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A Study of the Spectral Theory of the Orr–Sommerfeld Equation for Plane Couette Flow

Eine Abhandlung, vorgelegt mit dem Ziele der Habilitation für das Fach Mathematik an der Technischen Universität München, im September 1999

Dank Zähigkeit verliert eine dünne Schicht von Flüssigkeitsteilchen nicht die Bodenhaftung

Gewidmet dem Andenken meines Vaters,
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Contents

List of Figures

Kapitel 0

Zusammenfassung und Einführung

Die Strömung einer zähen Flüssigkeit wird durch die Navier–Stokes–Gleichungen beschrieben. In Fällen einfacher Geometrie pflegen diese Gleichungen zeitunabhängige (stationäre) Lösungen zu haben, sogenannt laminare Strömungen, und es treten keine Turbulenzen auf. Beobachtet werden diese Strömungen in der Praxis aber nur, wenn sie stabil gegenüber den unvermeidlichen Störungen wie Rauhigkeit der Gefäßwände oder äußere Erschütterungen sind. Hierfür gilt in der Theorie der dynamischen Systeme die Regel, daß eine Lösung einer (nichtlinearen) Differentialgleichung dann asymptotisch stabil ist, wenn das Spektrum (häufig Eigenwerte) des in der linearisierten Gleichung $\partial_t u = Au$ auftretenden Operators A in der linken Halbebene liegt. Diese Regel ist ein Theorem für gewöhnliche Differentialgleichungen, bedarf aber weiterer Voraussetzungen, um auch für partielle Differentialgleichungen zu gelten. Typischerweise ist die nötige Zusatzvoraussetzung, daß das Spektrum sogar in einem Sektor mit Öffnungswinkel $< \pi$ in der linken Halbebene liegt.

Die Gleichungen der Hydrodynamik unterliegen einem Skalenverhalten, bei welchem eine Kombination aus einer typischen Längenabmessung, einer typischen Strömungsgeschwindigkeit und der Viskosität der Flüssigkeit eine dimensionslose Zahl ergibt, die Reynoldszahl. Diese ist nach physikalischer Erfahrung charakteristisch für das qualitative Strömungsverhalten, wobei bei niedrigen Reynoldszahlen laminare Strömungen auftreten, bei hohen Reynoldszahlen hingegen turbulente.

Als konkretes Beispiel entscheidet das (verallgemeinerte) Eigenwertproblem der Orr–Sommerfeld–Gleichung

$$\begin{aligned} &\Big[\Big(\varepsilon(\partial_y^2-\alpha^2) - i\alpha U(y) - \lambda\Big)(\partial_y^2-\alpha^2) + i\alpha U''(y)\Big]\, w(y) = 0\ , \\ &w(0) = 0 = w(1)\ , \quad w'(0) = 0 = w'(1)\ , \\ &\alpha\colon \text{Wellenzahl}\ ,\ \ \varepsilon^{-1}\colon \text{Reynoldszahl}\ ,\ \ \lambda\colon \text{Spektralparameter} \end{aligned} \tag{OS}$$

über die dynamische Stabilität der Grundströmung $\boldsymbol{u}(x,y,z) = U(y)\boldsymbol{e}_x$ einer inkompressiblen zähen Flüssigkeit zwischen zwei unendlich ausgedehnten, parallelen Platten. Der Fall $U(y) = y(1-y)$ entspricht dabei einer (durch einen Druckgradienten erzwungenen) Strömung zwischen festen Platten, während der Fall $U(y) = y$,

$$\Big(\varepsilon(\partial_y^2-\alpha^2) - i\alpha y - \lambda\Big)(\partial_y^2-\alpha^2)w(y) = 0 \tag{OSebC}$$

einer Scherströmung zwischen Platten entspricht, die sich mit konstanter Geschwindigkeit gegeneinander bewegen. Gegenstand der vorliegenden Arbeit ist ausschließlich der Fall der Scherströmung, das sogenannte ebene Couette–Problem. Dessen Stabilität für beliebige Reynoldszahlen wurde bereits 1973 durch Romanov mathematisch streng nachgewiesen.

In der Einleitung gebe ich eine Ausarbeitung von Romanovs Stabilitätsbeweis, in der insbesondere die (scheinbare) Diskrepanz zwischen dem (rein linearisiert <u>definierten</u>) Stabilitätsbegriff der hydrodynamischen Literatur einerseits und dem in der Theorie dynamischer Systeme üblichen

Begriff der asymptotischen Stabilität des vollen (nichtlinearen) Systems andererseits, der hiervon durch die Zusatzvoraussetzung der Sektorbedingung getrennt ist, überbrückt wird. (Diese Überbrückung ist implizit bei Romanov vorhanden, wird dort aber nicht besonders thematisiert.)

Mit dem Beweis, daß im Falle der OS–Gleichung für die ebene Couette–Strömung alle Eigenwerte in der linken Halbebene liegen, und zwar für jede Reynoldszahl, und dem nachfolgenden Beweis, daß dies im gegebenen Falle in der Tat asymptotische Stabilität impliziert, ist die Fragestellung aber nicht zu einem befriedigenden Abschluß gelangt. Selbstverständlich sagt die asymptotische Stabilität einer Lösung noch nicht viel über deren praktische Stabilität aus, denn der Einzugsbereich der stabilen Lösung kann so extrem klein sein, daß in allen praktischen Fällen auch die kleinsten realistischen Störungen die Lösung sehr leicht aus diesem Einzugsbereich heraustreiben. (Unter dem Einzugsbereich ist dabei die Gesamtheit der Anfangszustände zu verstehen, die im Laufe der Zeit in einer geeigneten Topologie gegen die stabile Lösung konvergieren.) Darauf, daß dieses Phänomen durch zeitlich transiente Vergrößerung von Anfangsstörungen besonders verstärkt werden kann, wurde in jüngster Zeit (z.B. durch Trefethen [28]) hingewiesen. Mechanismus dieses transienten Verhaltens ist die Nicht–Normalität des linearisierten Navier–Stokes–Operators. (Nicht–)Normalität versteht sich hier bzgl. des physikalisch sinnvollen Skalarprodukts auf einem relevanten Funktionenraum.

Praktische Instabilität wird im ebenen Couette–Problem für hohe Reynoldszahlen (ab etwa 350) tatsächlich beobachtet. Die mathematische Natur der dabei beobachteten (turbulenten oder „sogenannte turbulenten“) Lösungen harrt eines tieferen Verständnisses. Hierbei mag die Interpretation numerischer und experimenteller Ergebnisse ein mathematisches Verständnis wohl voranbringen, vielleicht aber auch in gewissem Umfange zur Voraussetzung haben, da ja die unvermeidbare Diskrepanz zwischen den endlichen Experimenten und dem Modell unendlich ausgedehnter Platten nur durch interpretierende Beurteilung überbrückt werden kann. Der Versuch, solche mathematischen Erklärungsansätze zu entwickeln, mag an sich schon ein schwieriges Unterfangen sein, er trifft aber schon im Anfangsstadium auf die Schwierigkeit, daß das Spektrum der OS–Gleichung in seiner Gesamtheit, jenseits der Frage, ob es in der linken Halbebene liege, nicht vollständig verstanden ist, verstandene Teile hingegen in einer unüberschaubaren Literatur verstreut sind, in der Begriffe wie „bekannt“, „wohlbekannt“, „wohl bekannt“ oder „bewiesen“ je nach Fachrichtung unterschiedliche Bedeutungen annehmen.

Das Hauptziel der vorliegenden Arbeit ist es, das Spektrum der OS–Gleichung in seiner Gesamtheit geometrisch zu beschreiben und die Resultate weitestmöglich mathematisch streng zu beweisen. Dabei spielen insbesondere die Airy–Funktionen Ai und Bi, welche Lösungen der gewöhnlichen Differentialgleichung $w''(z) = zw(z)$ sind, oder Linearkombinationen davon eine Rolle, denn es kommt (unvermeidlicherweise) die charakteristische Gleichung

$$\Delta(\delta,t,\varepsilon) := \begin{vmatrix} \int_{-1}^{1} \mathrm{Ai}(\delta+itz)\sinh(4\varepsilon t^3 z)\,dz & \int_{-1}^{1} \mathrm{Ai}(\delta+itz)\cosh(4\varepsilon t^3 z)\,dz \\ \int_{-1}^{1} \mathrm{Bi}(\delta+itz)\sinh(4\varepsilon t^3 z)\,dz & \int_{-1}^{1} \mathrm{Bi}(\delta+itz)\cosh(4\varepsilon t^3 z)\,dz \end{vmatrix} = 0 \qquad \text{(CharEq)}$$

zum Einsatz, die dem Spektralproblem vermöge

$$2t = (\alpha/\varepsilon)^{1/3}\ , \quad \lambda = -\varepsilon(4it^3 + 4t^2\delta) - (8\varepsilon t^3)^2$$

äquivalent ist und von der (im Sinne des Prinzips von der Erhaltung der Schwierigkeit) Synge 1938 schreibt:

> The deduction of general results from [this equation] is well-nigh impossible.

(Zitiert nach Drazin/Reid [5], S. 214). (Die hier eingeführte Variable t hat nichts mit der Zeit zu tun.) Die in Frage kommenden Methoden sind problembedingt weitgehend vorgegeben: Abgrenzung von Gebieten im Parameterraum, Satz von Rouché oder verwandte topologische Methoden und Behandlung verschiedener Grenzfälle durch asymptotische Entwicklungen der Airy–Funktionen. Üblicherweise interpretiert man das Spektralproblem in dem Sinne, daß periodische

Randbedingungen in den Richtungen unendlicher Ausdehnung der Platten gestellt werden, so daß nach Fourieranlayse der Wellenzahlparameter α in (OS) diskrete Werte annimmt, abhängig vom (willkürlichen) Periodizitätsgitter. In der charakteristischen Gleichung (CharEq) hingegen ist es natürlich, die Wellenzahl als kontinuierlichen Parameter zu betrachten (und kein Gitter auszuzeichnen). Die Information für den Fall diskreter Werte der Wellenzahl gewinnt man durch Spezialisierung von α leicht zurück.

In Kapitel 1 erläutere ich die Genese der Orr–Sommerfeld–Gleichung aus den Navier–Stokes–Gleichungen. Einen Überblick über die wichtigsten Resultate zur OS–Gleichung gebe ich in Kapitel 2. Für spätere Verwendung stelle ich in Kapitel 3 die wesentlichen Eigenschaften der Airy–Funktionen zusammen, soweit sie ganz unabhängig von der OS–Gleichung von Interesse sind. Speziellere Eigenschaften der Airy–Funktionen gebe ich mit Beweisen in Abschnitt 5.2.1 an.

Als Fundament der Analyse von (CharEq) kann man den Grenzfall kleiner Wellenzahlen sehen, in dem als Lösungen nur reelle δ vorkommen können und das Verhalten für alle Reynoldszahlen (einschließlich geeignet interpretierter Grenzfälle 0 und ∞) qualitativ gleich ist. Dieser Teil der Untersuchung ist Gegenstand von Kapitel 4, insbesondere Abschnitt 4.3 und wird weiter unten noch näher ausgeführt. Qualitativ gleich für alle Reynoldszahlen ist auch der Bereich großer Wellenzahlen (die Asymptotik $t \to \infty$ bei beschränktem $\operatorname{Re}\delta$). Hier treten komplexe Lösungsäste auf (d.h., δ ist komplex), deren asymptotisches Verhalten einfachen Bedingungen genügt. Quantitativ zeigt sich die Singularität beider Limiten ($\varepsilon \to 0$ und $\varepsilon \to \infty$) in Sprungunstetigkeiten des asymptotischen Verhaltens:

<table>
<tr><th>Grenzfall</th><th>$\varepsilon = 0$</th><th>$0 < \varepsilon < \infty$</th><th>$\varepsilon = \infty$</th></tr>
<tr><td>reeller Ast:
$t \to 0,\ \delta \to \infty$
$x := \lim\limits_{t\to 0} \delta^{1/2} t$
erfüllt:</td><td colspan="2">$\sin x\,(x\cos x - \sin x) = 0$</td><td>$\sin x \cos x = 0$</td></tr>
<tr><td>komplexer Ast:
$t \to \infty,\ \operatorname{Im}\delta > 0$
$y := e^{-2\pi i/3} \lim\limits_{t\to\infty} (\delta - it)$
erfüllt:</td><td>$\int_y^\infty \operatorname{Ai}(z)\,dz = 0$</td><td colspan="2">$\operatorname{Ai}(y) = 0$</td></tr>
</table>

Wie die von $t = \infty$ kommenden komplexen und die von $t = 0$ kommenden reellen Äste zusammenhängen, entscheidet sich ausschließlich in einem kritischen Bereich, der innerhalb der durch die Bedingung $|t - \sqrt{3}\delta| < const$ gegebenen Menge liegt. Dies liegt daran, daß die erwähnten Äste von anderen Ästen durch a–priori–Schranken für Lösungen abgetrennt werden, welche in den Parameterraum „Wände“ einziehen, die das Lösungskontinuum nicht durchdringen kann. Soweit diese Wände einzelne Äste isolieren, kann von diesen kein anderer Ast abzweigen. und zwar deshalb, weil die Anzahl der Lösungen δ von $\Delta(\delta, t, \varepsilon) = 0$ in beschränkten Gebieten als Funktion von t konstant ist, solange keine Lösungen durch den Rand eindringen oder verschwinden. Die analytische Abhängigkeit von den Parametern geht hier entscheidend ein. Der Umstand, daß t nur reeller Werte fähig ist, zeichnet t gegenüber δ als Parameter für die Äste aus, und die Analytizität erlaubt es, auf dem Lösungskontinuum beliebig in Richtung wachsender oder fallender Werte von t voranzuschreiten: Wenn der Satz über implizite Funktionen dies wegen Verletzung seiner Voraussetzungen nicht mehr garantiert, tritt als eine (analytische) Verallgemeinerung der Weierstraßsche Vorbereitungssatz an seine Stelle und garantiert die Fortsetzung des Lösungskontinuums jenseits des kritischen Punktes auf einem abzweigenden Ast. Auf diese Weise ist jede Lösung von (CharEq) topologisch mit den (vergleichsweise) leicht zu klassifizierenden reellen Lösungen im Bereich $t < O(|\delta|^{1/4})$ verbunden.

Diese Prinzipien, nämlich das Verfolgen von Lösungsästen bis in den kritischen Bereich, wobei diese von anderen Ästen durch Wände getrennt sind, ist sowohl für den Fall $\varepsilon > 0$ (Abschnitt 5.3) als auch für den Fall $\varepsilon = 0$ (Kapitel 6) ein entscheidendes Hilfsmittel.

Die Analyse im kritischen Bereich $|t - \sqrt{3}\delta| < const$ ist naturgemäß die schwierigste, da sie alle auftretenden Verzweigungen des Lösungskontinuums, abhängig von der Reynoldszahl ε^{-1} erklären muß. Sie wird mit asymptotischen Methoden durchgeführt, was den Nachteil hat, daß die Beschreibung zunächst nur bis auf eventuell endlich viele Ausnahmeäste (d.h. außerhalb eines Kompaktums) gilt. Numerisch scheinen diese Ausnahmen gar nicht aufzutreten, und es sollte möglich sein, durch geeignete Intervallarithmetik und mit quantitativen Restgliedabschätzungen der asymptotischen Reihen dies streng zu zeigen. Ich habe einen solchen Versuch nicht unternommen und beschränke mich auf die asymptotische Analyse. (Siehe insbesondere Abschnitt 6.3.2.)

Analytisch erfreulich elegant läßt sich der Grenzfall $\varepsilon = \infty$ (Reynoldszahl 0) abhandeln, da er auf die Gleichung $\mathrm{Bi}(\delta + it)/\,\mathrm{Ai}(\delta + it) = \mathrm{Bi}(\delta - it)/\,\mathrm{Ai}(\delta - it)$ führt. Entscheidend dabei ist ein Studium der Niveaulinien der Funktion $z \mapsto |\mathrm{Bi}(z)/\,\mathrm{Ai}(z)|$, wobei die Kenntnis der Nullstellen von Ai und Bi wesentlich eingeht (Abschnitt 5.2 und Figur auf Seite 62). Störungstheoretisch ist von dort der Fall großer ε zugänglich. Genaugenommen ist nicht ε, sondern das Produkt $4\varepsilon t^3$, das ja in die Gleichung (CharEq) eingeht, entscheidend, also genau die Wellenzahl. Der formale Grenzwert $\varepsilon \to \infty$, der zunächst physikalisch wenig interessant erscheinen mag, erweist sich also zugleich für jedes feste $\varepsilon > 0$ physikalisch als Grenzwert großer Wellenzahlen. So liegt für jedes $\varepsilon > 0$ (d.h. jede endliche Reynoldszahl, wie groß auch immer) außerhalb eines (natürlich von ε abhängigen) Kompaktums eine Störung des Falles der Reynoldszahl 0 vor. Daher können sich Änderungen der Topologie des Lösungskontinuums in Abhängigkeit von der Reynoldszahl nur für $\varepsilon \to 0$ häufen. Dies ist die wesentliche Erkenntnis aus Abschnitt 5.3. Mit asymptotischen Methoden läßt sich nun der Fall $\varepsilon = 0$ untersuchen, und man stellt fest, daß gegenüber dem Falle großer ε in der Tat unendliche viele topologische Änderungen stattgefunden haben.

Das Studium des Falles $\varepsilon = 0$ ist Gegenstand von Kapitel 6. Im wesentlichen sind die Änderungen des Spektrums gegenüber dem Falle großer ε von der folgenden Art: Wenn man die reellen Lösungsäste nach ihrer natürlichen Anordnung und zugleich ihrem asymptotischen Verhalten (wie in der obigen Tabelle angegeben) durchnumeriert, so sind für große ε benachbarte Äste paarweise verbunden: 1 mit 2, 3 mit 4 usw. Für $\varepsilon = 0$ hingegen sind die Äste in Vierergruppen verbunden: 1 mit 4 und 2 mit 3, sodann 5 mit 8 und 6 mit 7, usw. Dies wurde schon von Gallagher [6] numerisch beobachtet und treffend als “mode crossing” bezeichnet; eine analytische Erklärung hat er nicht gegeben. Die führenden Terme der Asymptotik im kritischen Bereich finden sich bereits bei Reid [22]. Aber erst der Vergleich mit den Resultaten für große ε und die Erkenntnis, daß für jedes $\varepsilon > 0$ “mode crossing” höchstens endlich viele Moden betrifft, erklärt die Diskrepanz zwischen gelegentlich numerisch beobachtetem und gelegentlich nicht beobachtetem “mode crossing”, die Reid [22] abschließend als einen noch etwas obskuren Punkt thematisiert.

Numerische Evidenz spricht dafür, daß beim Verkleinern von ε nur solche Bifurkationen des Spektrums auftauchen, die dann bis $\varepsilon = 0$ auch erhalten bleiben, daß also die Verzweigungsrichtung bzgl. ε stets eine einseitige ist und daher die Gesamtheit stattgefundener Verzweigungen im Grenzfalle $\varepsilon = 0$ bewahrt ist. Dies mathematisch streng zu zeigen (evtl. bis auf Kompakta) bleibt Ziel weiterführender Untersuchungen, die noch nicht Gegenstand der vorgelegten Schrift sind. Desgleichen die Untersuchung beweisbarer qualitativer Eigenschaften der zugehörigen Eigenfunktionen.

Zumindest für kleine Werte von (δ, t), etwa bis 10, und für $\varepsilon \lesssim 0.01$ bereitet die Untersuchung der charakteristischen Gleichung mit einem Verfahren steilsten Abstiegs keine numerischen Probleme. Eine ad–hoc Prädiktor–Korrektor–Strategie für die Schrittweite auf den zu berechnenden Lösungskurven, über deren Anfang nahe ∞ man aus der Asymptotik gut Bescheid weiß, erlaubt auch dem nicht numerisch spezialisierten Mathematiker die Berechnung von Eigenwerten mit Mathematica, auch durch Verzweigungspunkte hindurch. Für große (δ, t) machen die oszillierenden Integrale Schwierigkeiten, aber dort greift sowieso das Verständnis mit Hilfe asymptotischer Methoden.

Zwei Besonderheiten sind noch zu erwähnen:

Die Untersuchung des Verhaltens für kleine t und der Beweis, daß dort nur reelle Lösungen δ auftauchen, führt auf eine Reihendarstellung der charakteristischen Gleichung, deren erstes Glied allein relevant ist. Tatsächlich handelt es sich bei dieser Reihe um eine konvergente Reihe, die durch Umordnung der vollständigen Taylorentwicklung von (CharEq) in allen Parametern entsteht, wobei überraschenderweise die Neuzusammenfassung der Koeffizienten ziemlich explizit möglich ist. Diese Reihenumordnung involviert eine ganze Menge kombinatorischer Überlegungen, die in Kapitel 4 ausgeführt werden. Die Determinante in (CharEq) führt dabei auf höhere Wronskische $\mathrm{Ai}^{(n)}\,\mathrm{Bi}^{(m)} - \mathrm{Bi}^{(n)}\,\mathrm{Ai}^{(m)}$. Das Verhalten der Funktion Δ wird in diesem Bereiche wesentlich durch die neue Variable δt^2 bestimmt, deren Signifikanz aus (CharEq) selbst noch nicht zu erkennen ist. Die Reihendarstellung von Δ (Theorem 4.7) scheint, soweit ich sehe, in der Literatur völlig neu zu sein.

Zweitens ist, da alle involvierten Funktionen Lösungen linearer gewöhnlicher Differentialgleichungen sind, auch der Ausdruck Δ Lösung gewisser linearer (partieller oder gewöhnlicher) Differentialgleichungen geeignet hoher Ordnung. Diese Gleichungen werden hergeleitet und zur Verfügung gestellt (Kapitel 7). Inwieweit sie nützlich sind, bleibt abzuwarten.

Von den beiden letztgenannten Nebenresultaten abgesehen sind die verwendeten Methoden geläufig, und auch die verschiedenen Grenzfälle wurden wiederholt betrachtet. Ich habe dafür jeweils (frühe) Referenzen angegeben, ohne den hoffnungslosen Versuch zu wagen, eine komplette Bibliographie zusammenzustellen. Nach meinem besten Wissen ist aber unter der Rahmenbedingung mathematischer Strenge eine ähnlich weitgehende Analyse des Spektrums in der Literatur nicht vorhanden. Überdies scheinen auch etablierte Resultate wie das von Romanov weniger bekannt zu sein, so daß eine auf das spezielle Problem (OSebC) fokussierte Synopse auch unter diesem Aspekt Nutzen verspricht.

Ein Verständnis der topologischen Struktur der Äste des Spektrums weckt Hoffnungen auf Kontinuitätsmethoden zur Untersuchung der Eigenfunktionen. Die Eigenwerte, die dem Grenzfall $t \to 0$, $\delta \to \infty$ auf den reellen Ästen entsprechen, sind nämlich selbst reell, und dort verhalten sich die Eigenfunktionen ähnlich wie bei Sturm-Liouville–Problemen. Hier bieten sich weitere Untersuchungen an.

Chapter 0

Abstract and Introduction

The flow of a viscous fluid is described by the Navier–Stokes equations. If the geometry is simple, these equations use to have time–independent (stationary) solutions, so called laminar flows, and turbulence does not occur. However these flows can be observed in reality only if they are stable with respect to the unavoidable perturbations like a coarse surface of the container or exogenic vibrations. In the theory of dynamical systems, there is the rule that a solution to a (nonlinear) differential equation is asymptotically stable, if the spectrum (often consisting of eigenvalues) of the operator A occurring in the linearized equation $\partial_t u = Au$ lies in the left half plane. This rule is a theorem in the case of ordinary differential equations, but needs additional assumptions in order to be valid for partial differential equations as well. Typically, the extra assumption needed is that the spectrum should even be contained in a sector whose vertex angle is $< \pi$ and which is contained in the left half plane.

The equations of hydrodynamics enjoy a scaling property: from a certain combination of a typical length scale, a typical flow velocity and the viscosity of the fluid, one may form a dimensionless number, the Reynolds number. According to the experience of physics, the Reynolds number characterizes the qualitative behaviour of the flow such that laminar flows occur for small Reynolds numbers, whereas turbulent ones occur for large Reynolds numbers.

As a concrete example, the (generalized) eigenvalue problem for the Orr–Sommerfeld equation

$$\begin{aligned} &\Big[\Big(\varepsilon(\partial_y^2-\alpha^2)-i\alpha U(y)-\lambda\Big)(\partial_y^2-\alpha^2)+i\alpha U''(y)\Big]\, w(y)=0\ , \\ &w(0)=0=w(1)\ , \quad w'(0)=0=w'(1)\ , \\ &\alpha:\text{wave number}\ ,\ \ \varepsilon^{-1}:\text{Reynolds number}\ ,\ \ \lambda:\text{spectral parameter} \end{aligned} \tag{OS}$$

determines the dynamical stability of the laminar flow $\boldsymbol{u}(x,y,z) = U(y)\boldsymbol{e}_x$ of an incompressible viscous fluid between parallel plates that extend indefinitely. Here, the case $U(y) = y(1-y)$ corresponds to a flow between fixed plates (caused by a pressure gradient), whereas the case $U(y) = y$,

$$\Big(\varepsilon(\partial_y^2-\alpha^2)-i\alpha y-\lambda\Big)(\partial_y^2-\alpha^2)w(y)=0 \tag{OSplC}$$

describes a shear flow between plates that are moving with respect to each other with constant velocity. The present work is concerned exclusively with the case of a shear flow, the so called plane Couette problem. The stability of this flow for arbitrary Reynolds numbers was proven by Romanov in full mathematical rigor as early as 1973.

