur, and that they occur one after another for $j = 1, 2, 3, \ldots$, with no changes of topology being subsequently undone again. A rigorous proof of this at least for large j is currently under investigation.

The figure shows the first few real and complex branches for $\varepsilon = 0$ ($\mathcal{R} = \infty$), projected in either the $(\operatorname{Re}\delta, t)$ or the $(\operatorname{Im}\delta, t)$ plane. The complex branches are dotted. The dashed lines ($t = \sqrt{3}\delta$ and $t = |\operatorname{Im}\delta|$) help to display the asymptotic behaviour.



6. Outlook

Next, a description of the corresponding eigenfunctions would be desirable. The method used to study the spectrum has this perspective in mind. In certain limiting cases, the eigenfunctions can be understood more easily: in particular, under the limit $t \to 0$, $2\sqrt{\delta}t \to \omega_n$, the αy term in (1) can be neglected, and the limiting eigenfunctions can be chosen as real. It is hoped that the classification of branches permits continuity methods to be applied to the eigenfunctions.

Acknowledgements

This work is part of my thesis of Habilitation at the Technische Universität München. It is a pleasure to acknowledge the continuing support of Professor Scheurle who opened this new field to me.

7. References

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