In the introduction, I am offering an extended resume of Romanov's stability proof which bridges in particular the (apparent) discrepancy between the stability concept of hydrodynamics (which is defined by linearization) on one hand and the concept of asymptotic stability of the full non-linear system, common in the theory of dynamical systems, on the other hand; this latter is separated from linearized stability by the extra assumption of sectoriality. (This gap is bridged implicitly in Romanov's work, although he does not particularly stress the point.)

However, the proof that all eigenvalues for the Orr–Sommerfeld equation of plane Couette flow lie in the left half plane, for any Reynolds number, plus the proof that for the case at hand, this implies indeed asymptotic stability, does not conclude the question in a satisfactory way. Clearly, asymptotic stability of a solution does not necessarily tell much about the practical stability of that solution, because the basin of attraction of the stable solution may be that tiny that in practice even the very smallest among realistic perturbations can easily drive the solution out of that basin of attraction. (The basin of attraction is defined as the set of all initial states that converge to the stable solution in an appropriate topology, as the time gets large.) Recently, e.g., Trefethen [28] has pointed out that this phenomenon can be reinforced by an amplification of initial perturbations that is transient in time. Non–normality of the linearized Navier–Stokes operator is a mechanism for this transient behaviour; here, (non–)normality is to be understood with respect to such a scalar product on a relevant function space as is natural from a physical point of view.

Practical instability at high Reynolds numbers (beginning at approximately 350) is actually observed in the plane Couette problem. A mathematical characterization of the observed solutions (turbulent or so called "turbulent") still lacks a deeper understanding. In that pursuit, the interpretation of numerical and experimental results may well foster a mathematical understanding, but may, on the other hand, also ground on such an understanding, given that the gap between finite experiments and the model of indefinitely extended plates can only be bridged by careful judgement and interpretation. Any attempt to develop rudiments of a mathematical explanation, difficult as the enterprise may anyway be, is hampered in its very beginning by the difficulty that the spectrum of the Orr–Sommerfeld equation as a whole, beyond the question whether it is confined to the left half plane, is not completely understood, whereas parts that are understood are scattered in a vast literature in which concepts like "known", "well-known", "seems to be known" or "proven" change their meaning dependent on which scientific community is involved.

The present work aims at a geometric description of the spectrum of the Orr–Sommerfeld equation as a whole, and at proving the results in full rigor, if at all possible. In particular, the Airy functions Ai and Bi or linear combinations thereof, in other words, the solutions of the ODE $w''(z) = zw(z)$, play a basic role, because one necessarily has to solve the characteristic equation

$$\Delta(\delta,t,\varepsilon) := \begin{vmatrix} \int_{-1}^{1} \mathrm{Ai}(\delta+itz)\sinh(4\varepsilon t^3 z)\,dz & \int_{-1}^{1} \mathrm{Ai}(\delta+itz)\cosh(4\varepsilon t^3 z)\,dz \\ \int_{-1}^{1} \mathrm{Bi}(\delta+itz)\sinh(4\varepsilon t^3 z)\,dz & \int_{-1}^{1} \mathrm{Bi}(\delta+itz)\cosh(4\varepsilon t^3 z)\,dz \end{vmatrix} = 0\ , \qquad \text{(CharEq)}$$

which is equivalent to the spectral problem through the relation

$$2t = (\alpha/\varepsilon)^{1/3}\ , \quad \lambda = -\varepsilon(4it^3 + 4t^2\delta) - (8\varepsilon t^3)^2$$

and about which Synge wrote in 1938:

> The deduction of general results from [this equation] is well-nigh impossible.

(Quotation following Drazin/Reid [5], p. 214). (The variable t introduced here has nothing to do with time.) The problem determines the methods available for its solution, namely delimiting domains in parameter space, using Rouché's theorem or related topological methods, and discussion of certain limiting cases by means of asymptotic expansions of the Airy functions. The spectral problem is commonly understood with respect to periodic boundary conditions in the directions of indefinite extension of the plate, and by Fourier analysis, this attributes discrete values to the wave number parameter, which are however dependent on the arbitrary lattice of periodicity. In the characteristic equation (CharEq) however, the wave number is used as a continuous parameter, and no lattice is selected. The information for the case of discrete values of the wave number can trivially be recovered by specialization.

In Chapter 1, I explain the genesis of the Orr–Sommerfeld equation from the Navier–Stokes equations. A review of some fundamental results concerning the OS equation is then given in Chapter 2. For the convenience of subsequent use, I collect those fundamental properties about the Airy functions in Chapter 3 which are of interest independent of the OS equation. More special properties of the Airy functions will be given and proved in Section 5.2.1.

The limiting case of small wave numbers may be seen as the foundation on which the analysis of (CharEq) rests. In this case, only real δ can occur as solutions, and the behaviour is qualitatively the same for all Reynolds numbers (including properly defined limiting cases of Reynolds numbers 0 and ∞). This part of the investigation is carried out in Chapter 4, in particular Section 4.3 and will be explained in more detail below. To have the same qualitative behaviour for all Reynolds numbers is also a feature of the limiting case of large wave numbers (more precisely, the asymptotics as $t \to \infty$ with $\operatorname{Re}\delta$ bounded). In this limit, there occur complex solution branches (i.e., δ is complex) whose asymptotic behaviour is governed by simple conditions. Quantitatively, however, the singularity of both limits ($\varepsilon \to 0$ and $\varepsilon \to \infty$) shows as a jump discontinuity of the asymptotic behaviour:

<table>
<tr><th>Limiting case</th><th>$\varepsilon = 0$</th><th>$0 < \varepsilon < \infty$</th><th>$\varepsilon = \infty$</th></tr>
<tr><td>real branch:
$t \to 0,\ \delta \to \infty$
$x := \lim\limits_{t\to 0} \delta^{1/2} t$
satisfies:</td><td colspan="2">$\sin x\,(x\cos x - \sin x) = 0$</td><td>$\sin x \cos x = 0$</td></tr>
<tr><td>complex branch:
$t \to \infty,\ \operatorname{Im}\delta > 0$
$y := e^{-2\pi i/3} \lim\limits_{t\to\infty} (\delta - it)$
satisfies:</td><td>$\int_y^\infty \operatorname{Ai}(z)\,dz = 0$</td><td colspan="2">$\operatorname{Ai}(y) = 0$</td></tr>
</table>

It is only a certain critical domain, contained in the set $|t - \sqrt{3}\delta| < const$, that determines how the complex branches coming from $t = \infty$ connect to the real branches coming from $t = 0$. The reason is that the branches are separated from other branches by a–priori bounds for solutions, so there are "walls" in the parameter space that cannot be crossed by the solution continuum. As long as these walls isolate single branches, no other branch can bifurcate from an isolated one, because the number of solutions δ to $\Delta(\delta, t, \varepsilon) = 0$ in bounded domains is constant as a function of t, as long as solutions cannot enter or leave through the boundary. The analytic dependence on all parameters is used here. The fact that t takes on only real values distinguishes t as a convenient parameter on the branches, and analyticity permits to follow the solution continuum indefinitely with ever decreasing or ever increasing values of t: As soon as this is no longer guaranteed by the implicit function theorem (because its hypotheses are violated), the Weierstraß preparation theorem is available as a generalization, and it guarantees that it is possible to continue the solution set beyond the critical point on a bifurcating branch. This way, <u>every</u> solution to (CharEq) is topologically connected with the real solutions in the domain $t < O(|\delta|^{1/4})$, which can be classified comparatively easily.

These basic tools, namely to follow the solution branches until they enter the critical domain, during which process the branches are separated from each other by walls, is a crucial ingredient both in the case $\varepsilon > 0$ (Section 5.3) and in the case $\varepsilon = 0$ (Chapter 6).

Obviously, the analysis within the critical domain $|t - \sqrt{3}\delta| < const$ is the most difficult, because it has to explain all bifurcations of the solution continuum that may occur as the Reynolds number ε^{-1} changes. This analysis is carried out by means of asymptotic methods, of which a disadvantage is that a–priori, the description is valid up to possibly finitely many exceptional branches (i.e. outside some compact set). Numerically, these exceptions do not appear to occur. It should be possible to prove this using interval arithmetics and quantitative bounds for the error terms. I have however not made such an attempt, but restrict the discussion to the asymptotic analysis. (See in particular Section 6.3.2.)

The limiting case $\varepsilon = \infty$ (Reynolds number 0) can be treated with elegance, because it leads to the equation $\mathrm{Bi}(\delta+it)/\mathrm{Ai}(\delta+it) = \mathrm{Bi}(\delta-it)/\mathrm{Ai}(\delta-it)$. It is a corollary to a study of the level lines of the function $z \mapsto |\mathrm{Bi}(z)/\mathrm{Ai}(z)|$, and a knowledge of the zeros of Ai and Bi is crucial here (Section 5.2 and the figure on page 62). The case of large ε is accessible from there by means of perturbation theoretic arguments. More precisely, it is not ε, but the product $4\varepsilon t^3$, namely the wave number, that plays a decisive role and also enters in (CharEq). The formal limit $\varepsilon \to \infty$, of little physical interest as it may seem at first glance, turns out to be, for every fixed $\varepsilon > 0$, the limit of large wave numbers. Therefore, for any $\varepsilon > 0$ (i.e., for any finite Reynolds number whatever large it may be), we have a perturbation of the the case of Reynolds number zero outside some compact set, which of course depends on ε. Hence changes in the topology of the solution continuum under changes of the Reynolds number can only accumulate at $\varepsilon = 0$. This is the key observation from Section 5.3. The case $\varepsilon = 0$ itself can be analyzed by means of asymptotic methods, and it turns out that indeed infinitely many changes in topology have taken place in comparison to the case of large ε.

A detailed study of the case $\varepsilon = 0$ is carried out in Chapter 6. Essentially, the changes of the spectrum as compared to the case of large ε are as follows: suppose that the real solution branches are numbered in their natural order, which is at the same time given by their asymptotic behaviour as specified in the table above. Then, for large ε, neighbouring branches are connected pairwise: 1 to 2, 3 to 4, etc. In contrast, for $\varepsilon = 0$, the branches are connected in groups of four: 1 to 4 and 2 to 3; next 5 to 8 and 6 to 7; etc. This observation has already been made by Gallagher [6] numerically, and quite appropriately he called it mode crossing; he did not give an analytic explanation. The leading terms of the asymptotic behaviour in the critical domain can already be found in Reid [22]. But it is only in comparison with the results for large ε and by understanding that for any $\varepsilon > 0$ only finitely many modes have crossed that the discrepancy between occasional observations and non-observations of mode crossing is lucidly explained. This discrepancy is addressed by Reid in the conclusion of his paper as a somewhat obscure point.

Numerical evidence suggests that only such bifurcations occur when decreasing ε that remain intact until $\varepsilon = 0$, in other words, that the direction of bifurcations is always the same and the totality of bifurcations that have taken place is preserved in the limiting case $\varepsilon = 0$. It will be an object of further pursuit to prove this rigorously (possibly up to compact sets), but this matter is not yet part of the present work. Similarly, research on provable qualitative properties of eigenfunctions is still pending.

At least for small values of (δ, t), up to approximately 10, and for $\varepsilon \lesssim 0.01$, the numerical study of the characteristic equation using the method of steepest descent presents no difficulties. An ad–hoc predictor corrector strategy for the stepwidth on the solution curves to be determined makes it possible also for the mathematician who is not specialized in numerical methods to calculate the eigenvalues with the help of Mathematica, crossing branch points included. The start of these curves near infinity is easily accessible asymptotic information. For large values of (δ, t), one has trouble with the oscillatory integrals, but this regime is anyway amenable to an understanding through asymptotic methods.

Two peculiarities deserve mentioning:

A study of the behaviour for small t and the proof that only real solutions δ occur in this case leads to a series expansion of the characteristic equation of which only the first term is relevant. As a matter of fact, this series is convergent, namely it arises from a complete Taylor expansion with respect to all parameters of (CharEq), and surprisingly the rearrangement of coefficients can be done rather explicitly. This rearrangement of the series involves a bunch of combinatorial arguments, which are detailed out in Chapter 4. The determinant involved in (CharEq) leads to higher Wronskians $\mathrm{Ai}^{(n)}\,\mathrm{Bi}^{(m)} - \mathrm{Bi}^{(n)}\,\mathrm{Ai}^{(m)}$. In the domain under discussion, the behaviour of the function Δ is characterized by the new variable δt^2 of which the significance cannot be recognized in (CharEq). To the best of my knowledge, the series representation of Δ given in Theorem 4.7 seems to be completely new in the literature.

Secondly, as all involved functions solve linear ordinary differential equations, the expression Δ itself also satisfies certain linear (partial or ordinary) differential equations of sufficiently high order. These equations are derived and made available (Chapter 7). Time will show how useful they can be.

Apart from these two spinoff results, the methods used are common, and the different limiting cases have been the object of repeated study. I have given (early) references, without venturing into a hopeless attempt to collect a complete bibliography. To the best of my knowledge, however, no similarly thorough analysis of the spectrum as a whole under the constraint of analytic rigor is available in the literature. Moreover, even established results like Romanov's appear to be less well-known, which makes a synopsis focused on that particular problem (OSplC) hopefully useful.

An understanding of the topological structure of the branches of the spectrum gives some hope that continuity methods may help in the investigation of the eigenfunctions. Those eigenvalues that correspond to the limiting case $t \to 0$, $\delta \to \infty$ on the real branches are real numbers themselves, and there, the eigenfunctions behave similarly as in the case of Sturm Liouville problems. Further research may issue from this point.

Chapter 1

The Genesis of the Orr–Sommerfeld Equation

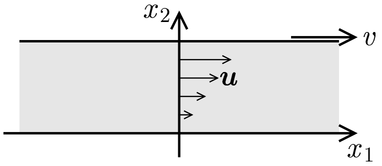
The Navier–Stokes equations for an incompressible flow in the absence of external forces are

$$\begin{aligned} &\boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\boldsymbol{\nabla})\boldsymbol{u} + \rho^{-1}\boldsymbol{\nabla}p = \boldsymbol{0} \\ &\mathrm{div}\boldsymbol{u} = 0\ . \end{aligned} \tag{1.1}$$

Here, $\boldsymbol{u}$ denotes the velocity field of the flow, referred to a coordinate system that is fixed in space, ν is the so-called kinematic viscosity (equal the dynamic viscosity divided by the density ρ) and p denotes the pressure. All of these are functions of space $\boldsymbol{x}$ and time t. The Laplacian Δ acts componentwise on $\boldsymbol{u}$.

For a viscous fluid in a bounded domain of space, one has boundary conditions for $\boldsymbol{u}$ that specify that the fluid sticks to the wall. There are no boundary conditions for the pressure. They are neither natural from the point of view of physics nor are they needed for a well-posed mathematical problem; the extra condition $\mathrm{div}\boldsymbol{u} = 0$ gives just enough information to dispense with a boundary condition for p.

We are here concerned with the flow between parallel walls at $x_2 = 0$ and $x_2 = h$. These walls extend infinitely in the (x_1, x_3) plane. The upper wall is moving with constant speed v in the x_1 direction. A solution matching this situation is



$$\boldsymbol{u}(\boldsymbol{x}) = \frac{x_2}{h} v\boldsymbol{e}_1\ , \quad p \equiv \mathit{const} \tag{1.2}$$

This is of course not the unique solution (e.g., a quadratic profile together with a pressure having constant gradient in x_1 direction would be another solution), but it is the unique stationary (time independent) and bounded solution with $\boldsymbol{u}$ in x_1 direction and constant pressure p, as can be easily seen.

We can introduce dimensionless variables $\tilde{\boldsymbol{x}} = \boldsymbol{x}/h$, $\tilde{t} = vt/h$, $\tilde{\boldsymbol{u}} = \boldsymbol{u}/v$, $\tilde{p} = p/v^2$, $\varepsilon = \nu/(hv)$ as usual (ε^{-1} is the Reynolds number). Moreover it is our purpose to consider the linearization of (1.1) about the solution (1.2) $\boldsymbol{u}(x_1, x_2, x_3) = (v/h)x_2\boldsymbol{e}_1$. This linearization, for the variables $\boldsymbol{w}$, q as variations of $\boldsymbol{u}$, p, reads (we immediately drop the tilde again):

$$\begin{aligned} &\boldsymbol{w}_t - \varepsilon\Delta\boldsymbol{w} + x_2\partial_1\boldsymbol{w} + w_2\boldsymbol{e}_1 + \rho^{-1}\boldsymbol{\nabla}q = \boldsymbol{0}\ , \\ &\mathrm{div}\boldsymbol{w} = 0\ , \\ &\boldsymbol{w}(\cdot, 0, \cdot) = \boldsymbol{w}(\cdot, 1, \cdot) = 0\ . \end{aligned} \tag{1.3}$$

To get a sensible problem one must either pose boundary conditions at infinity (say assume boundedness of $\boldsymbol{w}$ and q) or assume some (typically periodic) boundary conditions with respect to x_1 and x_3. Granted such boundary conditions, it should be said first that for initial data $\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0$ in the Sobolev class H^1 and satisfying $\operatorname{div}\boldsymbol{u}_0 = 0$, equation (1.1) has a unique solution existent for some time (Thm. VII.1.1 in [27]).

The stability problem consists of showing that the spectrum of the operator

$$\mathbf{A} : \boldsymbol{w} \mapsto \mathbf{P}(\varepsilon\Delta\boldsymbol{w} - x_2\partial_1\boldsymbol{w} - w_2\boldsymbol{e}_1) \tag{1.4}$$

in an appropriate Hilbert space of divergence free vector fields defined on $\mathbb{R} \times [0,1] \times \mathbb{R}$ (and possibly periodic with respect to a lattice $\Gamma \in \mathbb{R} \times \{0\} \times \mathbb{R}$) lies in the negative half plane $\operatorname{Re}\lambda < 0$. Here, $\mathbf{P}$ is a projection which selects from a given vector field its divergence free part. Of course, it is required to be more specific than that; we will go into the details in Sections 2.4 and 2.5. If one assumes periodic boundary conditions, then the operator $\mathbf{A}$ has compact resolvent, hence pure point spectrum. It inherits this property from the Laplacian by perturbative methods.

From the point of view of general abstract spectral theory, the linearized Navier–Stokes operator is a nice perturbation of the Laplacian and does not present too bad difficulties, in spite of the lack of self-adjointness. The difficulty lies in determining exactly the location of the spectrum. Questions concerning the completeness of eigenfunctions are of course not trivial due to the lack of self-adjointness, but are of no concern here. Recently, results in this direction have been obtained by Tretter and Langer [29],[15]. See also [4].

A mathematician reading literature on hydrodynamic stability may get the impression that the engineers are just defining stability as having spectrum in the left half plane, possibly in ignorance of the fact that in infinite dimensions, one should have some stronger spectral condition, say a cone condition $\operatorname{Re}\lambda < 0$, $|\operatorname{Im}\lambda| < C|\operatorname{Re}\lambda|$, to ensure stability of the solution to the non-linear equation. However, it turns out that a (weaker) cone condition, with a cone whose vertex may lie in the right half plane, is easily satisfied, due to the general structure of the Navier–Stokes equations, and independent of the geometric nature of the flow. See Section 2.5

Given this fact, for a cone condition with the vertex at 0 to hold, it is then clearly necessary and sufficient that the spectrum lies entirely in the left half plane. Therefore, the whole issue is indeed to get such a half plane estimate. This observation is probably not new to the experts, but is usually not pointed out explicitly.

The stability problem for the plane Couette flow has been solved by Romanov [23], and he not only proves the half plane estimate, but does give a stability proof for the solution with respect to the non-linear equations, and quite explicitly and elementarily so, without reference to some general theorem on linearized stability for PDEs. The cone estimate is implicit in the method he uses, and we will abstract his proof below. Kurth [14] has given a direct proof for the asymptotic stability of the plane Couette flow by means of energy estimates and Gronwall's inequality, without reference to linearized stability.

The linearized stability of the plane Couette flow concerns very small perturbations, with no estimate how small they have to be. On the other hand, experiments seem to suggest that the plane Couette flow becomes practically unstable for large Reynolds numbers in spite of the asymptotic stability.

Recently, progress towards an understanding of this instability was made by Trefethen et al. [28]. The explanation given by these authors is based on the observation that for non-normal operators, eigenvectors for different eigenvalues need not be orthogonal. Therefore, even though the eigenvalues have negative real part, the linearized dynamics may already be responsible for a (transient) growth of a tiny initial perturbation before that perturbation would then decay exponentially as $t \to \infty$. But this transient growth may already move the system far away from the equilibrium (in our case the stationary solution), into a region of phase space where nonlinear terms can no longer be neglected. These results are a mixture of numeric observations and heuristics supported by smartly chosen low dimensional model problems. For the time being, they lack the full mathematical rigor, promising as the approach may still appear.

In any case, it still seems desirable to have as full as possible an understanding of the spectrum of equation (1.3), and it is the purpose of the present work to attack this task. As we have just mentioned and will detail out later, we are only concerned with eigenvalues.

The eigenvalue problem for (1.3), i.e.,

$$\begin{aligned} &\lambda \boldsymbol{w} - \varepsilon\Delta\boldsymbol{w} + x_2\partial_1\boldsymbol{w} + w_2\boldsymbol{e}_1 = \text{ a gradient field} \\ &\mathrm{div}\boldsymbol{w} = 0 \\ &\boldsymbol{w}(\cdot,0,\cdot) = \boldsymbol{w}(\cdot,1,\cdot) = 0 \end{aligned} \tag{1.5}$$

can be reduced to the corresponding problem for an ODE, as has been lucidly explained by Squire [25]. We sketch his arguments in a more modern language. Taking the curl (say, in the sense of distributions, to avoid questions of differentiability) eliminates the pressure q from (1.3). Then, under the (partial) Fourier transform

$$\begin{aligned} \hat{\boldsymbol{w}}_{k_1k_3}(x_2) &= \frac{1}{\sqrt{2\pi}}\int\!\!\int \exp[-ik_1x_1 - ik_3x_3]\, \boldsymbol{w}(x_1,x_2,x_3)\, dx_1\, dx_3 \\ \boldsymbol{w}(x_1,x_2,x_3) &= \frac{1}{\sqrt{2\pi}}\int\!\!\int \exp[-ik_1x_1 - ik_3x_3]\, \hat{\boldsymbol{w}}_{k_1k_3}(x_2)\, dk_1\, dk_3 \ , \end{aligned} \tag{1.6}$$

equation (1.3) is further transformed into

$$\begin{aligned} &\lambda(\boldsymbol{\nabla}\times\hat{\boldsymbol{w}}) - \varepsilon\Delta(\boldsymbol{\nabla}\times\hat{\boldsymbol{w}}) + ik_1x_2(\boldsymbol{\nabla}\times\hat{\boldsymbol{w}}) + ik_1\boldsymbol{e}_2\times\hat{\boldsymbol{w}} + (\boldsymbol{\nabla}\hat{w}_2)\times\boldsymbol{e}_1 = \boldsymbol{0} \\ &\boldsymbol{\nabla} = (ik_1, \partial_2, ik_3)^T\ , \quad \Delta = -k_1^2 - k_3^2 + \partial_2^2 \\ &ik_1\hat{w}_1 + \partial_2\hat{w}_2 + ik_3\hat{w}_3 = 0 \\ &\hat{\boldsymbol{w}}(0) = \hat{\boldsymbol{w}}(1) = 0\ , \end{aligned} \tag{1.7}$$

or, in components,

$$\begin{aligned} &(\lambda - \varepsilon(-k_1^2 - k_3^2 + \partial_2^2) + ik_1x_2)(\partial_2\hat{w}_3 - ik_3\hat{w}_2) + ik_1\hat{w}_3 = 0 \\ &(\lambda - \varepsilon(-k_1^2 - k_3^2 + \partial_2^2) + ik_1x_2)(ik_3\hat{w}_1 - ik_1\hat{w}_3) + ik_3\hat{w}_2 = 0 \\ &(\lambda - \varepsilon(-k_1^2 - k_3^2 + \partial_2^2) + ik_1x_2)(ik_1\hat{w}_2 - \partial_2\hat{w}_1) - ik_1\hat{w}_1 - \partial_2\hat{w}_2 = 0 \\ &ik_1\hat{w}_1 + \partial_2\hat{w}_2 + ik_3\hat{w}_3 = 0 \\ &\hat{\boldsymbol{w}}(0) = \hat{\boldsymbol{w}}(1) = 0\ . \end{aligned}$$

If we consider (1.6) for the spectral problem with periodicity lattice Γ, the $\int\int \ldots dx_1dx_3$ extends over the torus $\mathbb{R}^2/\Gamma$, or equivalently, over a fundamental domain of Γ in $\mathbb{R}^2$, and the $\int\int \ldots dk_1dk_3$ is replaced by a sum over $(k_1,k_3) \in \Gamma^*$, where $\Gamma^* := \{\boldsymbol{k} \mid \langle\boldsymbol{k};\boldsymbol{x}\rangle \in 2\pi\mathbb{Z}\ (\forall \boldsymbol{x} \in \Gamma)\ \}$ is the dual lattice.

$\hat{w}_1$ and $\hat{w}_3$ can be algebraically eliminated from the x_2 component and the divergence condition:

$$\begin{aligned} i(k_1^2 + k_3^2)(\lambda - \varepsilon(-k_1^2 - k_3^2 + \partial_2^2) + ik_1x_2)\hat{w}_1 &= \\ = -ik_3^2\hat{w}_2 - k_1(\lambda &- \varepsilon(-k_1^2 - k_3^2 + \partial_2^2) + ik_1x_2)\partial_2\hat{w}_2 \\ i(k_1^2 + k_3^2)(\lambda - \varepsilon(-k_1^2 - k_3^2 + \partial_2^2) + ik_1x_2)\hat{w}_3 &= \\ = ik_1k_3\hat{w}_2 - k_3(\lambda &- \varepsilon(-k_1^2 - k_3^2 + \partial_2^2) + ik_1x_2)\partial_2\hat{w}_2\ . \end{aligned}$$

Inserting this in the x_1 and x_3 components

$$\begin{aligned} \partial_2(\lambda - \varepsilon(-k_1^2 - k_3^2 + \partial_2^2) + ik_1x_2)\hat{w}_3 &= ik_3(\lambda - \varepsilon(-k_1^2 - k_3^2 + \partial_2^2) + ik_1x_2)\hat{w}_2 \\ \partial_2(\lambda - \varepsilon(-k_1^2 - k_3^2 + \partial_2^2) + ik_1x_2)\hat{w}_1 &= ik_1(\lambda - \varepsilon(-k_1^2 - k_3^2 + \partial_2^2) + ik_1x_2)\hat{w}_2 - \partial_2\hat{w}_2\ , \end{aligned}$$

we conclude finally the Orr–Sommerfeld equation for $\hat{w}_2$, namely

$$\begin{aligned}&\Big(\lambda - \varepsilon(-k_1^2 - k_3^2 + \partial_2^2) + ik_1x_2\Big)(\partial_2^2 - k_1^2 - k_3^2)\hat{w}_2 = 0\ ,\\ &\hat{w}_2(0) = 0 = \hat{w}_2(1)\ , \quad \partial_2\hat{w}_2(0) = 0 = \partial_2\hat{w}_2(1)\ ,\end{aligned} \tag{1.8}$$

where the new boundary conditions follow from the divergence condition in (1.7) with $\hat{w}_{1|3} = 0$ on the boundary. These equations have already been obtained by Orr [20], even for flow profiles more general than the linear one. We will write now $y := x_2$ and $w := \hat{w}_2$.

It has already been observed by Squire [25] that the transformation

$$\tilde{\varepsilon} = \varepsilon\sqrt{k_1^2 + k_3^2}\Big/k_1\ , \quad \alpha = \sqrt{k_1^2 + k_3^2}\ , \quad \tilde{\lambda} = \lambda\alpha/k_1 \tag{1.9}$$

simplifies (1.8) into

$$\begin{aligned}&\Big(\tilde{\varepsilon}(\partial_y^2 - \alpha^2) - i\alpha y - \tilde{\lambda}\Big)(\partial_y^2 - \alpha^2)w(y) = 0\\ &w(0) = 0 = w(1)\ , \quad w'(0) = 0 = w'(1)\ .\end{aligned} \tag{1.10}$$

Squire's transformation is usually phrased as follows: The question of stability (of a planar flow) with respect to all 3-dimensional perturbations for all Reynolds numbers up a given one is equivalent to the question of stability with respect to all 2-dimensional perturbations for all Reynolds number up to that same one. If stability for some Reynolds number implies stability for all smaller Reynolds numbers (as is usually observed in nature and thus supposed in physics, but apparently not proved with mathematical rigor), then for a given Reynolds number, linearized stability is determined already by 2-dimensional perturbations

Actually, this observation is applicable only to a small number of instances: The plane Couette flow (where it is superseded by Romanov's result that this flow is actually stable); the plane Poiseuille flow (with a quadratic velocity profile), where it does give some information. There is a one parameter family of profiles superimposing a drift due to the movement of the upper plane with the quadratic profile due to a pressure gradient.

The Orr–Sommerfeld equation is usually considered for arbitrary velocity profiles $U(y)$,

$$\Big[\Big(\tilde{\varepsilon}(\partial_y^2 - \alpha^2) - i\alpha U(y) - \tilde{\lambda}\Big)(\partial_y^2 - \alpha^2) + i\alpha U''(y)\Big]\, w(y) = 0\ , \tag{1.11}$$

and Squire's transformation does apply in this generality, but, except for quadratic U, these velocity profiles do not arise from the linearization of Navier–Stokes and should rather be considered as a heuristic model whose connection to Navier–Stokes needs some explanation(cf. eqn (25.12) in [5]).

A few things are remarkable: (1.10) is of order 4 rather than order 6, as might have been expected (having eliminated from three 2nd order equations). This is a nice feature of the simple planar geometry.

The differential operator factorizes. This happens only for the linear velocity profile (1.2), whereas the corresponding Orr–Sommerfeld equation for quadratic velocity profile would not enjoy this property. We are going to exploit this factorization.

The kernel of the first factor in (1.10) can be expressed in terms of Airy functions. Namely, using the substitutions

$$\begin{aligned}\alpha &= 8\tilde{\varepsilon}t^3\ , \quad &\delta &= -it - \tfrac{\tilde{\lambda}}{4\varepsilon t^2} - 16\tilde{\varepsilon}t^4\\ 2t &= (\alpha/\tilde{\varepsilon})^{1/3}\ , \quad &\lambda &= -\tilde{\varepsilon}(4it^3 + 4t^2\delta) - (8\tilde{\varepsilon}t^3)^2\ ,\end{aligned} \tag{1.12}$$

we find that that kernel is spanned by $\mathrm{Ai}(\delta - i(2y-1)t)$ and $\mathrm{Bi}(\delta - i(2y-1)t)$. The (self adjoint) operator $D_0 := \partial_y^2 - \alpha^2$ defined on $W_0^{2,2}[0,1]$, the space of $W^{2,2}$ functions with boundary conditions $w(0) = 0 = w(1)$, maps its domain bijectively onto $L^2[0,1]$. But the range of the smaller (symmetric, but not self adjoint) operator $D_{00} := \partial_y^2 - \alpha^2$ with the boundary conditions $w(0) = 0 = w(1)$, $w'(0) = 0 = w'(1)$ has codimension two: If $D_{00}w = \alpha\,\mathrm{Ai}(\ldots) + \beta\,\mathrm{Bi}(\ldots)$ for some $w \neq 0$, then $\alpha\,\mathrm{Ai}(\ldots) + \beta\,\mathrm{Bi}(\ldots) \perp \ker D_{00}^* = \ker D$, where $D = \partial_y^2 - \alpha^2$ with <u>no</u> boundary conditions, and thus $\ker D = \operatorname{span}\{\sinh \alpha y, \cosh \alpha y\}$. For this to happen, it is necessary and sufficient that the Gram determinant $\begin{vmatrix} \langle \mathrm{Ai} \,|\, \sinh \rangle & \langle \mathrm{Ai} \,|\, \cosh \rangle \\ \langle \mathrm{Bi} \,|\, \sinh \rangle & \langle \mathrm{Bi} \,|\, \cosh \rangle \end{vmatrix}$ vanishes. Writing $1 - 2y =: z$, we come to the eigenvalue condition

$$\begin{vmatrix} \displaystyle\int_{-1}^{1} \mathrm{Ai}(\delta + itz) \sinh(4\varepsilon t^3 z)\, dz & \displaystyle\int_{-1}^{1} \mathrm{Ai}(\delta + itz) \cosh(4\varepsilon t^3 z)\, dz \\ \displaystyle\int_{-1}^{1} \mathrm{Bi}(\delta + itz) \sinh(4\varepsilon t^3 z)\, dz & \displaystyle\int_{-1}^{1} \mathrm{Bi}(\delta + itz) \cosh(4\varepsilon t^3 z)\, dz \end{vmatrix} = 0\,.$$

The main part of the present work consists of a study of this condition. We shall hardly have to return to the original coordinates, except for giving physical names to mathematical symbols. However, a reader who wants to get a graphical impression on how δ relates to λ is advised to glance ahead: Figures 6.2 – 6.4 on pages 84 – 85 should give a good impression.

Without the factorization of the OS operator, i.e., (1.11) instead of (1.8), one would have a 4×4 determinant with less explicit entries. A crucial ingredient to the study of the determinant condition is an asymptotic analysis as various parameters go to infinity, and for that purpose, the special structure is less important. The purpose of this work is not only to carry out the asymptotic estimates with full mathematical rigor, giving explicit O-symbols and proving why domains of the parameter space that are not covered by the asymptotics are actually not relevant for the problem. It is also to enhance them with arguments that control the behaviour of the solutions in compact sets of the parameters (δ, t, ε). This latter was possible in some cases (Chapter 5), whereas parts of the problem of the behaviour in compact domains would probably only yield to interval arithmetic in Chapter 6.

Of course, numerical calculations of the spectrum defined by the determinant condition have been made for different ranges of the variables by various authors, and asymptotic arguments have been employed to justify them; but it is sometimes not easy to draw the borderline between rigor and heuristics (given that the focus in such an applied problem was often not a mathematical one). As a consequence, even the established results seem to have fallen into partial oblivion. A thorough synopsis which has the spectrum in its totality in view has apparently not been available yet.

Chapter 2

Review of the Orr–Sommerfeld Equation

2.1 The Gram Determinant

As was motivated in the introduction, we are going to study the (generalized) eigenvalue problem (1.10) for the Orr–Sommerfeld equation, which, under the substitutions (1.12), leads to an equation $\Delta(\delta, t, \varepsilon) = 0$, where

$$\Delta(\delta, t, \varepsilon) := \begin{vmatrix} \int_{-1}^{1} \mathrm{Ai}(\delta + itz) \sinh(4\varepsilon t^3 z)\, dz & \int_{-1}^{1} \mathrm{Ai}(\delta + itz) \cosh(4\varepsilon t^3 z)\, dz \\ \int_{-1}^{1} \mathrm{Bi}(\delta + itz) \sinh(4\varepsilon t^3 z)\, dz & \int_{-1}^{1} \mathrm{Bi}(\delta + itz) \cosh(4\varepsilon t^3 z)\, dz \end{vmatrix} \tag{2.1}$$

or, equivalently,

$$\Delta(\delta, t, \varepsilon) = \frac{1}{2} \begin{vmatrix} \int_{-1}^{1} \mathrm{Ai}(\delta + itz) \exp(4\varepsilon t^3 z)\, dz & \int_{-1}^{1} \mathrm{Ai}(\delta + itz) \exp(-4\varepsilon t^3 z)\, dz \\ \int_{-1}^{1} \mathrm{Bi}(\delta + itz) \exp(4\varepsilon t^3 z)\, dz & \int_{-1}^{1} \mathrm{Bi}(\delta + itz) \exp(-4\varepsilon t^3 z)\, dz \end{vmatrix} . \tag{2.2}$$

Using (3.7) and (3.9) below, this is easily transformed into

$$\Delta(-\delta, t, \varepsilon) = -i \times$$
$$\begin{vmatrix} \int_{-1}^{1} \mathrm{Ai}\Big((\delta + itz)e^{i\pi/3}\Big) \exp(4\varepsilon t^3 z)\, dz & \int_{-1}^{1} \mathrm{Ai}\Big((\delta + itz)e^{i\pi/3}\Big) \exp(-4\varepsilon t^3 z)\, dz \\ \int_{-1}^{1} \mathrm{Ai}\Big((\delta + itz)e^{-i\pi/3}\Big) \exp(4\varepsilon t^3 z)\, dz & \int_{-1}^{1} \mathrm{Ai}\Big((\delta + itz)e^{-i\pi/3}\Big) \exp(-4\varepsilon t^3 z)\, dz \end{vmatrix} . \tag{2.3}$$

Or else, using (3.10) below, into

$$\Delta(\delta, t, \varepsilon) = e^{\pm i\pi/6} \times$$
$$\begin{vmatrix} \int_{-1}^{1} \mathrm{Ai}(\delta + itz) \exp(4\varepsilon t^3 z)\, dz & \int_{-1}^{1} \mathrm{Ai}(\delta + itz) \exp(-4\varepsilon t^3 z)\, dz \\ \int_{-1}^{1} \mathrm{Ai}\Big((\delta + itz)e^{\pm 2i\pi/3}\Big) \exp(4\varepsilon t^3 z)\, dz & \int_{-1}^{1} \mathrm{Ai}\Big((\delta + itz)e^{\pm 2i\pi/3}\Big) \exp(-4\varepsilon t^3 z)\, dz \end{vmatrix} . \tag{2.4}$$

Let us now review some relevant classical work on this problem.

2.2 Orr 1907, Sommerfeld 1908, Hopf 1914

As the name subsequently given to it suggests, the Orr–Sommerfeld equation (1.10) has already been derived and studied by Orr [20] in 1907 (page 81, equations (2),(3) and p. 95, equations (1),(2)). Orr starts by reviewing controversies about the stability question. He gives a reasonably complete study of the equation with modified boundary conditions $w(0) = 0 = w(1)$, $(\partial_y^2 - \alpha^2)w(0) = 0 = (\partial_y^2 - \alpha^2)w(1)$. (See his Art. 15, p 99ff.) The reason why this is a much simpler case, in present day language, is that then the OS operator factorizes not only as a differential algebraic expression, but also in the operator theoretic sense. The OS operator with the modified boundary conditions is the product of two second order operators, each with Dirichlet boundary conditions, where the right factor is invertible and only the left factor contains the eigenvalue parameter.

It is worth noting that the equation resulting from the modified boundary conditions does have some relevance for the original problem. Although Orr does not state this, the very same equation arises from the physical boundary conditions in the limit of infinite viscosity or slow motion ($\varepsilon \to \infty$). We discuss this limit in detail in Section 5.2.

For the correct boundary conditions, Orr also has the Gram determinant condition (2.2) (p. 118, eqn (70)), but he is not able to resolve the stability question in that case.

Sommerfeld [24] clearly formulated the program of studying the linearized equation and thus $\Delta = 0$ (2.1) on the International Congress of Mathematicians in Rome in 1908. This program has been the basis of hydrodynamic stability for the plane viscous flow at least since then, but carrying it out in full rigor is the real challenge.

Induced by Sommerfeld, L. Hopf carried out such a study [10]; he started from the case of small wave numbers and seems to be aware at least intuitively that all solution branches arise from the ones obtained for small wave numbers, but he does not give a mathematical argument: On p. 37, he says (author's translation):

> It is another question whether the roots discussed here are indeed all roots of Sommerfeld's transcendent. One should however not assume that for larger αR [larger α/ε in the notation of the present work – JD] there occur oscillations that have no analog for small αR [...]

We are going to discuss the small wave number limit in Section 4.2. We shall derive a complete <u>convergent</u> series in reasonably explicit form, whose leading term is just the term considered by Hopf.

The two limits $\varepsilon \to \infty$ and $t \to 0$ do not commute, and care must be taken in the region where ε is large and t is small. Neither Orr nor Hopf seem to treat this question, and it actually cannot be treated based on leading terms alone, without discussion of the order of magnitude of the error terms. Such a detailed discussion does not give unexpected contributions to the spectrum, but without it, a study of the global structure of this spectrum should not be considered as complete. (See Section 5.3.) The typical scaling behaviour that delimits "ε large" from "t small" can be seen from Theorem 4.7 and Lemma 5.11, namely ε large means $\varepsilon^2 t^6 \gg \delta t^2$, whereas t small means $\varepsilon^2 t^6 \ll \delta t^2$.

2.3 Trefethen et al 1990s

The work by Trefethen et al. [28] has cast doubt on the appropriateness of an approach through the spectrum alone. His punchline is that the ε-pseudospectrum

$$\Lambda_\varepsilon(L) := \{\omega \mid \|(\omega - L)^{-1}\| \geq 1/\varepsilon\}$$

is more relevant than the spectrum $\Lambda_0(L)$, and that the geometry of $\Lambda_\varepsilon(L)$ may be different from a "small" neighbourhood of $\Lambda_0(L)$. A detailed discussion of the relevance of pseudospectra can be found in [21]. See also the comments made on page 12.

This present work is not about pseudospectra. The author still considers the thorough analysis of the spectrum as a worthwhile enterprise, given the amount of work that builds on the spectrum and the scattering of different partial results in a vast literature.

2.4 Spaces

We use the obvious notations $C^\infty(\Omega) = C^\infty(\Omega \to \mathbb{R})$ for the space of (real valued) smooth functions in the domain $\Omega \in \mathbb{R}^3$, and $C_0^\infty(\Omega) = C_0^\infty(\Omega \to \mathbb{R})$ for the space of smooth functions with compact support in Ω. Here, Ω will be a Lipschitz domain (not necessarily bounded). For the corresponding spaces of vector valued functions, we write $C^\infty(\Omega \to \mathbb{R}^3)$ or $C_0^\infty(\Omega \to \mathbb{R}^3)$. (For aesthetic reasons, the author considers it as a desirable habit to use an arrow rather than the more common comma or semicolon.) We suppress (Ω) or $(\Omega \to \mathbb{R}^3)$ from the notation, if this information is clear from the context.

The subscript s will denote spaces of solenoidal (i.e., divergence free) vector fields, e.g., $C_\mathrm{s}^\infty(\Omega \to \mathbb{R}^3) := \{\boldsymbol{f} \in C^\infty(\Omega \to \mathbb{R}^3) \mid \operatorname{div}\boldsymbol{f} = 0\}$. On all these spaces, the norms

$$\|\boldsymbol{f}\|_{W^{k,p}} := \Big(\sum_{|\alpha|\le k}\sum_{i=1}^{3} |D^\alpha f_i|^p\Big)^{1/p} \quad \text{or} \quad \|f\|_{W^{k,p}} := \Big(\sum_{|\alpha|\le k} |D^\alpha f|^p\Big)^{1/p}$$

are defined ($1 \le p < \infty$). The Sobolev spaces $W^{k,p}$, $W_0^{k,p}$, $W_{0,\mathrm{s}}^{k,p}$, $W_\mathrm{s}^{k,p}$ may be defined as the completions of C^∞, C_0^∞, $C_{0,\mathrm{s}}^\infty$, C_s^∞ with respect to $\|\cdot\|_{W^{k,p}}$. This includes the special case $L^p = W^{0,p}$, and the Hilbert spaces $H^k := W^{k,2}$, with whatever subscripts.

Equivalently, $W^{k,p}$ is the space of locally integrable functions which, together with its distributional derivatives up to order k, are still functions whose p^{th} power is integrable. (See Thm III,3.16 in [2]).

Of course, $L_0^2 = L^2$, but in $L_{0,\mathrm{s}}^2$, the subscript 0 is relevant: it is shown in I§1 of Temam's book [26] for bounded Lipschitz domains Ω (with only minor modifications in the unbounded case) that there is an orthogonal decomposition

$$L^2(\Omega \to \mathbb{R}^3) = L_{0,\mathrm{s}}^2(\Omega \to \mathbb{R}^3) \overset{\perp}{\oplus} \boldsymbol{\nabla}\Big(H^1(\Omega \to \mathbb{R})\Big) \tag{2.5}$$

([26], Thm. 1.4); and also that $L_{0,\mathrm{s}}^2(\Omega \to \mathbb{R}^3)$ can be characterized as the space of L^2 vector fields whose divergence (in the sense of distributions) vanishes, and whose "normal component on the boundary $\partial\Omega$" vanishes as well. The latter condition relies on a theorem guaranteeing the following: the mapping which attributes to a C^1 vector field on $\bar{\Omega}$ the normal component of its restriction to $\partial\Omega$ can be naturally extended in a continuous way to the space of those L^2 vector fields whose divergence is also L^2. It is in the sense of this trace map that the vanishing of the normal component of a vector field in L_s^2 is to be understood.

Moreover,

$$H_{0,\mathrm{s}}^1(\Omega \to \mathbb{R}^3) = \{\boldsymbol{f} \in H_0^1(\Omega \to \mathbb{R}^3) \mid \operatorname{div}\boldsymbol{f} = 0\}\ .$$

2.5 Operator Theory and Romanov's Stability Proof

It is interesting to go into some details of Romanov's stability proof [23]. As he refers to some facts from abstract operator theory in Kreĭn's book [13], his paper is not quite self-contained. We give an exposition of his proof that does fill in the results from Kreĭn's theory in a motivated form and highlighting the key structure behind Kreĭn's results. At the same time, we give references to Kato's [12] approach to operator theory, which may be more familiar to the Western reader. Apparently, Romanov's result is less well known than it deserves, and the reason may be that his proof cannot be appreciated as natural and fully rigorous without going through the first chapter of Kreĭn's book.

Implicit in the results as used by Romanov is an understanding of the relevant questions of qualitative spectral theory that are often ignored in the quest for eigenvalues alone. This will substantiate the claim made in the introduction (page 12) that the hydrodynamic stability question is really only a question of whether the spectrum lies in a half plane, and that further results (like a sector condition) are then automatic.

We here take for granted the fact that all eigenvalues of the Orr–Sommerfeld equation (1.10) lie in some half plane $\operatorname{Re}\lambda \le \omega < 0$ (see Section 2.7 for this) and focus on the other features of the proof.

Letting $\bar{\boldsymbol{u}}$ a stationary solution of the incompressible Navier–Stokes equations (1.1), i.e.,

$$\begin{aligned} &\boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\boldsymbol{\nabla})\boldsymbol{u} + \rho^{-1}\boldsymbol{\nabla}p = \boldsymbol{0} \\ &\operatorname{div}\boldsymbol{u} = 0\ . \end{aligned}$$

in some (bounded or unbounded) domain $\Omega \subset \mathbb{R}^3$, one can write $\boldsymbol{w} = \boldsymbol{u} - \bar{\boldsymbol{u}}$,

$$\begin{aligned} &\boldsymbol{w}_t = \mathbf{A}\boldsymbol{w} - \mathbf{P}(\boldsymbol{w}\boldsymbol{\nabla})\boldsymbol{w} \quad \text{where} \\ &\mathbf{A} : \boldsymbol{w} \mapsto \mathbf{P}\left[\nu\Delta\boldsymbol{w} - (\bar{\boldsymbol{u}}\boldsymbol{\nabla})\boldsymbol{w} - (\boldsymbol{w}\boldsymbol{\nabla})\bar{\boldsymbol{u}}\right]\ ,\ \ L^2_{0,\mathrm{s}} \supset W^{2,2} \cap W^{1,2}_{0,\mathrm{s}} \to L^2_{0,\mathrm{s}} \end{aligned} \tag{2.6}$$

and $\mathbf{P}$ denotes the orthogonal projection onto the divergence free vector fields in the sense of the decomposition (2.5) above.

<u>Perturbation Theoretic Arguments</u>

The operator $\mathbf{A}$ will inherit most spectral properties from the (self-adjoint) Dirichlet Laplacian, of which it is a (non-self-adjoint, but relatively compact) perturbation.

Δ, considered as an operator in $L^2(\Omega \to \mathbb{R}^3)$, with domain of definition $C_0^\infty(\Omega \to \mathbb{R}^3)$, is essentially self-adjoint and nonpositive, and its closure has domain of definition $(H^2 \cap H_0^1)(\Omega \to \mathbb{R}^3)$. Given that, it is straightforward to show that

$$\|(\lambda - \Delta)^{-1}\| \le |\lambda|^{-1} \text{ for } \operatorname{Re}\lambda \ge 0\ .$$

These estimates hold for the restriction to the invariant subspace of solenoidal vector fields as well. On this subspace, we have $\mathbf{P}\Delta = \Delta$; therefore the basic estimates will be made for Δ rather than $\mathbf{P}\Delta$, and for vector fields of any divergence, and will then be appropriately restricted. For any perturbation operator $\mathbf{B}$ with relative Δ-bound 0, i.e.,

$$\mathcal{D}(\mathbf{B}) \supset \mathcal{D}(\Delta) \ \text{ and } \ \forall\eta > 0\, \exists C_\eta\, \forall \boldsymbol{f} \in \mathcal{D}(\Delta) : \|\mathbf{B}\boldsymbol{f}\| \le C_\eta\|\boldsymbol{f}\| + \eta\|\Delta\boldsymbol{f}\| \tag{2.7}$$

($\mathcal{D}$ denotes the domain of definition), one concludes

$$\|\mathbf{B}(\lambda - \Delta)^{-1}\| \le C_\eta\|(\lambda - \Delta)^{-1}\| + \eta\|(\lambda + (\Delta - \lambda))(\lambda - \Delta)^{-1}\| \le C_\eta/|\lambda| + \eta(1 + |\lambda|/|\lambda|)$$

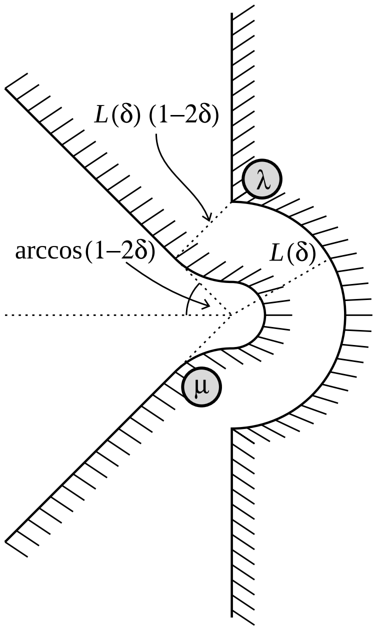
for $\operatorname{Re}\lambda \ge 0$. Given any $\delta > 0$, this norm can be made smaller than δ, provided $|\lambda|$ is sufficiently large ($\ge L(\delta) := 2C_{\delta/4}/\delta$). Then, we conclude that

$$\|(\lambda - \Delta + \mathbf{B})^{-1}\| = \|(\lambda - \Delta)^{-1}(\mathbf{1} + \mathbf{B}(\lambda - \Delta)^{-1})^{-1}\| \le ((1 - \delta)|\lambda|)^{-1} \tag{2.8}$$

for $\operatorname{Re}\lambda \ge 0$, $|\lambda| \ge L(\delta)$.

For any operator $\mathbf{C}$, the same argument with $\mathbf{B}$ replaced by $(\mu-\lambda)\mathbf{1}$ shows that

$$\|(\mu-\mathbf{C})^{-1}\| \le \frac{\|(\lambda-\mathbf{C})^{-1}\|}{1-\|(\lambda-\mathbf{C})^{-1}\|\cdot|\mu-\lambda|}$$



for $|\mu-\lambda| < 1/\|(\lambda-\mathbf{C})^{-1}\|$. Consequently, for $\mathbf{C} := \nu^{-1}\mathbf{A} = \Delta - \mathbf{B}$, where $\mathbf{B}\boldsymbol{w} = \nu^{-1}\mathbf{P}[(\bar{\boldsymbol{u}}\boldsymbol{\nabla})\boldsymbol{w} + (\boldsymbol{w}\boldsymbol{\nabla})\bar{\boldsymbol{u}}]$, we have

$$\|(\mu-\mathbf{C})^{-1}\| \le \frac{1}{(1-\delta)|\lambda|\left(1-\frac{(1-2\delta)|\lambda|}{(1-\delta)|\lambda|}\right)} = \frac{1}{\delta|\lambda|} \le \frac{1}{2\delta^2|\mu|} \qquad (2.9)$$

for every μ satisfying $|\mu-\lambda| < (1-2\delta)|\lambda|$ with some $|\lambda| > L(\delta)$, $\operatorname{Re}\lambda \ge 0$. This particular $\mathbf{B}$ indeed satisfies (2.7): in fact, for $f \in (H^2 \cap H_0^1)(\Omega) \subset H^2(\mathbb{R}^3)$ (with the imbedding effected through continuation by zero), one immediately has

$$\|\boldsymbol{\nabla} f\|_{L^2} \le \frac{1}{4\eta}\|f\|_{L^2} + \eta\|\Delta f\|_{L^2}\,, \quad \text{because} \quad |\boldsymbol{k}| \le \frac{1}{4\eta} + \eta|\boldsymbol{k}|^2$$

and because of the Fourier transform on $L^2(\mathbb{R}^3 \to \mathbb{R})$. The same estimate carries over to vector valued $\boldsymbol{f} \in (H^2 \cap H_0^1)(\Omega \to \mathbb{R}^3)$ and to the subspace of solenoidal vector fields.

The domain of admissible values for (2.9) is depicted in the inset figure, and it is clear from the construction that

$$\forall \phi > 0\ \exists M_\phi > 0, \lambda_0 = \lambda_0(\phi) > 0 : \|(\mu-\mathbf{A})^{-1}\| \le \frac{M_\phi}{|\mu-\lambda_0|} \quad \text{for} \quad |\arg(\lambda_0-\mu)| \ge \phi\,. \qquad (2.10)$$

The spectrum lies therefore in a sector with arbitrarily small opening angle, but with its vertex possibly in the right half plane. This completes the claim brought forward in the introduction on page 12 that a sector estimate (with vertex 0) for the spectrum is immediate, if a half plane estimate is available, because, based on the structure of the differential operators alone, and with no reference to the geometry, a sector estimate with <u>some</u> vertex is always available.

It is known (sec. 5.1 of Henry [9]) that for equilibria of (infinite dimensional) dynamical systems asymptotic stability follows if the spectrum of the linearization lies in an open cone in the left half plane. The general proof will naturally go along the lines of Romanov's proof given below.

The factorization $(\lambda-\Delta+\mathbf{B})^{-1} = (\lambda-\Delta)^{-1}(\mathbf{1}+\mathbf{B}(\lambda-\Delta)^{-1})^{-1}$ used in (2.8) also shows that $\mathbf{A}$ has a compact resolvent, if the same holds for Δ (which happens, if Ω is bounded or is made bounded by imposing a periodicity constraint as discussed after (1.4)). Then the spectrum is a pure point spectrum, and we need only look for eigenvalues.

Semigroup Arguments after Kreĭn; the Linearized Equation

The linearized equation to (2.6), namely $\boldsymbol{w}_t = \mathbf{A}\boldsymbol{w}$, is uniquely solvable for all initial data $\boldsymbol{w}_0 \in L^2_{\mathrm{s}}$ and all times $t > 0$. This property is inherited from the simpler case with Δ instead of $\mathbf{A}$, because it follows from the resolvent estimates, too: we do not repeat the whole existence theory for semigroups from [13] (I, §1.2–1.4, §3.3–3.4); let it suffice to say that solutions to $\boldsymbol{w}_t = \mathbf{A}\boldsymbol{w}$, $\boldsymbol{w}(0) = \boldsymbol{w}_0$ can be constructed for "smooth" enough initial data $\boldsymbol{w}_0 \in \mathcal{D}(\mathbf{A}^l)$ by means of the Laplace transform method, which (first formally) leads to

$$\boldsymbol{w}(t) = -\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t}(\mathbf{A}-\lambda)^{-1}\boldsymbol{w}_0\, d\lambda\,,$$

and that the convergence of that integral depends on l. For existence and uniqueness, the significant property is a resolvent estimate $\|(\mathbf{A}-\lambda)^{-1}\| \le M(1+|\lambda|)$ for $\operatorname{Re}\lambda \ge \omega_0$ (Thm. I,1.4 and 3.2 of [13]). If $\mathbf{A}$ is a closed operator (and it is closable, if $\|(\mathbf{A}-\lambda)^{-1}\| < \infty$ for some λ), then existence and uniqueness imply continuous dependence on initial data. The solution will now be written as $\boldsymbol{w}(t) =: e^{t\mathbf{A}}\boldsymbol{w}_0$, where the time evolution operators $e^{t\mathbf{A}}$ are bounded and form a (strongly continuous) semigroup.

The better estimate $\|(\mathbf{A}-\lambda)^{-1}\| \le M/|\lambda-\omega|$ for $\operatorname{Re}\lambda \ge \omega$ implies analytic dependence of the solutions on t. Essentially, this is true, because such an estimate (with the sector bound for the spectrum that comes with it as shown above) permits to pull the path of integration in the inverse Laplace transform to the left and thus get good convergence properties for the integral. (See [13], I,§3.5 or [12], IX,§1.6.) More generally, this kind of reasoning implies the estimate

$$\left\|\left(\frac{d}{dt}\right)^k e^{t\mathbf{A}}\right\| = \left\|\mathbf{A}^k e^{t\mathbf{A}}\right\| \le O(e^{\omega t}t^{1-(k+1)/\beta}) \ \text{ for } \ (k,\beta) \in \mathbb{N}_0 \times]0,1] \setminus \{(0,1)\} \tag{2.11}$$

provided the spectrum of $\mathbf{A}$ lies to the left of $\operatorname{Re}\lambda = \omega$ and the resolvent estimate $\|(\lambda-\mathbf{A})^{-1}\| \le O((1+|\operatorname{Im}\lambda|)^{-\beta})$ holds for $\operatorname{Re}\lambda \ge \omega$ (Thm. I3.4 and Rmk. I3.2 of [13]).

Variation of Constants; Search for the Right Banach Space

It is natural to transform the non-linear equation (2.6) into an integral equation by the variation of constants, treating the nonlinearity as an inhomogeneity:

$$\boldsymbol{w}(t) = \mathcal{T}(\boldsymbol{w})(t) := e^{\mathbf{A}t}\boldsymbol{w}_0 - \int_0^t e^{\mathbf{A}(t-s)}P(\boldsymbol{w}(s)\cdot\boldsymbol{\nabla})\boldsymbol{w}(s)\,ds\,. \tag{2.12}$$

This is to be solved by fixed point iteration. Let us discuss the appropriate space: the simplest choice would be $BC^0(I \to X)$ with I some time interval $[0,T]$ or $[0,\infty[$, and X an appropriate Banach space of functions of the space variable. Differentiability with respect to time could then be deduced iteratively from the integral equation. An $L^p(I \to X)$ type of space would make it much more difficult to cope with the non-linearity.

On any sensible Banach space X, the differentiation involved by $(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}$ will be unbounded, and it can only be compensated for by the smoothing effect of the semigroup $e^{\mathbf{A}(t-s)}$. Without such compensation, the iteration procedure would be doomed to failure from the very beginning. However, for t and hence $t-s$ small, this smoothing effect will be less pronounced, and therefore we must expect a singularity as $t \to 0+$ and be prepared to include an appropriate weight into the BC^0 space:

A first derivative d/dx will amplify contributions e^{ikx} with wave number k by just a factor ik. The time evolution from a second order derivative operator d^2/dx^2 after time t will damp that same contribution with a factor e^{-k^2t}. Letting $\tau := t^{1/2}k$, the absolute value of the total amplification will be $ke^{-tk^2} = t^{-1/2}\tau e^{-\tau^2} < \frac{1}{2}t^{-1/2}$ uniformly in time and wave number. The exponent $-\frac{1}{2}$ has arisen from merely balancing the number of derivatives in the generator of the semigroup (i.e., two) and their number in the perturbation term (i.e., one).

Simple minded Fourier transforms do not match general domains Ω; the reasoning should rather involve eigenfunctions of $\mathbf{A}$ than the e^{ikx}, which are eigenfunctions of Δ. But as the key feature in our heuristics amounts to a scaling argument that counts derivatives, we might want to replace $(\boldsymbol{u}\cdot\boldsymbol{\nabla})$ by something like $\mathbf{A}^{1/2}$, whatever this means. This motivates the use of fractional powers of an operator, which is one of the key focuses in Kreĭn's approach to operator theory. Various orders of differentiability can thus be expressed purely in terms of the operator under study.

The fact that these fractional powers are constructed in an abstract spectral theoretic setting hides the possibly embarrassing (but completely irrelevant) fact that $\mathbf{A}^{1/2}$ is not a differential operator at all and will act on functions in a nonlocal unintuitive way.

If the spectrum of $\mathbf{A}$ satisfies $\operatorname{Re}\lambda \le \omega < 0$, we expect an appropriate decay and look therefore for solutions in a space that has the exponential decay $e^{\omega t}$ built in. (If successful, then local uniqueness will guarantee that a solution thus found is the solution. See Thm. VII.1.1 in [27] for local uniqueness.)

To sum up the discussion, equation (2.12) will be treated in the space

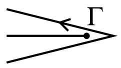
$$\mathbf{X} = \Big\{ \boldsymbol{f}(\cdot,\cdot) \in C^0\Big(]0,\infty[\to H^1_{0,\mathrm{s}}(\Omega \to \mathbb{R}^3)\Big) \;\Big|\; |||\boldsymbol{f}||| := \sup_t t^\alpha e^{|\omega| t} \|\mathbf{A}_0^\beta \boldsymbol{f}(t,\cdot)\|_{L^2} < \infty \Big\} \qquad (2.13)$$

with initial value $\boldsymbol{w}_0$ satisfying $\|\mathbf{A}_0^{\beta_0} \boldsymbol{w}_0\|_{L^2} < \infty$, and with $\alpha \in \,]0,1[$, $\beta > \frac{1}{2}$, $\beta_0 \le \beta$ yet to be specified.

Fractional Powers after Kreĭn, Interpolation and Sobolev Estimates

For $\lambda_0 > 0$ sufficiently large, $\mathbf{A}_0 := \lambda_0 - \mathbf{A}$ has no spectrum on the negative real axis, and the estimate $\|(\mathbf{A}_0 - \lambda)^{-1}\| \le M_\phi / |\lambda - \lambda_0| \le \tilde{M}_\phi / |\lambda|$ for $|\arg \lambda| > \phi$ permits us to define ([13], I§5)

$$\begin{aligned} \mathbf{A}_0^{-\alpha} \boldsymbol{f} &:= \frac{1}{2\pi i} \int_\Gamma \lambda^{-\alpha} (\mathbf{A}_0 - \lambda)^{-1} \boldsymbol{f}\, d\lambda = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{-\alpha} (\mathbf{A}_0 + \lambda)^{-1} \boldsymbol{f}\, d\lambda \\ \mathbf{A}_0^{\alpha} \boldsymbol{f} &= \qquad \mathbf{A}_0^{\alpha-1} \mathbf{A}_0 \boldsymbol{f} \qquad = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{\alpha-1} (\mathbf{A}_0 + \lambda)^{-1} \mathbf{A}_0 \boldsymbol{f}\, d\lambda\ , \end{aligned} \qquad (2.14)$$



where Γ is a path that goes around the negative axis with positive orientation (and thus around the spectrum of $\mathbf{A}_0$ with negative orientation), and where the real integrals assume $\alpha \in \,]0,1[$.

The analogous definition (2.14) for $(1-\Delta)^{1/2}$ on $(H^2 \cap H^1_0)(\Omega) \subset H^2(\mathbb{R}^3)$ – scalar or vector valued – amounts to the same as the one through Fourier analysis, and Fourier analysis shows that

$$\begin{aligned} &\|\boldsymbol{f}\|_{H^1} = \|(1-\Delta)^{1/2} \boldsymbol{f}\|_{L^2}\ , \quad \|\boldsymbol{f}\|_{H^2} \le 3\|(1-\Delta)\boldsymbol{f}\|_{L^2} \\ &\text{for } \ \boldsymbol{f} \in (H^2 \cap H^1_0)(\Omega) \subset H^2(\mathbb{R}^3)\ . \end{aligned} \qquad (2.15)$$

Fourier analysis also shows that one has

$$\begin{aligned} &\|(1-\Delta)^{1/2}(1-\Delta+\lambda)^{-1}\|_{L^2\to L^2} \le (1+\lambda)^{-1/2}\ , \\ &\|(1-\Delta)(1-\Delta+\lambda)^{-1}\|_{L^2\to L^2} \le 1 \\ &\text{for } \ \lambda \ge 0\ , \quad \mathcal{D}(\Delta) = (H^2 \cap H^1_0)(\Omega) \text{ or } H^2(\mathbb{R}^3)\ . \end{aligned} \qquad (2.16)$$

The quite believable interpolation estimate

$$\|\mathbf{A}_0^\alpha \boldsymbol{f}\| \le c_\alpha \|\boldsymbol{f}\|^{1-\alpha}\, \|\mathbf{A}_0 \boldsymbol{f}\|^\alpha \le c_\alpha \Big(\varepsilon^\alpha \|\boldsymbol{f}\| + \varepsilon^{\alpha-1} \|\mathbf{A}_0 \boldsymbol{f}\| \Big) \ \text{ for } \ 0 < \alpha < 1 \ \text{ and any } \ \varepsilon > 0$$

follows from these definitions ([13], Thm. I.5.2). Together with the estimate (2.11) for the semigroup one concludes ([13], I.§7.3)

$$\begin{aligned} \|\mathbf{A}_0^\alpha e^{t\mathbf{A}} \boldsymbol{f}\| &\le c_\alpha \|e^{t\mathbf{A}} \boldsymbol{f}\|^{1-\alpha}\, \|\mathbf{A}_0 e^{t\mathbf{A}} \boldsymbol{f}\|^\alpha \le c_\alpha \Big(\varepsilon^\alpha \|e^{t\mathbf{A}} \boldsymbol{f}\| + \varepsilon^{\alpha-1}\, \|\mathbf{A}_0 e^{t\mathbf{A}} \boldsymbol{f}\| \Big) \\ &\le M c_\alpha \Big(\varepsilon^\alpha + \varepsilon^{\alpha-1} \|\mathbf{A}_0 \mathbf{A}^{-1}\|\, O(e^{\omega t} t^{-1}) \Big) \|\boldsymbol{f}\| \le O(t^{-\alpha} e^{\omega t}) \|\boldsymbol{f}\| \end{aligned} \qquad (2.17)$$

where we have chosen $\varepsilon := 1/t$ in the final step.

The rest is smart shuffling around of Hölder and Sobolev estimates (c denotes varying constants):

$$\begin{array}{ccccc} \|\boldsymbol{w}\|_{L^p} & \overset{\text{Hölder}}{\le} & c\,\|\boldsymbol{w}\|_{L^2}^{1-\theta}\, \|\boldsymbol{w}\|_{L^r}^\theta & \overset{\text{Sobolev}}{\le} & c\,\|\boldsymbol{w}\|_{L^2}^{1-\theta}\, \|\boldsymbol{w}\|_{W^{1,2}}^\theta \\ \frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{r} & & -\frac{3}{r} = 1 - \frac{3}{2} & & \Longrightarrow \theta = \frac{3}{2} - \frac{3}{p} \end{array} \qquad (2.18)$$

$$\begin{array}{ccccc} \|\mathbf{P}(\boldsymbol{v}\boldsymbol{\nabla})\boldsymbol{w}\|_{L^2} & \overset{\text{Hölder}}{\le} & c\,\|\boldsymbol{v}\|_{L^p}\, \|\boldsymbol{w}\|_{W^{1,q}} & \overset{\text{Sobolev}}{\le} & c\,\|\boldsymbol{v}\|_{W^{1,r}}\, \|\boldsymbol{w}\|_{W^{1,q}} \\ \frac{1}{2} = \frac{1}{p} + \frac{1}{q} & & -\frac{3}{p} = 1 - \frac{3}{r} & & \text{want } r = q \\ & & & & \text{get } q = \frac{12}{5} \end{array} \qquad (2.19)$$

The $W^{1,p}$ norms can be estimated in terms of the operators $(1-\Delta)^\alpha$ or, equivalently $\mathbf{A}_0^\alpha$, for $0<\alpha<1$ (see Romanov's Lemma 5):

$$\begin{aligned}
\|(\mathbf{1}-\Delta)^{-\alpha}\boldsymbol{w}\|_{W^{1,p}} &\le \frac{\sin\alpha\pi}{\pi}\int_0^\infty \lambda^{-\alpha}\,\|(\mathbf{1}-\Delta+\lambda)^{-1}\boldsymbol{w}\|_{W^{1,p}}\,d\lambda \\
{\scriptstyle (2.18)} &\le c_\alpha \int_0^\infty \lambda^{-\alpha}\,\|(\mathbf{1}-\Delta+\lambda)^{-1}\boldsymbol{w}\|_{W^{1,2}}^{1-\theta}\,\|(\mathbf{1}-\Delta+\lambda)^{-1}\boldsymbol{w}\|_{W^{2,2}}^{\theta}\,d\lambda \\
{\scriptstyle (2.15)} &\le c_{\alpha,p} \int_0^\infty \lambda^{-\alpha}\,\|(\mathbf{1}-\Delta)^{1/2}(\mathbf{1}-\Delta+\lambda)^{-1}\boldsymbol{w}\|_{L^2}^{1-\theta}\,\|(\mathbf{1}-\Delta)(\mathbf{1}-\Delta+\lambda)^{-1}\boldsymbol{w}\|_{L^2}^{\theta}\,d\lambda \\
{\scriptstyle (2.16)} &\le c_{\alpha,p} \int_0^\infty \lambda^{-\alpha}(1+\lambda)^{(\theta-1)/2}\,d\lambda\,\|\boldsymbol{w}\|_{L^2} \le c_{\alpha,p}\,\|\boldsymbol{w}\|_{L^2} \qquad \text{if } \tfrac{1+\theta}{2}<\alpha<1\,.
\end{aligned}$$

Consequently,

$$\|\boldsymbol{w}\|_{W^{1,p}} \le c_{\alpha,p}\,\|(\mathbf{1}-\Delta)^\alpha\boldsymbol{w}\|_{L^2} \le c_{\alpha,p}\,\|\mathbf{A}_0^\alpha\boldsymbol{w}\|_{L^2} \quad\text{for}\quad p>2\,,\ \tfrac{5}{4}-\tfrac{3}{2p}<\alpha<1\,. \tag{2.20}$$

The second of these inequalities relies on the fact that $(1-\Delta)^\alpha$ and $\mathbf{A}_0^\alpha=(\lambda_0-\mathbf{A})^\alpha$ have the same domain of definition and induce equivalent norms on it, provided $\alpha>\frac12$. This equivalence of norms is explained in [13], I§7.4. Of course, the constant in the comparison of $\|(1-\Delta)^\alpha\boldsymbol{w}\|_{L^2}$ and $\|(\lambda_0-\mathbf{A})^\alpha\boldsymbol{w}\|_{L^2}$ depends on the viscosity ν (and on λ_0, which itself had been chosen right before equation (2.14) implicitly dependent on ν. Essentially, the reason for this equivalence of norms is that $(\lambda_0-\mathbf{A})^\alpha$ and $\nu(1-\Delta)^\alpha$ differ by an operator that is relatively bounded with respect to $(1-\Delta)^\kappa$, for $\kappa>\alpha+\frac12-1$, because $(\lambda_0-\mathbf{A})$ and $\nu(1-\Delta)$ differ by an operator that is relatively bounded with respect to $(1-\Delta)^{1/2}$.

Apply the Contraction Principle

Equation (2.12) is to be solved by fixed point iteration $\boldsymbol{w}_{n+1}=\mathcal{T}(\boldsymbol{w}_n)$ in the Banach space $\mathbf{X}$, and we need to establish the contraction property in a subset invariant under $\mathcal{T}$. This is the routine part, and the only reason why it is carried out here is to follow the choices for the appropriate Banach space (which is given by Romanov without comment).

Using (2.19), (2.20) and (2.17) in (2.12) yields

$$\begin{aligned}
&\|\mathbf{A}_0^\beta\mathcal{T}(\boldsymbol{w}_1)(t)-\mathbf{A}_0^\beta\mathcal{T}(\boldsymbol{w}_2)(t)\|_{L^2}\le\\
&\qquad \le C\textstyle\int_0^t(t-s)^{-\beta}e^{-|\omega|(t-s)}\|\mathbf{A}_0^\beta(\boldsymbol{w}_1-\boldsymbol{w}_2)(s)\|_{L^2}\,\|\mathbf{A}_0^\beta(\boldsymbol{w}_1+\boldsymbol{w}_2)(s)\|_{L^2}\,ds
\end{aligned}$$

provided $\beta>\frac58$. Therefore

$$\begin{aligned}
&t^\alpha e^{|\omega|t}\|\mathbf{A}_0^\beta\mathcal{T}(\boldsymbol{w}_1)(t)-\mathbf{A}_0^\beta\mathcal{T}(\boldsymbol{w}_2)(t)\|_{L^2}\le\\
&\qquad \le Ct^\alpha\textstyle\int_0^t(t-s)^{-\beta}e^{-|\omega|s}s^{-2\alpha}\,ds\,|||\boldsymbol{w}_1-\boldsymbol{w}_2|||\,(2|||\boldsymbol{w}_1|||+|||\boldsymbol{w}_1-\boldsymbol{w}_2|||)
\end{aligned}$$

and

$$\begin{aligned}
&|||\mathcal{T}(\boldsymbol{w}_1)-\mathcal{T}(\boldsymbol{w}_2)|||\le K\,|||\boldsymbol{w}_1-\boldsymbol{w}_2|||\,(2|||\boldsymbol{w}_1|||+|||\boldsymbol{w}_1-\boldsymbol{w}_2|||)\quad\text{with}\\
&K=C\sup\nolimits_t t^{1-\alpha-\beta}\textstyle\int_0^1(1-\sigma)^{-\beta}\sigma^{-2\alpha}e^{-|\omega|\sigma t}\,d\sigma \qquad\qquad \Big|\ \tfrac1p+\tfrac1q=1\\
&\quad\le C\sup\nolimits_t t^{1-\alpha-\beta}\left(\textstyle\int_0^1(1-\sigma)^{-p\beta}\sigma^{-2p\alpha}\,d\sigma\right)^{1/p}t^{-1/q}\left(1-e^{-q|\omega|t}\right)^{1/q}\\
&\quad<\infty\qquad\text{provided }\ -1<1-\alpha-\beta-\tfrac1q<0
\end{aligned}$$

This makes it a feasible choice to take $\alpha=\frac14$, $\beta=\frac34$. One takes $\beta_0=\beta-\alpha=\frac12$ and gets $|||\mathcal{T}(\mathbf{0})|||\le\sup_t t^\alpha e^{|\omega|t}\|\mathbf{A}_0^\alpha e^{\mathbf{A}t}\|_{L^2\to L^2}\|\mathbf{A}_0^{\beta_0}\boldsymbol{w}_0\|_{L^2}\le C\|\mathbf{A}_0^{\beta_0}\boldsymbol{w}_0\|_{L^2}$ by (2.17). Assuming $\|\mathbf{A}_0^{\beta_0}\boldsymbol{w}_0\|_{L^2}$ sufficiently small, one gets convergence of $\mathcal{T}^n(\mathbf{0})$ in $\mathbf{X}$.

The boundedness of Ω did not enter into the argument; in particular, the Sobolev estimates were used with the exact exponent, i.e., in their scaling invariant form for functions on (or extensible to) $\mathbb{R}^3$.

2.6 Joseph's and Synge's Bounds

A very simple approach towards stability, which alas gives only partial results, has been carried out by Joseph [11], based on ideas going back to Synge. The method is explained concisely in Section 26.2 of [5] and also easily accessible from the original work, and we can be very short:

Take equation (1.10),

$$w^{(4)} - 2\alpha^2 w'' + \alpha^4 w = \frac{1}{\varepsilon}(i\alpha y + \lambda)(w'' - \alpha^2 w) \ ,$$

multiply by $\bar{w}$ (the complex conjugate of w) and integrate by parts (using the boundary conditions). The result is:

$$\begin{aligned} \operatorname{Im}\lambda &= -\alpha \frac{\int_0^1 y|w'|^2 + \alpha^2 \int_0^1 y|w|^2}{I_1^2 + \alpha^2 I_0^2} \\ \operatorname{Re}\lambda &= -\varepsilon \frac{I_2^2 + 2\alpha^2 I_1^2 + \alpha^4 I_0^2}{I_1^2 + \alpha^2 I_0^2} + \alpha \frac{\operatorname{Im}\int_0^1 w'\bar{w}}{I_1^2 + \alpha^2 I_0^2} \\ \text{where} \ \ I_k^2 &:= \textstyle\int_0^1 |w^{(k)}|^2 \end{aligned} \tag{2.21}$$

Using $I_2^2 \geq 4\pi^2 I_1^2$, $I_1^2 \geq \pi^2 I_0^2$, $I_2^2 \geq (4.73\ldots)^4 I_0^2 > (3\pi/2)^4 I_0^2$ and $2|\alpha \operatorname{Im}\int_0^1 w'\bar{w}| \leq \gamma I_1^2 + \alpha^2 I_0^2/\gamma$ (for any $\gamma > 0$), one easily gets $\operatorname{Re}\lambda < 0$ for sufficiently large ε (i.e., small Reynolds numbers). Alas, the argument is not sufficient for the general case.

It is pointed out in [21] that the failure to get a negative bound for $\operatorname{Re}\lambda$ by this method is inevitable, if the numerical range of the Orr–Sommerfeld operator (which is a superset of the convex hull of the spectrum) extends beyond the imaginary axis. And this will necessarily happen, if the semigroup generated by that operator has a transient growth in norm, as is suspected and shown numerically in the works along the lines of Trefethen. In determining the numerical range, one gets the set of those λ given by (2.21) where w runs through all of the domain of definition of the Orr–Sommerfeld operator, not only through the (unknown) eigenfunctions. For such w, one can clearly obtain $\operatorname{Im}\int_0^1 w'\bar{w} > 0$, and thus $\operatorname{Re}\lambda > 0$ for sufficiently small ε.

Let us make another interesting remark: the information for $\operatorname{Im}\lambda$ is usually ignored from the very beginning. However, using $0 \leq y \leq 1$, one immediately obtains $|\operatorname{Im}\lambda| \leq |\alpha|$, or in the coordinates (1.12) $|t + \operatorname{Im}\delta| \leq 2|t|$. At least, we get a very simple sectoriality proof (for eigenvalues, which is sufficient in the case of periodic boundary conditions for the x_1, x_3 variables) out of the argument: Letting $\gamma = 1$, one finds an upper estimate for $\operatorname{Re}\lambda$: $\operatorname{Re}\lambda \leq -\varepsilon(\pi^2 + \alpha^2) + \frac{1}{2}$ (which could by optimized by more careful choice of parameters). The spectrum is contained not only in a sector, but even in a parabola!

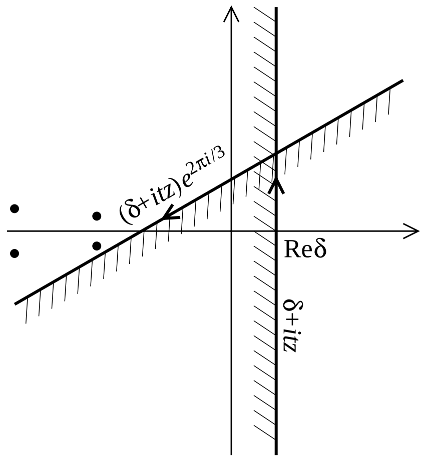
2.7 Orr–Sommerfeld in Romanov's Stability Proof

Romanov's estimate consists of two parts, each valid for a different range of parameters. The first is Joseph's bound, which has just been discussed. The second (for small ε) uses the characteristic equation in the form

$$\begin{vmatrix} \int_{-1}^{1} \operatorname{Ai}\Big((\delta + itz)e^{2i\pi/3}\Big) \exp(4\varepsilon t^3 z)\, dz & \int_{-1}^{1} \operatorname{Ai}\Big((\delta + itz)e^{2i\pi/3}\Big) \exp(-4\varepsilon t^3 z)\, dz \\ \int_{-1}^{1} \operatorname{Ai}\Big((\delta + itz)e^{-2i\pi/3}\Big) \exp(4\varepsilon t^3 z)\, dz & \int_{-1}^{1} \operatorname{Ai}\Big((\delta + itz)e^{-2i\pi/3}\Big) \exp(-4\varepsilon t^3 z)\, dz \end{vmatrix} = 0 \ .$$

The basic feature is that the Airy functions in the first row are large for positive z and small for negative z, whereas the converse holds for the ones in the second row. (This is immediate from the asymptotics.) An analogous behaviour applies to the exponentials with respect to the columns.

So we hope that each diagonal term is larger in absolute value than its competitor to the left or right. To get this proven, one has to control the cancellations in the integral, which are due to the fast varying argument of the Airy functions, whereas for small ε, the exponentials play a less important role. The main contribution comes therefore from an integral of the Airy functions. This is where zeros of $\int^\infty \mathrm{Ai}$ need to be avoided. The figure shows the arguments of Ai along the paths of integration, some zeros of $\int^\infty \mathrm{Ai}$, and the hatching displays the region in which δ has to lie to avoid the zeros and thus make the estimate work.



It should not be necessary to repeat the technicalities from Romanov's paper, having given the qualitative ideas behind the calculation. It remains to show that the two cases cover all values of the parameters, and to give an estimate of the exact location of the zeros of $\int^\infty \mathrm{Ai}$. This concerns only the small zeros, because asymptotically, the zeros lie near the negative real axis.

This calculation is the only place where numerics enters into Romanov's proof, and it does so in a very transparent and simple way (combined with analytic a-priori estimates), immediately amenable to interval arithmetic, so that his proof may be considered as analytically rigorous.

The asymptotic estimates referred to here are classical and can be found in Section 3.3.

2.8 Heisenberg, Wasow, Lin

Much confusion has arisen from attempts to approximate the viscous equation for large Reynolds numbers by means of the inviscid equation. This is typically done for the OS equation (1.11) with general velocity profile U, namely

$$\Big[\Big(\varepsilon(\partial_y^2-\alpha^2)-i\alpha U(y)-\lambda\Big)(\partial_y^2-\alpha^2)+i\alpha U''(y)\Big]\,w(y)=0\ , \tag{2.22}$$

not worrying about the fact that only very special profiles U actually arise from linearizing Navier–Sokes about actual solutions in the way described in Chapter 1. A formal approach leads to series expansions in negative powers of α/ε or $(\alpha/\varepsilon)^{1/2}$, i.e., in t^{-3} or $t^{-3/2}$ in the notation of the present work. These expansions were already considered by Heisenberg [8]. The situation is thoroughly explained in chapter 8 of Lin's book [16]. See also [17]. Much of the difficulty is caused by the fact that the formal limit of (2.22), namely

$$\Big[(U(y)-i\lambda/\alpha)(\partial_y^2-\alpha^2)-U''(y)\Big]\,w(y)=0\ , \tag{2.23}$$

with the boundary conditions $w(0)=0=w(1)$ has a singularity at $y\in U^{-1}(i\lambda/\alpha)$ and therefore solutions with branch points. The question of which branches serve as approximants to what is quite nontrivial and has been explained by Wasow [32]. However, the case $U(y)=y$ is somewhat special as the singularity disappears, but nontrivial solutions to (2.23) with the Dirichlet boundary conditions do not exist in this case. This is also what should be expected. An inviscid fluid will simply not follow the moving upper plate, because the no slip condition on the boundary is adverse to the definition of an inviscid fluid. So a shear flow cannot develop at all. The eigenvalues for plane Couette are not related to eigenvalues of an inviscid problem (cf. [18]).

Therefore, for the particular problem under discussion in the present work, the results just abstracted briefly may be less relevant.

Chapter 3

Well–Known Facts about the Airy Functions

3.1 Differential Equation and Series

The Airy differential equation

$$w''(z) = z\,w(z) \tag{3.1}$$

has entire functions as solutions. For <u>any</u> solution Ai, the initial value problem is solved by the power series

$$\begin{aligned}\mathrm{Ai}(z) &= \mathrm{Ai}(0)\sum_{k=0}^{\infty}\frac{(3k-2)!!!}{(3k)!}z^{3k} + \mathrm{Ai}'(0)\sum_{k=0}^{\infty}\frac{(3k-1)!!!}{(3k+1)!}z^{3k+1}\\ \mathrm{Ai}'(z) &= \mathrm{Ai}(0)\sum_{k=0}^{\infty}\frac{(3k+1)!!!}{(3k+2)!}z^{3k+2} + \mathrm{Ai}'(0)\sum_{k=0}^{\infty}\frac{(3k-1)!!!}{(3k)!}z^{3k}\ ,\end{aligned} \tag{3.2}$$

where we have employed the notation $n!!!$ in analogy to the common $n!!$. It is defined by $n!!! = n(n-3)!!!$ and $2!!! = 2$, $1!!! = 0!!! = 1$.

In the following, we will reserve Ai for the solution with the initial conditions $\mathrm{Ai}(0) = 3^{-2/3}/\Gamma(2/3)$, $\mathrm{Ai}'(0) = -3^{-1/3}/\Gamma(1/3)$ and Bi for the solution with the initial conditions $\mathrm{Bi}(0) = 3^{-1/6}/\Gamma(2/3)$, $\mathrm{Bi}'(0) = 3^{1/6}/\Gamma(1/3)$, as is common usage. An arbitrary solution will be denoted by Ci (but subject to certain specifications which may vary at different invocations of the symbol Ci).

Ai is a solution that satisfies $\lim_{x\to+\infty}\mathrm{Ai}(x) = 0$, and up to a constant factor the only one. That the Ai defined by the given initial data has the claimed decay property, follows form observing that

$$\mathrm{Ai}(x) := \frac{1}{\pi}\int_0^{\infty}\cos(t^3/3 + xt)\,dt \tag{3.3}$$

indeed satisfies the claimed initial value problem. The details can be found, e.g., in Olver's book [19, ch. 2§ 8]. We note that the value of the Wronskian is

$$W \equiv \mathrm{Ai}'(z)\,\mathrm{Bi}(z) - \mathrm{Ai}(z)\,\mathrm{Bi}'(z) = \frac{-1}{\pi}\ .$$

The substitution $\zeta = \frac{2}{3}z^{3/2}$, $w(z) = \zeta^{1/3}u(\zeta)$, transforms Airy's ODE into a special case of the modified Bessel ODE, namely into

$$\zeta^2u''(\zeta) + \zeta u'(\zeta) - (\zeta^2 + 1/9)u(\zeta) = 0\ . \tag{3.4}$$

In particular,

$$\begin{aligned} \mathrm{Ai}(z) &= \pi^{-1}(z/3)^{1/2}K_{1/3}(\zeta) & \mathrm{Ai}(-z) &= \left(z^{1/2}/3\right)(J_{1/3}(\zeta)+J_{-1/3}(\zeta)) \\ \mathrm{Bi}(z) &= (z/3)^{1/2}(I_{1/3}(\zeta)+I_{-1/3}(\zeta)) & \mathrm{Bi}(-z) &= (z/3)^{1/2}(J_{-1/3}(\zeta)-J_{1/3}(\zeta)) \end{aligned} \tag{3.5}$$

where I, J, K are the modified Bessel or Bessel functions commonly thus denoted: See [1] or [30]. Among the extensive theory of Bessel functions, there are two types of results for the Airy functions which we will need: Firstly, some integrals:

$$\int_0^\infty \mathrm{Ai}(x)\,dx = 1/3 \;, \qquad \int_0^\infty \mathrm{Ai}(-x)\,dx = 2/3 \;, \qquad \int_0^\infty \mathrm{Bi}(-x)\,dx = 0 \,. \tag{3.6}$$

The formulas proved in Chapter XIII, section 13.2 and 13.21 of [30] specialize to the ones in (3.6).

The other application is the extensive knowledge about asymptotic expansions of Bessel functions, which immediately produces the corresponding expansions in Section 3.3 below.

3.2 Threefold Symmetry

The following are from [1] again, and their proofs are immediate: Both sides of the alleged equalities are solutions of Airy's ODE with the same initial data.

$$\mathrm{Ai}(z) + e^{2\pi i/3}\,\mathrm{Ai}(ze^{2\pi i/3}) + e^{-2\pi i/3}\,\mathrm{Ai}(ze^{-2\pi i/3}) = 0 \tag{3.7}$$

$$\mathrm{Bi}(z) + e^{2\pi i/3}\,\mathrm{Bi}(ze^{2\pi i/3}) + e^{-2\pi i/3}\,\mathrm{Bi}(ze^{-2\pi i/3}) = 0 \tag{3.8}$$

$$\mathrm{Bi}(z) = e^{\pi i/6}\,\mathrm{Ai}(ze^{2\pi i/3}) + e^{-\pi i/6}\,\mathrm{Ai}(ze^{-2\pi i/3}) \tag{3.9}$$

$$\mathrm{Bi}(z) \pm i\,\mathrm{Ai}(z) = 2e^{\pm\pi i/6}\,\mathrm{Ai}(ze^{\pm 2\pi i/3}) \tag{3.10}$$

A trivial variant of this is

$$\mathrm{Ai}(z) \pm i\,\mathrm{Bi}(z) = 2e^{\pm i\pi/3}\,\mathrm{Ai}(e^{\mp 2\pi i/3}z) \,. \tag{3.11}$$

3.3 Asymptotic Expansions

The following asymptotic formulas are taken from Abramowitz–Stegun [1], slightly rewritten in some cases. We give their number as reference. The coefficients c_k, d_k and the number ζ occurring in these formulas are:

$$c_k = \frac{(6k-1)!!}{(2k-1)!!\,6^{3k}\,k!} \qquad d_k = -\frac{6k+1}{6k-1}\,c_k \qquad \zeta = \frac{2}{3}z^{3/2}$$

k	0	1	2	3	4	5	6
c_k	1	$\frac{5}{72}$	$\frac{385}{10368}$	$\frac{85085}{2239488}$	$\frac{37182145}{644972544}$	$\frac{5391411025}{46438023168}$	$\frac{5849680962125}{20061226008576}$
d_k	1	$\frac{-7}{72}$	$\frac{-455}{10368}$	$\frac{-95095}{2239488}$	$\frac{-40415375}{644972544}$	$\frac{-5763232475}{46438023168}$	$\frac{-6183948445675}{20061226008576}$

(3.12)

Proofs for these expansions can be obtained through an integral representation. For Ai, this is carried out in Section 4.4 of [19]. The formulas for Bi result from those for Ai by means of the threefold symmetry formulas. For a proof in terms of Bessel functions, see Chapter 7 of [30].

$$\mathrm{Ai}(z) \sim \frac{z^{-1/4}}{2\pi^{1/2}} \exp[-\zeta] \sum_{k=0}^{\infty} (-1)^k c_k \zeta^{-k} \ , \quad -\pi < \arg z < \pi \qquad [1], 10.4.59 \tag{3.13}$$

$$\mathrm{Ai}'(z) \sim \frac{-z^{1/4}}{2\pi^{1/2}} \exp[-\zeta] \sum_{k=0}^{\infty} (-1)^k d_k \zeta^{-k} \ , \quad -\pi < \arg z < \pi \qquad [1], 10.4.61 \tag{3.14}$$

$$\begin{aligned} \mathrm{Ai}(-z) &\sim \frac{z^{-1/4}}{\pi^{1/2}} \left[\sin[\zeta + \pi/4] \sum_{k=0}^{\infty} (-1)^k c_{2k} \zeta^{-2k} - \cos[\zeta + \pi/4] \sum_{k=0}^{\infty} (-1)^k c_{2k+1} \zeta^{-2k-1} \right] \\ &= \frac{z^{-1/4}}{2\pi^{1/2}} \left[\exp[i\zeta] e^{-i\pi/4} \Big(\sum_{k=0}^{\infty} c_k (i\zeta)^{-k} \Big) + \exp[-i\zeta] e^{i\pi/4} \Big(\sum_{k=0}^{\infty} c_k (-i\zeta)^{-k} \Big) \right] \ , \\ &\qquad -2\pi/3 < \arg z < 2\pi/3 \qquad [1], 10.4.60 \end{aligned} \tag{3.15}$$

$$\begin{aligned} \mathrm{Ai}'(-z) &\sim \frac{-z^{1/4}}{\pi^{1/2}} \left[\cos[\zeta + \pi/4] \sum_{k=0}^{\infty} (-1)^k d_{2k} \zeta^{-2k} + \sin[\zeta + \pi/4] \sum_{k=0}^{\infty} (-1)^k d_{2k+1} \zeta^{-2k-1} \right] \\ &= \frac{-z^{1/4}}{2\pi^{1/2}} \left[\exp[i\zeta] e^{i\pi/4} \Big(\sum_{k=0}^{\infty} d_k (i\zeta)^{-k} \Big) + \exp[-i\zeta] e^{-i\pi/4} \Big(\sum_{k=0}^{\infty} d_k (-i\zeta)^{-k} \Big) \right] \ , \\ &\qquad -2\pi/3 < \arg z < 2\pi/3 \qquad [1], 10.4.62 \end{aligned} \tag{3.16}$$

$$\mathrm{Bi}(z) \sim \frac{z^{-1/4}}{\pi^{1/2}} \exp[\zeta] \sum_{k=0}^{\infty} c_k \zeta^{-k} \ , \quad -\pi/3 < \arg z < \pi/3 \qquad [1], 10.4.63 \tag{3.17}$$

$$\mathrm{Bi}'(z) \sim \frac{z^{1/4}}{\pi^{1/2}} \exp[\zeta] \sum_{k=0}^{\infty} d_k \zeta^{-k} \ , \quad -\pi/3 < \arg z < \pi/3 \qquad [1], 10.4.66 \tag{3.18}$$

$$\begin{aligned} \mathrm{Bi}(z) &\sim \frac{z^{-1/4}}{\pi^{1/2}} \left[\exp[\zeta] \sum_{k=0}^{\infty} c_k \zeta^{-k} + \frac{i}{2} \exp[-\zeta] \sum_{k=0}^{\infty} (-1)^k c_k \zeta^{-k} \right] \ , \\ &\qquad -\pi/3 < \arg z < \pi \qquad [1], 10.4.65 \qquad \text{for } -\pi < \arg z < \pi/3 \text{ change } i \text{ to } -i \end{aligned} \tag{3.19}$$

$$\begin{aligned} \mathrm{Bi}'(z) &\sim \frac{z^{1/4}}{\pi^{1/2}} \left[\exp[\zeta] \sum_{k=0}^{\infty} d_k \zeta^{-k} - \frac{i}{2} \exp[-\zeta] \sum_{k=0}^{\infty} (-1)^k d_k \zeta^{-k} \right] \ , \\ &\qquad -\pi/3 < \arg z < \pi \qquad [1], 10.4.68 \qquad \text{for } -\pi < \arg z < \pi/3 \text{ change } i \text{ to } -i \end{aligned} \tag{3.20}$$

$$\begin{aligned} \mathrm{Bi}(-z) &\sim \frac{z^{-1/4}}{\pi^{1/2}} \left[\cos[\zeta + \pi/4] \sum_{k=0}^{\infty} (-1)^k c_{2k} \zeta^{-2k} + \sin[\zeta + \pi/4] \sum_{k=0}^{\infty} (-1)^k c_{2k+1} \zeta^{-2k-1} \right] \\ &= \frac{z^{-1/4}}{2\pi^{1/2}} \left[\exp[i\zeta] e^{i\pi/4} \Big(\sum_{k=0}^{\infty} c_k (i\zeta)^{-k} \Big) + \exp[-i\zeta] e^{-i\pi/4} \Big(\sum_{k=0}^{\infty} c_k (-i\zeta)^{-k} \Big) \right] \ , \\ &\qquad -2\pi/3 < \arg z < 2\pi/3 \qquad [1], 10.4.64 \end{aligned} \tag{3.21}$$

$$\begin{aligned}\mathrm{Bi}'(-z) &\sim \frac{-z^{1/4}}{\pi^{1/2}}\left[\sin[\zeta+\pi/4]\sum_{k=0}^{\infty}(-1)^k d_{2k}\zeta^{-2k} - \cos[\zeta+\pi/4]\sum_{k=0}^{\infty}(-1)^k d_{2k+1}\zeta^{-2k-1}\right] \\ &= \frac{-z^{1/4}}{2\pi^{1/2}}\left[\exp[i\zeta]e^{-i\pi/4}\Big(\sum_{k=0}^{\infty} d_k(i\zeta)^{-k}\Big) + \exp[-i\zeta]e^{i\pi/4}\Big(\sum_{k=0}^{\infty} d_k(-i\zeta)^{-k}\Big)\right], \\ &\qquad -2\pi/3 < \arg z < 2\pi/3 \qquad [1], 10.4.67\end{aligned} \tag{3.22}$$

The asymptotic expansion give rise to immediate global upper estimates:

Lemma 3.1 *There exists a constant K such that for all $z \in \mathbb{C}$, it holds:*

$$|\mathrm{Ai}(z)| \le K \min\{1, |z|^{-1/4}\} \exp\Big[-\tfrac{2}{3}\,\mathrm{Re}(z^{3/2})\Big]$$

$$|\mathrm{Ai}'(z)| \le K \max\{1, |z|^{1/4}\} \exp\Big[-\tfrac{2}{3}\,\mathrm{Re}(z^{3/2})\Big]$$

$$|\mathrm{Bi}(z)| \le K \min\{1, |z|^{-1/4}\} \exp\Big|\tfrac{2}{3}\,\mathrm{Re}(z^{3/2})\Big|$$

$$|\mathrm{Bi}'(z)| \le K \max\{1, |z|^{1/4}\} \exp\Big|\tfrac{2}{3}\,\mathrm{Re}(z^{3/2})\Big|$$

In the case of Ai, Ai$'$, *the ambiguity of the sign of $z^{3/2}$ must be resolved by the determination* $\arg z \in [-\pi, \pi]$.

PROOF:
In the sector $\arg z \in\,]-\pi/3, \pi[$, as well as in the sector $\arg z \in\,]-\pi, \pi/3[$, the estimate for Bi follows from (3.19). It carries over to the whole complex plane due to (3.8). The same argument applies to Bi$'$.

Ai and Ai$'$ satisfy the very same estimates as Bi and Bi$'$, but (3.13) and (3.14) produce an immediate improvement in the sector $\arg z \in [-\pi/3, \pi/3]$. In the other sectors, the estimates given for Ai and Bi are the same, despite the first appearance. ■

It is possible to integrate the series (3.13), and after use of (3.7) also the series (3.15). The results are:

$$\int_z^{\infty} \mathrm{Ai}(x)\,dx \sim \frac{z^{-3/4}}{2\pi^{1/2}}\exp[-\zeta]\sum_{k=0}^{\infty}(-1)^k g_k\zeta^{-k}, \quad -\pi < \arg z < \pi \tag{3.23}$$

$$\begin{aligned}\int_{-\infty}^{-z} \mathrm{Ai}(x)\,dx &\sim \frac{z^{-3/4}}{\pi^{1/2}}\left[\cos[\zeta+\pi/4]\sum_{k=0}^{\infty}(-1)^k g_{2k}\zeta^{-2k} + \sin[\zeta+\pi/4]\sum_{k=0}^{\infty}(-1)^k g_{2k+1}\zeta^{-2k-1}\right] \\ &= \frac{z^{-3/4}}{2\pi^{1/2}}\left[\exp[i\zeta]e^{i\pi/4}\Big(\sum_{k=0}^{\infty} g_k(i\zeta)^{-k}\Big) + \exp[-i\zeta]e^{-i\pi/4}\Big(\sum_{k=0}^{\infty} g_k(-i\zeta)^{-k}\Big)\right], \\ &\qquad -2\pi/3 < \arg z < 2\pi/3\end{aligned} \tag{3.24}$$

where

$$g_k = \sum_{s=0}^{k} c_s \frac{(2k-1)!!}{(2s-1)!!} 2^{s-k}, \qquad \begin{array}{c||c|c|c|c|c} k & 0 & 1 & 2 & 3 & 4 \\ \hline g_k & 1 & \frac{41}{72} & \frac{9241}{10368} & \frac{5075225}{2239488} & \frac{5153008945}{644972544} \end{array}. \tag{3.25}$$

3.4 Zeros

We will need some results on the zeros of Airy functions. Their asymptotic behaviour follows from the previous section. However, we have chosen to put these results into Section 5.2.1, which is where they are needed.

Chapter 4

Higher Order Wronskians of Airy Functions

The key result of this chapter is Theorem 4.7 on page 43. It provides a convergent series expansion for the spectral condition, which is good for large δ. The evaluation of this series involves some careful combinatorics (Section 4.1). The leading term of the series is used to calculate the spectrum for small wave numbers and / or large δ. This corresponds to a limit in which the OS operator converges to a self adjoint operator.

The full spectrum arises by continuation and only simple bifurcations from this limit.

4.1 Combinatorial Formulas

We will need to expand Airy functions like $\mathrm{Ai}(\delta + itz)$ in Taylor series at δ, and to express the higher derivatives in terms of $\mathrm{Ai}(\delta)$ and $\mathrm{Ai}'(\delta)$. For this purpose, we collect all relevant facts now, ready for use in the sequel.

Lemma 4.1 *We can write*

$$\begin{aligned} \mathrm{Ai}^{(N)}(\delta) &= P_N(\delta)\,\mathrm{Ai}(\delta) + Q_N(\delta)\,\mathrm{Ai}'(\delta) \\ \mathrm{Bi}^{(N)}(\delta) &= P_N(\delta)\,\mathrm{Bi}(\delta) + Q_N(\delta)\,\mathrm{Bi}'(\delta) \end{aligned} \tag{4.1}$$

for the sequence of polynomials P_N, Q_N, given recursively by

$$\begin{aligned} &P_0 = 1\,, \quad Q_0 = 0 \\ &P_1 = 0\,, \quad Q_1 = 1 \\ &P_N = P'_{N-1} + \delta Q_{N-1} \\ &Q_N = Q'_{N-1} + P_{N-1} \end{aligned} \tag{4.2}$$

These polynomials can be written in the form

$$\begin{aligned} P_{2k+1}(\delta) &= \textstyle\sum_{j=0}^{*} p_j^{2k+1}\delta^{k-1-3j}\,, && (\,* = \lfloor (k-1)/3 \rfloor) \\ Q_{2k+1}(\delta) &= \textstyle\sum_{j=0}^{*} q_j^{2k+1}\delta^{k-3j}\,, && (\,* = \lfloor k/3 \rfloor\,) \\ P_{2r}(\delta) &= \textstyle\sum_{j=0}^{*} p_j^{2r}\delta^{r-3j}\,, && (\,* = \lfloor r/3 \rfloor\,) \\ Q_{2r}(\delta) &= \textstyle\sum_{j=0}^{*} q_j^{2r}\delta^{r-2-3j}\,, && (\,* = \lfloor (r-2)/3 \rfloor) \end{aligned} \tag{4.3}$$

with

$$\begin{aligned} p_j^{2k+1} &= \binom{k}{3j+1} P_{j+1}^{\text{odd}}(k) & \qquad q_j^{2k+1} &= \binom{k}{3j} Q_j^{\text{odd}}(k) \\ p_j^{2r} &= \binom{r}{3j} P_j^{\text{even}}(r) & \qquad q_j^{2r} &= \binom{r}{3j+2} Q_j^{\text{even}}(r) \end{aligned} \tag{4.4}$$

and $p_0^0 = 1$, $q_0^1 = 1$. *Here the* $(P|Q)_j^{(\text{even}|\text{odd})}$ *are themselves polynomials of degree* j. *The upper limit "∗" in (4.3) stands for the largest* j *such that no negative powers of* δ *occur and has been specified merely for easier reference.*

PROOF:
The formulas (4.1), (4.2) follow immediately by induction, using the Airy ODE $\mathrm{Ai}''(\delta) = \delta\,\mathrm{Ai}(\delta)$. The recursions in (4.2) imply the ones in (4.3). That only every third power of δ occurs can also be deduced from the fact that $\delta \mapsto \mathrm{Ai}(e^{2\pi i/3}\delta)$ also satisfies Airy's differential equation, and by comparing the analog of (4.1) for that function with the original (4.1) evaluated at $e^{2\pi i/3}\delta$. For the relevant values of j, k, r, the binomial coefficients in (4.4) do not vanish, so we can start by taking (4.4) as definition for the $(P|Q)_j^{(\text{even}|\text{odd})}$. We try to evaluate these terms inductively, which will show them to be polynomials.

Firstly, the recursions (4.3) are transformed into

$$\begin{aligned} p_j^{2k+1} &= q_j^{2k} + (k-3j)p_j^{2k} &, \quad &(k \geq 3j+1\ ;\ \text{else } p_j^{2k+1} = 0) \\ q_j^{2k+1} &= p_j^{2k} + (k+1-3j)q_{j-1}^{2k} &, \quad &(k \geq 3j\ ;\ \text{else } q_j^{2k+1} = 0) \\ p_j^{2r} &= q_j^{2r-1} + (r+1-3j)p_{j-1}^{2r-1} &, \quad &(r \geq 3j\ ;\ \text{else } p_j^{2r} = 0) \\ q_j^{2r} &= p_j^{2r-1} + (r-1-3j)q_j^{2r-1} &, \quad &(r \geq 3j+2\ ;\ \text{else } q_j^{2r} = 0) \end{aligned} \tag{4.5}$$

and therefore

$$\begin{aligned} P_{j+1}^{\text{odd}}(k) &= \frac{k-3j-1}{3j+2} Q_j^{\text{even}}(k) + (3j+1)P_j^{\text{even}}(k) \\ Q_j^{\text{odd}}(k) &= P_j^{\text{even}}(k) + 3j Q_{j-1}^{\text{even}}(k) \\ P_j^{\text{even}}(r) &= \frac{r-3j}{r} Q_j^{\text{odd}}(r-1) + \frac{3j(3j-1)}{r} P_j^{\text{odd}}(r-1) \\ Q_j^{\text{even}}(r) &= \frac{3j+2}{r} P_{j+1}^{\text{odd}}(r-1) + \frac{(3j+1)(3j+2)}{r} Q_j^{\text{odd}}(r-1) \end{aligned} \tag{4.6}$$

This allows to compute the functions P^{odd}, Q^{odd}, P^{even}, Q^{even} by induction with respect to j as follows: P_0^{odd} and Q_{-1}^{even}, which do not occur, are defined to be 0 for convenience. The following sequence of steps needs to be done for $j = 0$ to start the induction, and then successively for $j \geq 1$ assuming they are done for $j-1$.

Inserting the second equation of (4.6) into the third, we get

$$\begin{aligned} &kP_j^{\text{even}}(k) - (k-3j)P_j^{\text{even}}(k-1) = \\ &\qquad = (3j)(k-3j)Q_{j-1}^{\text{even}}(k-1) + (3j)(3j-1)P_j^{\text{odd}}(k-1) \qquad (j \geq 0) \end{aligned} \tag{4.7}$$

which will determine P_j^{even}.

Then obviously the second equation of (4.6) determines Q_j^{odd}.

Inserting the fourth equation of (4.6) into the first, we get

$$\begin{aligned} &kP_{j+1}^{\text{odd}}(k) - (k-3j-1)P_{j+1}^{\text{odd}}(k-1) = \\ &\qquad = (3j+1)(k-3j-1)Q_j^{\text{odd}}(k-1) + k(3j+1)P_j^{\text{even}}(k) \qquad (j \geq 0) \end{aligned} \tag{4.8}$$

which will determine $P^{\rm odd}_{j+1}$.

Then the fourth equation of (4.6) will determine $Q^{\rm even}_j$.

We now show that the above induction procedure (enhanced with the normalization condition $p^0_0 = 1$) determines the functions $(P|Q)^{\rm even|odd}_j(\cdot)$ uniquely on their respective domains of definition, and that they coincide with polynomials of the claimed degree there.

Firstly, (4.7) implies (with $p^0_0 = 1$) that $P^{\rm even}_0 \equiv 1$, and hence $Q^{\rm odd}_0 \equiv 1$. For $j \geq 0$, (4.8) is a linear first order difference equation for functions defined on $\mathbb{Z}$. It has a one dimensional affine solution space. The corresponding homogeneous equation $kf(k)-(k-3j-1)f(k-1) = 0$ has the solution $f(k) = f(0)(-1)^k\binom{3j}{k}$, which is supported outside the domain of definition of $P^{\rm odd}_{j+1}$. Therefore, the solution is unique on the original domain of definition, and if an ansatz $P^{\rm odd}_{j+1}(k) = \sum_{i=0}^{j+1} a_i k^i$, succeeds on (4.8) for any given polynomial of degree at most $j+1$ on the right hand side, we are done with this part of the induction step. A straightforward calculation shows that such an ansatz produces a linear system whose matrix is triangular, with the coefficients of $a_i k^i$ (i.e., on the diagonal) being $i + 3j + 1 \neq 0$. The same argument works for (4.7) when $j \geq 1$.

The fourth equation of (4.6) clearly determines $Q^{\rm even}_j$ on its domain of definition, but in order to show that $Q^{\rm even}_j$ is a polynomial, we need to show that the polynomial $R_{j+1}(r) := P^{\rm odd}_{j+1}(r) + (3j+1)Q^{\rm odd}_j(r)$ is divisible by $r+1$. The claim about the degree is then clear again. Adding to equation (4.8) the term $(3j+1)kQ^{\rm odd}_j(k)$ implies

$$kR_{j+1}(k) - (k-3j-1)R_{j+1}(k-1) = (3j+1)k\left[Q^{\rm odd}_j(k) + P^{\rm even}_j(k)\right]$$

which, as an equation of polynomials, can be used for $k = 0$, disregarding the the original domain of definition. We get $R_{j+1}(-1) = 0$, hence $R_{j+1}(k)$ is divisible by $k+1$. This completes the induction. ■

We give the first few polynomials for later reference:

$$\begin{array}{ll} & P^{\rm even}_0(r) = 1 \\ P^{\rm odd}_1(k) = k & P^{\rm even}_1(r) = -5+3r \\ P^{\rm odd}_2(k) = -4-8k+4k^2 & P^{\rm even}_2(r) = 100-150r+30r^2 \\ Q^{\rm odd}_0(k) = 1 & Q^{\rm even}_0(r) = 2 \\ Q^{\rm odd}_1(k) = 1+3k & Q^{\rm even}_1(r) = -20+20r \\ Q^{\rm odd}_2(k) = -20-30k+30k^2 & Q^{\rm even}_2(r) = 672-1232r+336r^2 \end{array} \tag{4.9}$$

The next lemma concerns higher order Wronskians. We define

$$[\![M|N]\!](\delta) := \mathrm{Ai}^{(M)}(\delta)\,\mathrm{Bi}^{(N)}(\delta) - \mathrm{Ai}^{(N)}(\delta)\,\mathrm{Bi}^{(M)}(\delta) \tag{4.10}$$

and

$$\begin{aligned} x^{\uparrow n} &:= x(x+1)\cdots(x+n-1) = \Gamma(x+n)/\Gamma(x)\ , \\ x^{\downarrow n} &:= x(x-1)\cdots(x+1-n) = \Gamma(x+1)/\Gamma(x+1-n)\ . \end{aligned} \tag{4.11}$$

Lemma 4.2 *For* Ai *and* Bi *any two linearly independent solutions of Airy's ODE and* $W := \mathrm{Ai}'(\delta)\,\mathrm{Bi}(\delta) - \mathrm{Ai}(\delta)\,\mathrm{Bi}'(\delta) = [\![1|0]\!]$ *their Wronskian (a nonzero constant), we obtain:*

$$[\![2k+1|2r]\!](\delta) = W\sum_{j=0}^{*} c^{k,r}_j\,\delta^{k+r-3j} \qquad (\text{where } * = \lfloor (k+r)/3 \rfloor)\ . \tag{4.12}$$

Here, $c^{k,r}_j$ *is a polynomial of joint degree* $4j$ *in* (k,r)*; more specifically,*

$$c^{k,r}_j = \frac{(k+r-j)!}{(k+r-3j)!}\,\tilde{c}^{k,r}_j\ , \tag{4.13}$$

where

$$\begin{aligned} \tilde{c}_j^{k,r} &= \sum_{l=0}^{j} \binom{k+r-l}{j-l} \frac{(-3)^{l-j}}{(2l)!} (k-r)^{\downarrow l} (k-r+1)^{\uparrow l} \\ &= \sum_{l=0}^{j} \binom{m-l}{j-l} \frac{(-3)^{l-j}}{(2l)!} \prod_{s=1}^{l} \left[2\Theta + (m-s)(m+s-1)\right] =: d_j(\Theta, m) \end{aligned} \tag{4.14}$$

with $m = k+r$, $\Theta = k(1-2r)$. In particular,

$$\tilde{c}_0^{k,r} = 1\ , \quad \tilde{c}_1^{k,r} = \frac{3k^2 - 6kr + 3r^2 + k - 5r}{6} \tag{4.15}$$

and the $\tilde{c}_j^{k,r}$ satisfy the following recursion:

$$\begin{aligned} 2j(2j-1)\tilde{c}_j^{k,r} &= -\frac{1}{3}\tilde{c}_{j-1}^{k,r}(j-k-r-1)(j-3k-3r-3) \\ &\quad +(2r-1)r\tilde{c}_{j-1}^{k,r-1} + (2k+1)k\tilde{c}_{j-1}^{k-1,r} \\ &\quad +\frac{1}{18}(j-k-r-1)(j-k-r-2)\tilde{c}_{j-2}^{k,r} \end{aligned} \tag{4.16}$$

PROOF:
Using (4.1), it is immediate that (4.12) is equivalent to

$$Q_{2k+1}(\delta)P_{2r}(\delta) - P_{2k+1}(\delta)Q_{2r}(\delta) = \sum_{j=0}^{*} c_j^{k,r}\delta^{k+r-3j} \tag{4.17}$$

and by (4.3), (4.4), this holds indeed with

$$\begin{aligned} c_j^{k,r} &:= \sum_{i=0}^{j} (p_i^{2r} q_{j-i}^{2k+1} - q_{i-1}^{2r} p_{j-i}^{2k+1}) \\ &= \sum_{i=0}^{j} \left[\binom{r}{3i}\binom{k}{3(j-i)} P_i^{\text{even}}(r) Q_{j-i}^{\text{odd}}(k) - \binom{r}{3i-1}\binom{k}{3(j-i)+1} Q_{i-1}^{\text{even}}(r) P_{j-i+1}^{\text{odd}}(k) \right] \end{aligned} \tag{4.18}$$

This already shows that $c_j^{k,r}$ is a polynomial of degree $4j$. (4.15) follows from (4.9) and (4.18). Although it can be easily seen from (4.18) that $c_j^{k,r}$ vanishes whenever $k+r < 3j$, this is not sufficient to infer that the given factors $(k+r-j)!/(k+r-3j)!$ can actually be split off. We therefore proceed by showing the recursion (4.16) first.

Using the convention that $c_j^{k,r} = 0$, if $j < 0$ or $j > k+r$ (the latter being a consequence for a corresponding convention for the p_j^{2r} etc.), we need not worry about the summation limits in (4.12) and can easily shift the index of summation. For any given j, recursion (4.16), or, more precisely, the recursion for $c_j^{k,r}$ into which (4.16) transforms by (4.13), is equivalent to a corresponding equation for generating functions, namely

$$\begin{aligned} &8\delta^{-1/2+k+r}\frac{d}{d\delta}\delta^{5/2}\frac{d}{d\delta}\delta^{-k-r}[\![2k+1|2r]\!] = \\ &\quad -2\delta^{-8k-8r-8}\frac{d}{d\delta}\delta^{8k+8r+9}\left(\frac{d}{d\delta}\right)^3[\![2k+1|2r]\!] \\ &\quad + 9(2r-1)(2r)\left(\frac{d}{d\delta}\right)^2[\![2k+1|2r-2]\!] \\ &\quad + 9(2k+1)(2k)\left(\frac{d}{d\delta}\right)^2[\![2k-1|2r]\!] \\ &\quad + \left(\frac{d}{d\delta}\right)^6[\![2k+1|2r]\!]\ . \end{aligned} \tag{4.19}$$

This equation can be checked as follows: Evaluate the terms in (4.19), using only the derivatives of orders $2k+1, 2k, 2k-1$ and $2r, 2r-1, 2r-2$ of the Airy functions. Higher derivatives are resolved by, e.g.,

$$\mathrm{Ai}^{(2k+2)}(\delta) = (\mathrm{Ai}'')^{(2k)}(\delta) = \partial_\delta^{2k}(\delta\,\mathrm{Ai}(\delta)) = 2k\,\mathrm{Ai}^{(2k-1)}(\delta) + \delta\,\mathrm{Ai}^{(2k)}(\delta)$$

Then a straightforward, though lengthy calculation confirms (4.19) and hence (4.16). In particular, this recursion defines polynomials $\tilde{c}_j^{k,r}$, which coincide with the $\tilde{c}_j^{k,r}$ defined by (4.13) for $k + r \geq 3j$.

The equivalence of (4.14) for $j = 0$ and $j = 1$ with (4.15) is immediate, and so is the equivalence of both lines in (4.14). The validity of the first line is proved by showing that this expression satisfies the same recursion (4.16) as $\tilde{c}_j^{k,r}$ does.

For this purpose, we start the summation from the top, i.e., we let $l = j - n$ or $l = j - 1 - n$ or $l = j - 2 - n$, in (4.14) for the terms corresponding to $\tilde{c}_j$, $\tilde{c}_{j-1}$, and $\tilde{c}_{j-2}$ respectively. Then the recursion (4.16) is equivalent to $f(j) = 0$, where

$$\begin{aligned} f(N) := \sum_{n=0}^{N} & \frac{(k+r-j+1)^{\uparrow n}(k-r+j-2-n)^{\downarrow 2j-4-2n}(-3)^{-2-n}}{(2j-2n)!n!} \times \Big\{ \\ & -18j(2j-1)(k-r+j-n)^{\downarrow 2}(k-r-j+n+1)^{\uparrow 2} \\ & -3(3k+3r+3-j)(k+r-j+n+1)(2j-2n)^{\downarrow 2}(k-r+j-1-n)(k-r-j+n+2) \\ & +9r(2r-1)(2j-2n)^{\downarrow 2}(k-r+j-n)^{\downarrow 2} \\ & +9k(2k+1)(2j-2n)^{\downarrow 2}(k-r-j+n+1)^{\uparrow 2} \\ & +\frac{1}{2}(k+r-j+n+1)^{\uparrow 2}(2j-2n)^{\downarrow 4}\Big\} \ . \end{aligned} \tag{4.20}$$

To this end, we show by induction on N that

$$\begin{aligned} f(N) = 2 & \frac{(k+r-j+1)^{\uparrow N+1}(k-r+j-2-N)^{\downarrow 2j-2N-4}(-3)^{j-N-2}}{(2j-2N-2)!\,N!} \times \\ & \Big\{-3(k-r)^{\uparrow 2}(4j-2N-3) + (2j-2N-3)^{\uparrow 2}\Big(\frac{5}{2}(k+r)+j+2N+3\Big)\Big\} \end{aligned} \tag{4.21}$$

This is a routine check, albeit a tedious one. On the other hand, given this formula, the denominator $(2j-2N-2)!$ guarantees $f(j) = 0$, as claimed. This establishes formula (4.14) and thus completes the proof of our lemma. ∎

It is one key feature that $c_j^{k,r}$ depends on (k, r) only through the combinations $m = k + r$ and $\Theta = k(1 - 2r)$. This could have been seen already from (4.16) without the explicit formula (4.14). Namely, letting $\bar{c}_j^{k,r} = \tilde{c}_j^{k,r+1/2}$, the recursion (4.16) propagates the symmetry $\bar{c}_j^{k,r} = \bar{c}_j^{r,k}$ from $j-2, j-1$ to j.

There is an apparent paradox here, which should not confuse the reader, namely the symmetry $\bar{c}_j^{k,r} = \bar{c}_j^{r,k}$ for coefficients arising from the anti-symmetric $[\![2k+1|2r]\!]$:

Equation (4.19) can be written as a recursion for $[\![M|N]\!]$ by letting $k = (M-1)/2$ and $r = N/2$, and the resulting formula is valid independent of the parity of M, N. On the other hand, if we write (4.12) in the form

$$[\![M|N]\!](\delta) = \sum_j \frac{(\frac{M+N-1}{2} - j)!}{(\frac{M+N-1}{2} - 3j)!}\,\bar{c}_j^{(M-1)/2,(N-1)/2}\,\delta^{(M+N-1)/2-3j}$$

we appear to have an equation whose left hand side is clearly antisymmetric, whereas its right hand side is symmetric with respect to $M \leftrightarrow N$.

The paradox is resolved by observing that the symmetry of $\bar{c}_j^{k,r}$ amounts to $c_j^{k,r+1/2} = c_j^{r,k+1/2}$, but equation (4.12) does not hold for half-integer k, r, unless j runs over appropriate non-integers and the c_j are appropriately defined in that case; namely, only integer powers of δ can occur in $[\![M|N]\!](\delta)$, whatever the parity of M, N is. Then, (4.16) still allows an induction, say, over $j \in \frac{1}{2} + \mathbb{N}$, and it will also propagate the appropriate antisymmetry $\bar{c}_{1/2}^{k,r} = -\bar{c}_{1/2}^{r,k}$, $\bar{c}_{3/2}^{k,r} = -\bar{c}_{3/2}^{r,k}$ from the initial values to higher values of j.

In the following section, the expression

$$H_{n,m,j} = \sum_{k=0}^{m} \sum_{l=0}^{n} \binom{2m+1}{2k+1} \binom{2n+1}{2l+1} \frac{c_j^{k,m-k}}{(2m+2n-2k-2l+1)(2k+2l+3)} \tag{4.22}$$

will play a crucial role, so we are going to evaluate it next: $H_{n,m,j}$ vanishes automatically for $m < 3j$. This follows from the genesis of $H_{n,m,j}$ back through (4.22), (4.18), (4.3). We observe empirically that in particular

$$\begin{aligned}
H_{n,m,0} &= \frac{4^{m+n+1}}{(2m+2n+3)(2m+2n+4)} \\
H_{n,m,1} &= -\frac{m(m-1)(m-2)(2m^2+(4n-3)m+(n+2)(2n-1))4^{m+n}}{6(m+n+1)(m+n+2)(2m+2n+1)(2m+2n+3)} \\
H_{n,m,2} &= \frac{m(m-1)(m-2)(m-3)(m-4)(m-5)P(n,m)4^{n+m-1}}{36(m+n)(m+n+1)(m+n+2)(2m+2n-1)(2m+2n+1)(2m+2n+3)}
\end{aligned} \tag{4.23}$$

with

$$\begin{aligned}
P(n,m) = \big(&4m^4 + (16n-20)m^3 + (24n^2-36n+71)m^2 \\
&+ (16n^3-12n^2-14n+5)m + (4n^4+4n^3-13n^2+5n-18)\big)
\end{aligned} \tag{4.24}$$

By observation, it appears that in general, we have

Lemma 4.3

$$H_{n,m,j} = (-)^j \binom{m}{3j} 4^{n+m+1} \frac{R_j(m,n)}{(2m+2n+4)(2m+2n+3)\cdots(2m+2n+3-2j)} , \tag{4.25}$$

where R_j is a polynomial of joint degree $2j$ in (m,n). Specifically, it is given by formula (4.49) below.

We are going to show this below, after giving the first special cases.

Namely,

$$R_0 = 1 , \qquad R_1 = \begin{bmatrix} 1 \\ m \\ m^2 \end{bmatrix}^T \begin{bmatrix} -2 & 3 & 2 \\ -3 & 4 & \\ 2 & & \end{bmatrix} \begin{bmatrix} 1 \\ n \\ n^2 \end{bmatrix} \tag{4.26}$$

$$R_2 = 10 \begin{bmatrix} 1 \\ m \\ m^2 \\ m^3 \\ m^4 \end{bmatrix}^T \begin{bmatrix} -18 & 5 & -13 & 4 & 4 \\ 5 & -14 & -12 & 16 & \\ 71 & -36 & 24 & & \\ -20 & 16 & & & \\ 4 & & & & \end{bmatrix} \begin{bmatrix} 1 \\ n \\ n^2 \\ n^3 \\ n^4 \end{bmatrix} \tag{4.27}$$

$$R_3 = 280 \begin{bmatrix} 1 \\ m \\ m^2 \\ m^3 \\ m^4 \\ m^5 \\ m^6 \end{bmatrix}^T \begin{bmatrix} -648 & 294 & -205 & 123 & -46 & -12 & 8 \\ 618 & -356 & 171 & 32 & -132 & 48 & \\ 2441 & -1107 & 804 & -408 & 120 & & \\ -2451 & 1328 & -552 & 160 & & & \\ 602 & -348 & 120 & & & & \\ -84 & 48 & & & & & \\ 8 & & & & & & \end{bmatrix} \begin{bmatrix} 1 \\ n \\ n^2 \\ n^3 \\ n^4 \\ n^5 \\ n^6 \end{bmatrix} \tag{4.28}$$

Proof of Lemma 4.3:
In order to prove (4.25), we repeat (4.22) and have to show for $j = 0, 1, 2, \dots$ the second equality of

$$\begin{aligned}
&(2m+2n+4)H_{n,m,j} = \\
&\quad = \sum_{k=0}^{m}\sum_{l=0}^{n} \binom{2m+1}{2k+1}\binom{2n+1}{2l+1} \frac{(2m+2n+4)c_j^{k,m-k}}{(2m+2n-2k-2l+1)(2k+2l+3)} \\
&\quad = \binom{m}{3j} \frac{(-1)^j 4^{m+n+1} R_j(m,n)}{(2m+2n+3)\cdots(2m+2n+3-2j)}
\end{aligned} \tag{4.29}$$

with $R_j(m,n)$ a polynomial of joint degree $2j$. The proof for $j = 0$ is simplest and will be the paradigm for the case of general j: $c_0^{k,r} = 1$. Using

$$\frac{2m+2n+4}{(2m+2n-2k-2l+1)(2k+2l+3)} = \frac{1}{2m+2n-2k-2l+1} + \frac{1}{2k+2l+3}$$

and the binomial theorem, we obtain:

$$\begin{aligned}
&4\sum_{k=0}^{m}\sum_{l=0}^{n} \binom{2m+1}{2k+1}\binom{2n+1}{2l+1} \frac{(2m+2n+4)\,X^{2k}Y^{2l+1}}{(2m+2n-2k-2l+1)(2k+2l+3)} = \\
&\quad = \int_0^1 \Bigg[\tau^{2m+2n+2} \frac{(1+X/\tau)^{2m+1} - (1-X/\tau)^{2m+1}}{X} \Big((1+Y/\tau)^{2n+1} - (1-Y/\tau)^{2n+1} \Big) + \\
&\qquad + \frac{(1+\tau X)^{2m+1} - (1-\tau X)^{2m+1}}{X} \Big((1+\tau Y)^{2n+1} - (1-\tau Y)^{2n+1} \Big) \Bigg]\, d\tau\,.
\end{aligned} \tag{4.30}$$

For $X = Y = 1$, the integral can be evaluated immediately, and the result is

$$2\int_0^1 \Big[(1+\tau)^{2m+2n+2} + (1-\tau)^{2m+2n+2} \Big]\, d\tau = 4\frac{2^{2m+2n+2}}{2m+2n+3}\,,$$

as was claimed, with $R_0(m,n) \equiv 1$. There are two interesting qualitative features in this simple calculation: The integrand is an even function of τ; this prevents a term $1/(2m+2n+3)$ without the power of 2 from showing up. Secondly, once we let $X = 1$, there is a symmetry under the exchange $\tau \leftrightarrow 1/\tau$; it is responsible for the cancellation of mixed terms like $(1+\tau)^{2m+1}(1-\tau)^{2n+1}$.

For general j, we continue to let $Y = 1$, but leave X still available for further manipulation. We use the particular form (4.13), (4.14) for $c_j^{k,r}$, namely

$$c_j^{k,m-k} = \frac{(m-j)!}{(m-3j)!}\, d_j\Big(k(2k+1-2m), m\Big)\,,$$

and let

$$\begin{aligned}
D_a(\tau, X) &:= \frac{(1+\tau X)^a - (1-\tau X)^a}{X}\,, \\
S_a(\tau, X) &:= (1+\tau X)^a + (1-\tau X)^a\,.
\end{aligned} \tag{4.31}$$

In the generating function in (4.30), extra coefficients $2k$ are produced by the operator $X\partial_X$. We then conclude from (4.29) and (4.30) that

$$\begin{aligned}
&4(2m+2n+4)H_{n,m,j} = \\
&\quad = \frac{(m-j)!}{(m-3j)!}\int_0^1 d_j(\mathbf{\Theta},m)\Big[D_{2n+1}(\tau,1)D_{2m+1}(\tau,X)+ \\
&\qquad\qquad + \tau^{2m+2n+2}D_{2n+1}(1/\tau,1)D_{2m+1}(1/\tau,X)\Big]\Big|_{X=1}\,d\tau
\end{aligned} \tag{4.32}$$

with

$$\mathbf{\Theta} := \frac{1}{2}X\partial_X(1-2m+X\partial_X) \tag{4.33}$$

in perfect correspondence with the Θ defined in Lemma 4.2. Define

$$\begin{aligned}
\mathcal{A}^{(t)}(X) &:= X^t\left[D_{2n+1}(\tau,1)\,D_{2(m-t)+1}(\tau,X)+\tau^{2(m-t)+1}S_{2n+1}(\tau,1)\,D_{2(m-t)+1}(1/\tau,X)\right] \\
\mathcal{A}^{(t+1)}(X) &:= X^t\left[D_{2n+1}(\tau,1)\,S_{2(m-t)-1}(\tau,X)+\tau^{2(m-t)-1}S_{2n+1}(\tau,1)\,S_{2(m-t)-1}(1/\tau,X)\right]
\end{aligned} \tag{4.34}$$

for t even.

The first of these definitions is motivated by the fact that the term under the integral (4.32) equals $d_j(\mathbf{\Theta},m)\mathcal{A}^{(0)}(X)|_{X=1}$, the second by the subsequent calculation. Namely, it is straightforward (although lengthy) to show that

$$\begin{aligned}
X\partial_X\,D_a(\tau,X) &= -D_a(\tau,X)+\tau a S_{a-1}(\tau,X) \\
X\partial_X\,S_a(\tau,X) &= \tau a X^2\,D_{a-1}(\tau,X) = a(S_a(\tau,X)-S_{a-1}(\tau,X)) \\
\mathbf{\Theta}\circ X^t &= X^t\circ\frac{1}{2}(X\partial_X+t)(X\partial_X+t+1-2m)
\end{aligned} \tag{4.35}$$

and therefore for both even and odd t separately:

$$2\mathbf{\Theta}\mathcal{A}^{(t)}(X) = (t-2m)(t-1)\mathcal{A}^{(t)}(X)-\tau(2m-2t)(2m-2t+1)\mathcal{A}^{(t+1)}(X)\,. \tag{4.36}$$

The preservation of the symmetry $\tau\leftrightarrow 1/\tau$ at $X=1$ under all differentiations is a key feature, which rests upon the fact that $X\partial_X$ only enters through the combination $\mathbf{\Theta}$. The inclusion of the factors X^t in (4.34) is a fine tuning that prevents the buildup of terms violating that symmetry at each invocation of $\mathbf{\Theta}$.

In view of (4.14), we enhance (4.36): it easily follows

$$\begin{aligned}
&[2\mathbf{\Theta}+(m-s)(m+s-1)]\,\mathcal{A}^{(t)} = \\
&\qquad = (m-t+s)(m-t-s+1)\mathcal{A}^{(t)}-\tau(2m+1-2t)(2m-2t)\mathcal{A}^{(t+1)}\,.
\end{aligned} \tag{4.37}$$

Therefore

$$\prod_{s=1}^{l}\left[2\mathbf{\Theta}+(m-s)(m+s-1)\right]\mathcal{A}^{(0)} = \sum_{t=0}^{l} g_t^l\,(-\tau)^t\mathcal{A}^{(t)} \tag{4.38}$$

with $g_0^0=1$ and

$$g_t^l = (m-t+l)(m-t-l+1)g_t^{l-1}+(2m+3-2t)(2m+2-2t)g_{t-1}^{l-1}\,, \tag{4.39}$$

from which we conclude by an easy induction that

$$g_t^l = \binom{l}{t}(2m+1)^{\downarrow 2t}(m-t)^{\downarrow l-t}(m+1)^{\uparrow l-t}\,. \tag{4.40}$$

Finally, we need to evaluate $\int_0^1 (-\tau)^t\, \mathcal{A}^{(t)}(1)\, d\tau$. One easily calculates

$$\begin{array}{ll} \mathcal{A}^{(t)}(1) = 2S_{B-2t}(1) & \text{for } t \text{ even} \\ \mathcal{A}^{(t)}(1) = 2D_{B-2t}(1) & \text{for } t \text{ odd} \end{array} \quad , \text{ with } B := 2m+2n+2\,. \tag{4.41}$$

Therefore, using $-\tau = [(1-\tau)-(1+\tau)]/2$,

$$\begin{aligned} &\int_0^1 (-\tau)^t\, \mathcal{A}^{(t)}(1)\, d\tau = \\ &= 2^{1-t} \int_0^1 \sum_{\kappa=0}^{t} \binom{t}{\kappa} (-1)^{t-\kappa} (1-\tau)^\kappa (1+\tau)^{t-\kappa} \Big((1+\tau)^{B-2t} \pm (1-\tau)^{B-2t} \Big)\, d\tau \\ &= 2^{1-t} \sum_{\kappa=0}^{t} \binom{t}{\kappa} (-1)^{t-\kappa} \int_0^1 \Big((1-\tau)^\kappa (1+\tau)^{B-t-\kappa} \pm (1-\tau)^{B-2t+\kappa} (1+\tau)^{t-\kappa} \Big)\, d\tau \\ &= 2^{1-t} \sum_{\kappa=0}^{t} \binom{t}{\kappa} (-1)^{t-\kappa} \int_0^1 \Big((1-\tau)^\kappa (1+\tau)^{B-t-\kappa} + (1+\tau)^\kappa (1-\tau)^{B-t-\kappa} \Big)\, d\tau\,, \end{aligned} \tag{4.42}$$

where in the last line, we have changed the index of summation from κ to $t-\kappa$ for the second term under the integral.

Using

$$\int_0^1 \Big[(1+\tau)^M (1-\tau)^\kappa + (1-\tau)^M (1+\tau)^\kappa \Big]\, d\tau = \frac{\kappa!\, 2^{M+\kappa+1}}{(M+\kappa+1)(M+\kappa)\cdots(M+1)} \tag{4.43}$$

and combining (4.32), (4.14), (4.38), (4.40), (4.42) and (4.43), we obtain

$$\begin{aligned} 4(2m+2n+4) H_{n,m,j} &= \frac{(m-j)!}{(m-3j)!} 2^{2m+2n+4} \times \\ &\times \sum_{l=0}^{j} \sum_{t=0}^{l} \sum_{\kappa=0}^{t} \binom{m-l}{j-l} \binom{l}{t} \binom{t}{\kappa} \kappa!\, \frac{(-3)^{l-j}}{(2l)!} 2^{-2t} (-1)^{t-\kappa} \times \\ &\times \frac{(2m+1)^{\downarrow 2t} (m-t)^{\downarrow l-t} (m+1)^{\uparrow l-t}}{(2m+2n-t+3)(2m+2n-t+2)\cdots(2m+2n-t-\kappa+3)}\,, \end{aligned} \tag{4.44}$$

or, equivalently

$$\begin{aligned} H_{n,m,j} = &\frac{m!}{(m-3j)!} \frac{4^{m+n+1}}{(2m+2n+4)(2m+2n+3)\cdots(2m+2n-2j+3)} \times \\ &\times \sum_{l=0}^{j} \sum_{t=0}^{l} \sum_{\kappa=0}^{t} \frac{l!}{(j-l)!\,(l-t)!\,(t-\kappa)!} \frac{(-3)^{l-j}}{(2l)!} (-1)^{t-\kappa} \times \\ &\times (m+1/2)^{\downarrow t} (m+1)^{\uparrow l-t} (2m+2n+3)^{\downarrow t} (2m+2n+3-2j)^{\uparrow 2j-t-\kappa}\,. \end{aligned} \tag{4.45}$$

This corresponds to (4.25), with the second two lines corresponding to $(-1)^j R_j(m,n)/(3j)!$. It is clear from (4.45), and could have been obtained with less than the explicit formula, that R_j is of degree $3j$ in (m,n), and of degree $2j$ in n alone. However, we want to argue that many terms cancel and the degree is only $2j$ in both variables together. It is the summation over t which brings about this effect. Therefore, using $s := t-\kappa$, $\lambda := l-\kappa$, we write (4.45) in the form

$$\begin{aligned} \frac{(-1)^j}{(3j)!} R_j(m,n) = &\sum_{l=0}^{j} \sum_{\kappa=0}^{l} \frac{l!(-3)^{l-j}}{(2l)!(j-l)!(l-\kappa)!} (m+1/2)^{\downarrow\kappa} (B+1)^{\downarrow\kappa} (B+1-2j)^{\uparrow 2j-\kappa-l} \times \\ &\times \sum_{s=0}^{\lambda} (-1)^s \binom{\lambda}{s} (m+1/2-\kappa)^{\downarrow s} (m+1)^{\uparrow\lambda-s} (B+1-\kappa)^{\downarrow s} (B+1-2\kappa-\lambda)^{\uparrow\lambda-s}\,. \end{aligned} \tag{4.46}$$

The coefficient of $m^\lambda B^\lambda$ in the product under the last sum equals 1, so this coefficient cancels, because $\sum_{s=0}^{\lambda}(-1)^s\binom{\lambda}{s} = 0$. In order to treat other coefficients similarly, we use the following reasonably well known lemma from the calculus of finite differences:

Lemma 4.4 *Given a sequence $x(s)$, $s = 0, 1, 2\ldots$, let $x^{(0)}(s) := x(s)$ and let $x^{(i+1)}(s) := x^{(i)}(s+1) - x^{(i)}(s)$ the sequences of successive forward differences. Then $x(s) = \sum_{r=0}^{*}\binom{s}{r}x^{(r)}(0)$.*

With this lemma, we obtain easily

$$\begin{aligned}(m+1/2-\kappa)^{\downarrow s}(m+1)^{\uparrow\lambda-s} &= \sum_{r=0}^{*}\binom{s}{r}(m+1)^{\uparrow\lambda-r}(1/2-\kappa-\lambda)^{\uparrow r} ,\\ (B+1-\kappa)^{\downarrow s}(B+1-2\kappa-\lambda)^{\uparrow\lambda-s} &= \sum_{\rho=0}^{*}\binom{s}{\rho}(B+1-2\kappa-\lambda)^{\uparrow\lambda-\rho}(1+\kappa)^{\uparrow\rho} .\end{aligned}$$

The sum over s can now be evaluated, and we obtain

$$\sum_{s=0}^{\lambda}(-1)^s\binom{\lambda}{s}\binom{s}{r}\binom{s}{\rho} = \frac{(-1)^\lambda\lambda!}{(\lambda-r)!\,(\lambda-\rho)!\,(\rho+r-\lambda)!} , \tag{4.47}$$

as can be easily seen by evaluating the identity

$$\left(\frac{\partial}{\partial x}\right)^r\left(\frac{\partial}{\partial y}\right)^\rho\Big|_{x=y=1}\sum_{s=0}^{\lambda}(-1)^s\binom{\lambda}{s}x^sy^s = \left(\frac{\partial}{\partial x}\right)^r\left(\frac{\partial}{\partial y}\right)^\rho\Big|_{x=y=1}(-1)^\lambda(xy-1)^\lambda .$$

This turns (4.46) into

$$\begin{aligned}\frac{(-1)^j}{(3j)!}R_j(m,n) &= \sum_{l=0}^{j}\sum_{\kappa=0}^{l}\frac{l!(-3)^{l-j}(-1)^{l-\kappa}}{(2l)!(j-l)!}(m+1/2)^{\downarrow\kappa}(B+1)^{\downarrow\kappa}(B+1-2j)^{\uparrow 2j-\kappa-l}\times\\ &\times\sum_{r=0}^{l-\kappa}\sum_{\rho=0}^{l-\kappa}\frac{(m+1)^{\uparrow l-\kappa-r}(1/2-l)^{\uparrow r}(B+1-\kappa-l)^{\uparrow l-\kappa-\rho}(1+\kappa)^{\uparrow\rho}}{(l-\kappa-r)!\,(l-\kappa-\rho)!\,(\rho+r-l+\kappa)!} .\end{aligned} \tag{4.48}$$

The "binomial" theorem $\sum_{\rho=0}^{r}\binom{n}{\rho}x^{\uparrow\rho}y^{\uparrow n-\rho} = (x+y)^{\uparrow n}$ permits to evaluate the sum over ρ in (4.48). We use that $l-\kappa \geq r$ and get our final result

$$\begin{aligned}\frac{3^j}{(3j)!}R_j(m,n) &= \sum_{l=0}^{j}\sum_{\kappa=0}^{l}\sum_{r=0}^{l-\kappa}\frac{l!\,3^l\,(-1)^\kappa}{(2l)!\,(j-l)!\,r!\,(l-\kappa-r)!}(1/2-l)^{\uparrow r}(1+\kappa)^{\uparrow l-\kappa-r}\times\\ &\times(m+1)^{\uparrow l-\kappa-r}(m+1/2)^{\downarrow\kappa}\,(2m+2n+3)^{\downarrow\kappa+r}(2m+2n+3-2j)^{\uparrow 2j-\kappa-l} .\end{aligned} \tag{4.49}$$

This ends the proof of Lemma 4.3. ■

Actually, this formula can be processed further: exchange the inner two sums and notice that the sum $\sum_{\kappa=0}^{l-r}$ produces again cancellations (which, however, will not reduce the total degree this time, but only simplify the formula a bit). Treating these cancellations as before by means of lemma 4.4 and applying the "binomial" theorem to the terms containing $B = 2m+2n+2$, and finally renaming the index of summation l into $j-l$, we come up with

$$\begin{aligned}(-1)^jR_j(m,n) = &\sum\sum\sum_{\substack{l,r,\nu\geq 0\\ l+r+\nu\leq j}}\frac{3^{-l}(-1)^{\nu+l}(3j)!}{4^{r+\nu}r!\,l!}\binom{2j-2l-2r-\nu}{\nu}\times\\ &\times(2m+2n+3)^{\downarrow r+\nu}(2m+2n+3-2j)^{\uparrow 2l+r}(m+1)^{\uparrow j-l-r-\nu} .\end{aligned} \tag{4.50}$$

Lemma 4.3 permits us to evaluate the sums over m,n in (4.53) rather explicitly in terms of trigonometric functions and rationals of $2\sqrt{\delta}t$ and hyperbolic functions and rationals of $8\varepsilon t^3$, and integrals of such. Even though the resulting formula for the integrals remains with a rather high complexity, the special structure of its dependence on (δ, t, ε) allows to control their behaviour. For many purposes, (4.53) can be treated as if it were a power series about $\delta = \infty$. The dependence of the coefficients of this "power series" on δ is tame enough for cutoff errors to be determined by the grading of the powers, but not influenced by the coefficients. The fact that not all of the grading given by δ^{-3j} is eaten up by the coefficients, is caused by the cancellation stated implicitly in the degree of R_j according to Lemma 4.3, and detailed out on the way from (4.45) to (4.49).

4.2 A Series Form of the Spectral Condition

In order to evaluate (2.1), i.e.,

$$\Delta(\delta,t,\varepsilon) := \begin{vmatrix} \int_{-1}^{1} \mathrm{Ai}(\delta+itz)\sinh(4\varepsilon t^3 z)\,dz & \int_{-1}^{1} \mathrm{Ai}(\delta+itz)\cosh(4\varepsilon t^3 z)\,dz \\ \int_{-1}^{1} \mathrm{Bi}(\delta+itz)\sinh(4\varepsilon t^3 z)\,dz & \int_{-1}^{1} \mathrm{Bi}(\delta+itz)\cosh(4\varepsilon t^3 z)\,dz \end{vmatrix} \tag{4.51}$$

we expand the integrands into Taylor series and integrate term by term:

$$\begin{aligned} \mathrm{Ai}(\delta+itz) &= \sum_{k=0}^{\infty} \mathrm{Ai}^{(2k+1)}(\delta)\,\frac{(itz)^{2k+1}}{(2k+1)!} + \sum_{r=0}^{\infty} \mathrm{Ai}^{(2r)}(\delta)\,\frac{(itz)^{2r}}{(2r)!} \\ \mathrm{Bi}(\delta+itz) &= \sum_{k=0}^{\infty} \mathrm{Bi}^{(2k+1)}(\delta)\,\frac{(itz)^{2k+1}}{(2k+1)!} + \sum_{r=0}^{\infty} \mathrm{Bi}^{(2r)}(\delta)\,\frac{(itz)^{2r}}{(2r)!} \\ \sinh(4\varepsilon t^3 z) &= \sum_{l=0}^{\infty} \frac{(4\varepsilon t^3 z)^{2l+1}}{(2l+1)!} \\ \cosh(4\varepsilon t^3 z) &= \sum_{s=0}^{\infty} \frac{(4\varepsilon t^3 z)^{2s}}{(2s)!} \end{aligned} \tag{4.52}$$

Using the higher order Wronskians (4.10), we get

$$\begin{aligned} &\frac{1}{4it\,4\varepsilon t^3}\Delta(\delta,t,\varepsilon) = \\ &\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-t^2)^{k+r}\,(16\varepsilon^2 t^6)^{l+s}\,[\![2k+1|2r]\!](\delta)}{(2k+1)!\,(2l+1)!\,(2r)!\,(2s)!\,(2r+2s+1)\,(2k+2l+3)} \\ &= W \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\lfloor m/3\rfloor} H_{n,m,j}\,\frac{(-\delta t^2)^m\,(16\varepsilon^2 t^6)^n}{(2m+1)!\,(2n+1)!}\,\delta^{-3j} \end{aligned} \tag{4.53}$$

(where we have used (4.12) and taken $m = k+r$, $n = l+s$) with

$$H_{n,m,j} = \sum_{k=0}^{m}\sum_{l=0}^{n} \binom{2m+1}{2k+1}\binom{2n+1}{2l+1} \frac{c_j^{k,m-k}}{(2m+2n-2k-2l+1)\,(2k+2l+3)}$$

as given in (4.22) above. Actually, the sum over j could be extended from 0 to ∞, because $H_{n,m,j} = 0$ for $m < 3j$.

Plug (4.25) into (4.53) and let $A := 64\varepsilon^2 t^6$ and $x = 4\delta t^2$. Moreover employ a partial fraction decomposition for the denominator of $H_{n,m,j}$:

$$\frac{1}{(2m+2n+4)\cdots(2m+2n+3-2j)} = \frac{-1}{(2j+1)!}\sum_{s=0}^{2j+1}(-1)^s\binom{2j+1}{s}\frac{1}{(2m+2n+4-s)}\,. \tag{4.54}$$

Then we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} H_{n,m,j}\,\frac{(-\delta t^2)^m\,(16\varepsilon^2 t^6)^n}{(2m+1)!\,(2n+1)!} = \\
&\quad = \frac{4(-1)^{j+1}}{(2j+1)!}\sum_{s=0}^{2j+1}(-1)^s\binom{2j+1}{s}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-x)^m}{(2m+1)!}\,\frac{A^n}{(2n+1)!}\binom{m}{3j}\frac{R_j(m,n)}{2m+2n+4-s} \\
&\quad = \frac{4(-1)^{j+1}}{(2j+1)!}\sum_{s=0}^{2j+1}(-1)^s\binom{2j+1}{s}\int_0^1 R_j(x\partial_x, A\partial_A)\binom{x\partial_x}{3j}\tau^{3-s}\frac{\sin(\tau\sqrt{x})}{\tau\sqrt{x}}\,\frac{\sinh(\tau\sqrt{A})}{\tau\sqrt{A}}\,d\tau \\
&\quad = \frac{4(-1)^j}{(2j+1)!}\int_0^1(1-\tau)^{2j+1}\tau^{2-2j}\,R_j(x\partial_x, A\partial_A)\binom{x\partial_x}{3j}\frac{\sin(\tau\sqrt{x})}{\tau\sqrt{x}}\,\frac{\sinh(\tau\sqrt{A})}{\tau\sqrt{A}}\,d\tau\ .
\end{aligned} \tag{4.55}$$

The convergence of the integrals in (4.55) near $\tau = 0$ is not manifest, but follows from the genesis of these integrals. It is obvious that for $j = 0$, the integral can be evaluated elementarily. But we will see that for $j \geq 1$, too, the powers of τ in the denominator are compensated by explicit powers of τ, not merely by cancellation of leading terms of the power series of trigonometric and hyperbolic functions. Therefore, the integrals can be evaluated in closed form for all j, and we will give such a closed form. We will pretend that in (4.55), the term $\sin(\tau\sqrt{x})\sinh(\tau\sqrt{A})/(\tau\sqrt{x}\,\tau\sqrt{A})$ is replaced by $\exp(\tau\sqrt{-x}+\tau\sqrt{A})/(\tau\sqrt{-x}\,\tau\sqrt{A})$. The true result can be reconstructed in the very end by projecting onto the subspace of functions which are even in both $\sqrt{-x}$ and $\sqrt{A}$.

To carry this evaluation through, let $u := \tau\sqrt{-x}$, $\boldsymbol{m} := x\partial_x = \frac{1}{2}u\partial_u$, and consider the vector space $\mathcal{U}$ spanned by the functions U_k given below. Let $\bar{\mathcal{U}}$ consist of the corresponding formal infinite sums (i.e., sequences). We use the symbol $\sum\!\!\!\!\mathfrak{f}$ for formal sums.

$$\begin{aligned}
&U_k(u) := u^{k-1}\exp u \qquad (k\geq 0) \\
&\mathcal{U} := \bigoplus_{k=0}^{\infty}\mathbb{R}\cdot U_k \qquad \hookrightarrow \qquad \bar{\mathcal{U}} := \prod_{k=0}^{\infty}\mathbb{R}\cdot U_k\ .
\end{aligned} \tag{4.56}$$

The operator $\boldsymbol{m}$ operates on $\mathcal{U}, \bar{\mathcal{U}}$ by

$$\boldsymbol{m}\cdot U_k = \frac{1}{2}\left[(k-1)U_k + U_{k+1}\right] \tag{4.57}$$

Similarly, letting $v := \tau\sqrt{A}$, $\boldsymbol{n} := A\partial_A = \frac{1}{2}v\partial_v$ and

$$\begin{aligned}
&V_k(v) := v^{k-1}\exp v \qquad (k\geq 0) \\
&\mathcal{V} := \bigoplus_{k=0}^{\infty}\mathbb{R}\cdot V_k \qquad \hookrightarrow \qquad \bar{\mathcal{V}} := \prod_{k=0}^{\infty}\mathbb{R}\cdot V_k\ ,
\end{aligned} \tag{4.58}$$

the operator $\boldsymbol{n}$ operates on $\mathcal{V}$, $\bar{\mathcal{V}}$ by

$$\boldsymbol{n}\cdot V_k = \frac{1}{2}\Big[(k-1)V_k + V_{k+1}\Big]\ . \tag{4.59}$$

We denote the functions U_k and V_k differently (although they are defined by the same formulas), because they live in different (albeit isomorphic) spaces $\mathcal{U}$, $\mathcal{V}$, in whose tensor product the calculation below lives. We are not discussing a topology of these spaces, nor are we relying on results of functional analysis here, but we have an ad-hoc version of the spectral theorem for the situation at hand, proved by explicit calculations:

Theorem 4.5 *Both $\boldsymbol{m}$ and $\boldsymbol{n}$ have simple eigenvalues $\lambda_i = (i-1)/2$, $i = 0, 1, 2\ldots$, with corresponding eigenvectors $e_i \in \bar{\mathcal{U}}$ and $f_i \in \bar{\mathcal{V}}$ respectively, given by*

$$e_i = \sum\!\!\!\!\mathfrak{f}_{a=i}^{\infty}(-1)^a\frac{U_a}{(a-i)!}\ ,\quad \boldsymbol{m}\cdot e_i = \lambda_i e_i \quad ; \qquad f_i = \sum\!\!\!\!\mathfrak{f}_{b=i}^{\infty}(-1)^b\frac{V_b}{(b-i)!}\ ,\quad \boldsymbol{n}\cdot f_i = \lambda_i f_i \tag{4.60}$$

The basis vectors U_k or V_k can be expressed in terms of the eigenvectors by

$$U_k = \sum_{i=k}^{\infty} (-1)^i \frac{e_i}{(i-k)!} \quad ; \quad V_k = \sum_{i=k}^{\infty} (-1)^i \frac{f_i}{(i-k)!} \tag{4.61}$$

Consequently,

$$\boldsymbol{m}^\nu \cdot U_0 = \sum_{a=0}^{\nu} \left[\sum_{i=0}^{a} (-1)^i \binom{a}{i} \lambda_i^\nu \right] (-1)^a \frac{U_a}{a!} \quad ; \quad \boldsymbol{n}^\nu \cdot V_0 = \sum_{b=0}^{\nu} \left[\sum_{i=0}^{b} (-1)^i \binom{b}{i} \lambda_i^\nu \right] (-1)^b \frac{V_b}{b!} \tag{4.62}$$

$$\binom{\boldsymbol{m}}{3j} R_j(\boldsymbol{m}, \boldsymbol{n}) \cdot U_0 \otimes V_0 = \sum_{\substack{a,b \geq 0 \\ 2j \leq a+b \leq 5j}} \sum_{i=0}^{a} \sum_{h=0}^{b} \frac{(-1)^{i+h+a+b}}{a!\, b!} \binom{a}{i} \binom{b}{h} \binom{\lambda_i}{3j} R_j(\lambda_i, \lambda_h) U_a \otimes V_b \tag{4.63}$$

In particular, $\binom{\boldsymbol{m}}{3j} R_j(\boldsymbol{m}, \boldsymbol{n}) \cdot U_0 \otimes V_0$ *is a linear combination of those* $U_a \otimes V_b$ *for which* $2j \leq a + b \leq 5j$.

All factors τ in the denominator of the integrand of the last line of (4.55) cancel explicitly, and this integral can be evaluated in terms of elementary functions.

Of course, the sums in (4.60) converge: $e_i = (-1)^i u^{i-1}$; but this fact is of no interest here.

Proof:
Checking (4.60) and (4.61) is straightforward; the sums that occur upon multiplications of the infinite matrices implicit in (4.60) and (4.61) are finite, because these matrices are upper triangular. The straightforward evaluation of (4.62) leads to $\sum_{l=0}^{\infty}$ instead of $\sum_{l=0}^{\nu}$, however. The fact that λ_i depends linearly on its index i is responsible that the terms in brackets vanish for $l > \nu$, as can again be seen by differentiating the binomial theorem. Or else, the vanishing is clear a-priori, as a consequence of (4.57), (4.59). Finally, (4.63) is immediate from (4.62), except that one first obtains the summation over $0 \leq a + b \leq 5j$ instead of $2j \leq a + b \leq 5j$. This missing detail is our following lemma: ∎

Lemma 4.6 *With* $\lambda_i = (i-1)/2$, *it holds* $\binom{\lambda_i}{3j} R_j(\lambda_i, \lambda_h) = 0$ *for for nonnegative integers* i, h *satisfying* $i + h < 2j$.

Proof:
The claim amounts to $R_j(m, n) = 0$ for $m \in \{\frac{-1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots j - \frac{3}{2}\}$, $n \in \{\frac{-1}{2}, 0, \frac{1}{2}, 1, \ldots j-1\}$, $m + n \leq j - \frac{3}{2}$. The case of integer m (odd i) is obviously dealt with by the binomial coefficient.

This can be seen most easily from (4.46); remember that there, $B + 1 = 2m + 2n + 3$, so we have here $B + 1 \leq 2j$.

Let now $m + \frac{1}{2} \in \{0, 1, \ldots, j-1\}$. Then the term $(m + \frac{1}{2})^{\downarrow\kappa}$ vanishes, unless $\kappa \leq m + \frac{1}{2}$ $(< j)$. But then, the term $(B + 1 - 2j)^{\uparrow 2j-\kappa-l}$ is not the empty product, starts with $B + 1 - 2j \leq 0$, and ends with a factor $B - \kappa - l \geq 2m + (-1) + 2 - \kappa - l \geq \kappa - l \geq 0$, i.e., this term vanishes. We have therefore proved the lemma. ∎

We will now show, based on (4.53), (4.55), Lemma 4.6 and the integration formula

$$\int_0^1 \tau^N \exp[\tau q]\, d\tau = \left(\frac{d}{dq}\right)^N \int_0^1 \exp[\tau q]\, d\tau = \frac{(-1)^{N+1} N!}{q^{N+1}} + (\exp q) \sum_{\sigma=0}^{N} \frac{(-1)^\sigma N^{\downarrow\sigma}}{q^{\sigma+1}} \tag{4.64}$$

that it holds:

Theorem 4.7 *Let $x := 4\delta t^2$ and $A := 64\varepsilon^2 t^6$. Then we have the locally (in $(\mathbb{C}\backslash\{0\})\times\mathbb{R}\times[0,\infty[\ni(\delta,t,\varepsilon)$) uniformly absolutely convergent series expansion*

$$\frac{\Delta(\delta,t,\varepsilon)}{16Wi\varepsilon t^4} = \sum_{j=0}^{\infty} \frac{\delta^{-3j}}{(A+x)^{4j+2}} \Big[P_j^{00}(x,A) + P_j^{\rm cc}(x,A)\cos\sqrt{x}\,\cosh\sqrt{A} + \\ + P_j^{\rm cs}(x,A)\cos\sqrt{x}\,\frac{\sinh\sqrt{A}}{\sqrt{A}} + P_j^{\rm sc}(x,A)\frac{\sin\sqrt{x}}{\sqrt{x}}\,\cosh\sqrt{A} + P_j^{\rm ss}(x,A)\frac{\sin\sqrt{x}}{\sqrt{x}}\,\frac{\sinh\sqrt{A}}{\sqrt{A}}\Big] \tag{4.65}$$

with polynomials P_j^{**} consisting of monomials of which the joint degrees with respect to (x,A) are within the range given by the table:

deg of mon's in	P_j^{00}	$P_j^{\rm cc}$	$P_j^{\rm cs}$	$P_j^{\rm sc}$	$P_j^{\rm ss}$
...from	$4j$	$4j$	$4j+1$	$4j+1$	$4j+1$
...to	$5j$	$5j+\lfloor j/2\rfloor$	$5j+\lfloor (j+1)/2\rfloor$	$5j+\lfloor (j+1)/2\rfloor$	$5j+\lfloor j/2\rfloor+1$
deg wrt $A\le$	$2j$	$4j$	$4j+1$	$4j$	$4j+1$

The apparent pole at $x=-A$ is compensated, in each order j, by a zero of order at least $4j+2$ of the term in brackets.

Note: P_j^{00} appears to have an explicit factor x^{3j}. We do not use or prove this observation. Apparently, this occurs for those parts of the sum (4.73) below for which (l,s) is fixed, but does not occur for smaller parts where some of the summations over (r,ν,c,h) are not yet carried out. As a consequence, the degree of P_j^{00} with respect to A would be only $2j$. We do prove this weaker result independently. It also appears (but we do not prove) that $P_j^{\rm cc}$ and $P_j^{\rm sc}$ have an explicit factor x, $P_j^{\rm sc}$ and $P_j^{\rm cs}$ a factor $A+x$.

Before going into the proof, a few remarks are appropriate: It seems enticing to take the series apart into five series from which coefficients like $\cos\sqrt{x}\cosh\sqrt{A}$ can be pulled out. We have no proof that this is permissible, but strong empirical evidence to the opposite. In any case, uniformity of the convergence near $x=-A$ would have to fail under such a process, because the single P_j^{**} do not have appropriate factors to cancel the powers of $A+x$.

There may be some hope that the series converges uniformly even as $x\to\infty$, provided factors $\cosh\operatorname{Im}\sqrt{x}\cosh\sqrt{A}$ are pulled out and a power of $x^{3j/2}$ is taken out of the bracket to join the δ^{-3j} (i.e. to produce a power series in $8t^3\delta^{-3/2}$). We have not decided whether this is actually the case.

Let us also spell out the first few terms of (4.65) explicitly, using $w := \sqrt{x} = 2t\sqrt{\delta}$ and $a :=$

$\sqrt{A} = 8\varepsilon t^3$: (they follow from the proof below)

$$
\begin{aligned}
&\frac{\Delta(\delta,t,\varepsilon)}{16Wi\varepsilon t^4} = \\
&\quad = \frac{4}{(a^2+w^2)^2}\left[2(1-\cos w\cosh a) + (a^2-w^2)\frac{\sin w}{w}\,\frac{\sinh a}{a}\right] + \\
&\quad + \frac{-4\delta^{-3}}{6(a^2+w^2)^6}\Bigg[3w^6\left(48a^2+3a^4-32w^2+6a^2w^2+3w^4\right) \\
&\qquad - \frac{3}{8}w^2\left(a^8+4a^6w^2+384a^2w^4+14a^4w^4-256w^6+20a^2w^6+9w^8\right)\cos w\cosh a \\
&\qquad - \frac{w^2(a^2+w^2)}{8}\begin{pmatrix}-3a^6-9a^4w^2+a^6w^2+135a^2w^4+3a^4w^4\\ -243w^6+3a^2w^6+w^8\end{pmatrix}\frac{\sin w}{w}\cosh a \\
&\qquad - \frac{a^2+w^2}{16}\begin{pmatrix}-15a^8-66a^6w^2+a^8w^2-60a^4w^4+2a^6w^4\\ -654a^2w^6+123w^8-2a^2w^8-w^{10}\end{pmatrix}\cos w\,\frac{\sinh a}{a} \\
&\qquad + \frac{3}{16}\begin{pmatrix}-5a^{10}-27a^8w^2+2a^{10}w^2-42a^6w^4+10a^8w^4+210a^4w^6\\ +28a^6w^6-945a^2w^8+28a^4w^8+105w^{10}+2a^2w^{10}-6w^{12}\end{pmatrix}\frac{\sin w}{w}\,\frac{\sinh a}{a}\Bigg] + \\
&\quad + \frac{4\delta^{-6}}{120(a^2+w^2)^{10}}[\ldots]
\end{aligned}
\tag{4.66}
$$

Proof of Theorem 4.7:
Using (4.53) and (4.55), we conclude

$$
\begin{aligned}
\frac{\Delta(\delta,t,\varepsilon)}{16Wi\varepsilon\, t^4} &= \sum_{j=0}^{\infty}\delta^{-3j}\,\frac{4(-1)^j}{(2j+1)!}\times \\
&\times\int_0^1 (1-\tau)^{2j+1}\tau^{2-2j}\,R_j(x\partial_x, A\partial_A)\binom{x\partial_x}{3j}\frac{\sin(\tau\sqrt{x})}{\tau\sqrt{x}}\,\frac{\sinh(\tau\sqrt{A})}{\tau\sqrt{A}}\,d\tau\ .
\end{aligned}
\tag{4.67}
$$

The (locally uniformly absolute) convergence of this sum is clear, because it arises from a power series convergent in $\mathbb{C}^3$ by carrying out the inner two summations.

Instead of the trigonometric and hyperbolic functions in this formula, we first use exponentials, and in the end we will recover the trigonometric and hyperbolic functions by projecting on the even subspace. We obtain a horrible 9-fold summation:

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} H_{n,m,j}\,\frac{(-\delta t^2)^m\,(16\varepsilon^2 t^6)^n}{(2m+1)!\,(2n+1)!} = \\
&\stackrel{(4.55)}{=} \frac{4(-1)^j}{(2j+1)!}\int_0^1 (1-\tau)^{2j+1}\tau^{2-2j}\,R_j(x\partial_x, A\partial_A)\binom{x\partial_x}{3j}\frac{\exp[\tau(\sqrt{A}+\sqrt{-x})]}{\tau^2\sqrt{-x}\sqrt{A}}\,d\tau = \\
&\stackrel{(4.63)}{=} \frac{4(-1)^j}{(2j+1)!}\sum_{\substack{a,b\geq 0\\ 2j\leq a+b\leq 5j}}\sum_{i=0}^{a}\sum_{h=0}^{b}\frac{(-1)^{i+h+a+b}}{a!\,b!}\binom{a}{i}\binom{b}{h}\binom{(i-1)/2}{3j}R_j\Big(\frac{i-1}{2},\frac{h-1}{2}\Big)\times \\
&\quad\times\int_0^1 (1-\tau)^{2j+1}\tau^{2-2j}(\tau\sqrt{-x})^{a-1}(\tau\sqrt{A})^{b-1}\exp[\tau(\sqrt{-x}+\sqrt{A})]\,d\tau = \\
&\stackrel{(4.50)}{=} \frac{4(3j)!}{(2j+1)!}\sum_{\substack{l,r,\nu\geq 0\\ l+r+\nu\leq j}}\frac{3^{-l}(-1)^{\nu+l}}{4^{r+\nu}r!\,l!}\binom{2j-2l-2r-\nu}{\nu}\sum_{\substack{a,b\geq 0\\ 2j\leq a+b\leq 5j}}\sum_{i=0}^{a}\sum_{h=0}^{b}\frac{(-1)^{i+h+a+b}}{a!\,b!}\times \\
&\quad\times\binom{a}{i}\binom{b}{h}\binom{(i-1)/2}{3j}(i+h+1)^{\downarrow r+\nu}(i+h+1-2j)^{\uparrow 2l+r}\left(\frac{i+1}{2}\right)^{\uparrow j-l-r-\nu}\times \\
&\quad\times\int_0^1 (1-\tau)^{2j+1}\tau^{2-2j}(\tau\sqrt{-x})^{a-1}(\tau\sqrt{A})^{b-1}\exp[\tau(\sqrt{-x}+\sqrt{A})]\,d\tau = \\
&\stackrel{(4.64)}{=} \frac{4(3j)!}{(2j+1)!}\sum_{\substack{l,r,\nu\geq 0\\ l+r+\nu\leq j}}\frac{3^{-l}(-1)^{\nu+l}}{4^{r+\nu}r!\,l!}\binom{2j-2l-2r-\nu}{\nu}\sum_{\substack{a,b\geq 0\\ 2j\leq a+b\leq 5j}}\sum_{i=0}^{a}\sum_{h=0}^{b}\frac{(-1)^{i+h+a+b}}{a!\,b!}\times \\
&\quad\times\binom{a}{i}\binom{b}{h}\binom{(i-1)/2}{3j}(i+h+1)^{\downarrow r+\nu}(i+h+1-2j)^{\uparrow 2l+r}\left(\frac{i+1}{2}\right)^{\uparrow j-l-r-\nu}\times \\
&\quad\times\sum_{s=0}^{2j+1}(-1)^s\binom{2j+1}{s}\left(\sqrt{-x}\right)^{a-1}\left(\sqrt{A}\right)^{b-1}\times \\
&\quad\times\left(\sum_{\sigma=0}^{s+a+b-2j}(-1)^\sigma\binom{s+a+b-2j}{\sigma}\frac{\sigma!\,\exp[\sqrt{-x}+\sqrt{A}]}{(\sqrt{-x}+\sqrt{A})^{\sigma+1}}+\mathrm{TWOE}\right).
\end{aligned}
\tag{4.68}
$$

Here, TWOE stands for "term without exponential", and we do not carry it through explicitly, because it just arises by omitting the summation over σ in favor of only the largest σ in that sum and replacing the exponential with -1.

The first simplification to be made is to carry out the summation over s, based on the formula

$$
\sum_{s=0}^{2j+1}(-1)^s\binom{2j+1}{s}\binom{s+a+b-2j}{\sigma} = -\binom{a+b-2j}{\sigma-2j-1}, \tag{4.69}
$$

which arises from evaluating $\partial_X^\sigma X^C\sum_{s=0}^{n}(-1)^s\binom{n}{s}X^s = \partial_X^\sigma X^C(1-X)^n$ at $n=2j+1$, $C=a+b-2j$. Next, it is convenient to let $c := a+b$, abandon b in favor of c and carry out the summation over a, based on

$$
\sum_{a=0}^{c}\binom{a}{i}\binom{c-a}{h}\frac{\left(\sqrt{-x}\right)^{a-1}\left(\sqrt{A}\right)^{c-a-1}}{a!\,(c-a)!} = \frac{\left(\sqrt{-x}\right)^{i-1}\left(\sqrt{A}\right)^{h-1}\left(\sqrt{-x}+\sqrt{A}\right)^{c-i-h}}{(c-i-h)!\,i!\,h!}. \tag{4.70}
$$

This formula is obtained by differentiating i times with respect to $\sqrt{-x}$ and h times with respect to $\sqrt{A}$ the binomial theorem for $\left(\sqrt{-x}+\sqrt{A}\right)^c$. In particular, for $i+h>c$, the sum vanishes.

The result is (with (4.55) and $\bar\sigma := \sigma - 2j$):

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} H_{n,m,j}\,\frac{(-\delta t^2)^m\,(16\varepsilon^2 t^6)^n}{(2m+1)!\,(2n+1)!} = \text{The even part of} \\
&\quad \frac{4}{(2j+1)!}\sum\sum\sum_{\substack{l,r,\nu\ge 0\\ l+r+\nu\le j}}\ \sum_{c=2j}^{5j}\ \sum_{\bar\sigma=0}^{c-2j}\ \sum\sum_{\substack{i,h\ge 0\\ i+h\le c}} \binom{c-2j}{\bar\sigma}(-1)^{i+h+\nu+l+\bar\sigma+c}\,\frac{3^{-l}4^{-r-\nu}}{r!\,l!}\times \\
&\quad \times\binom{2j-2l-2r-\nu}{\nu}\frac{(\bar\sigma+2j+1)!}{(c-i-h)!\,i!\,h!}(i+h+1)^{\downarrow r+\nu}(i+h+1-2j)^{\uparrow 2l+r}\times \\
&\quad \times\left(\frac{i+1}{2}-3j\right)^{\uparrow 4j-l-r-\nu}\frac{\left(\sqrt{-x}\right)^{i-1}\left(\sqrt{A}\right)^{h-1}}{\left(\sqrt{-x}+\sqrt{A}\right)^{-c+2j+i+h+\bar\sigma+2}}\exp[\sqrt{-x}+\sqrt{A}] + \text{TWOE}\,,
\end{aligned} \tag{4.71}$$

where TWOE will be evaluated below. The computational complexity of this formula can be reduced by a factor 2, if one observes that odd i do not contribute. Significant further simplification has eluded the author. However, the homogeneity degree in $(\sqrt{-x}, \sqrt{A})$, i.e., $c - 2j - \bar\sigma - 4$ for the single terms in the sum, can easily be seen to vary from -4 to $3j - 4$. The denominator $\sqrt{-x} + \sqrt{A}$ appears with the power $i + h + 2 - (c - 2j - \bar\sigma)$, which will be $5j + 2$ at most. However, some cancellation can be shown to occur: namely, with $k := i + h$, $\sum_{i+h\le c} = \sum_{k=0}^{c}\sum_{i=0}^{k}$, and

$$\sum_{i=0}^{k}\binom{k}{i}\left(\sqrt{-x}\right)^{i}\left(\sqrt{A}\right)^{k-i}\left(\frac{i+1}{2}-3j\right)^{\uparrow 4j-l-r-\nu} \tag{4.72}$$

is the result of a certain polynomial of degree $4j-l-r-\nu$ in $\sqrt{-x}\,\partial_{\sqrt{-x}}$ applied to $\left(\sqrt{-x}+\sqrt{A}\right)^{k}$. Therefore, if $k + 2 - (c - 2j - \bar\sigma) > 4j + 2$, we have in particular $k > 4j$; the sum over i leads to a factor of $\left(\sqrt{-x}+\sqrt{A}\right)^{k-4j}$, to cancel the denominator, which is at worst $(\sqrt{-x}+\sqrt{A})^{k+2}$. This accounts for the denominator given in Theorem 4.7. The degrees of the polynomials P_j^{**} given there are a consequence of the homogeneity degrees given above, except for the case of P_j^{00}, which we discuss next.

The term without exponentials can be read off easily from (4.68) by taking $\sigma = s + a + b - 2j$. This does not match with the summation over s for <u>fixed</u> σ in (4.69). We retain the sum over s, but evaluate the sum over a according to (4.70). We then get

$$\begin{aligned}
&\text{TWOE} = \text{The even part of} \\
&\quad \frac{4}{(2j+1)!}\sum\sum\sum_{\substack{l,r,\nu\ge 0\\ l+r+\nu\le j}}\ \sum_{c=2j}^{5j}\ \sum_{s=0}^{2j+1}\ \sum\sum_{\substack{i,h\ge 0\\ i+h\le c}}(-1)^{i+h+\nu+l+1}\,\frac{3^{-l}4^{-r-\nu}}{r!\,l!}\binom{2j-2l-2r-\nu}{\nu}\times \\
&\quad \times\binom{2j+1}{s}\frac{(s+c-2j)!}{(c-i-h)!\,i!\,h!}(i+h+1)^{\downarrow r+\nu}(i+h+1-2j)^{\uparrow 2l+r}\times \\
&\quad \times\left(\frac{i+1}{2}-3j\right)^{\uparrow 4j-l-r-\nu}\frac{\left(\sqrt{-x}\right)^{i-1}\left(\sqrt{A}\right)^{h-1}}{\left(\sqrt{-x}+\sqrt{A}\right)^{s+i+h+1-2j}}
\end{aligned} \tag{4.73}$$

The very same conclusions as before hold for the denominator, but now the homogeneity degree is $2j - s - 3$, which varies from -4 to $2j - 3$. Parity prevents single factors $\sqrt{-x}$ or $\sqrt{A}$ from occurring in the denominator.

The cancellation argument analogous to (4.72) for $\sum_{h=0}^{c-i}$ in (4.71) produces terms $\left(\sqrt{A}\right)^{-1}\Big/\left(\sqrt{-x}+\sqrt{A}\right)^{2j-2l-2r-\nu+\bar\sigma+2}$, so with respect to $\sqrt{A}$ alone, the highest order near ∞

is $\left(\sqrt{A}\right)^{-3}$. This accounts for the degrees of P_j^{**} with respect to A, except for P_j^{00}. Its degree is clear from (4.73), where the degree at ∞ of the single terms is $\left(\sqrt{A}\right)^{-s-i-2j-2}$, which is at most $\left(\sqrt{A}\right)^{-2j-2}$.

The proof of the information in the table is finished. The genesis of (4.65), e.g., formula (4.55), makes it clear that no singularity can occur at $x = -A$. ■

4.3 The Series from the Small Wave Number or Nearly Self–Adjoint Point of View

To begin with, the title of this section refers to the following observation: we intend to show that on solution branches on which $t < C|\delta|^{1/4}$ holds, δt^2 converges to a constant, whereas $t \to 0$. In this limit, using (1.12), the OS equation (1.10) reduces to an ordinary differential equation whose operator is self-adjoint, namely to the equation $(\partial_y^2 - \alpha^2)^2 w(y) = 0$.

The series representation of the spectral condition can be obtained by a completely different method, namely from the asymptotic expansions of the Airy functions as collected in Section 3.3. However, this approach gives the series only as a formal (asymptotic) series, whose convergence could not have been expected. This approach has the benefit of giving information also as $x \to \infty$, $A \to \infty$. Without some a-priori information of this kind, Theorem 4.7 would be of rather limited use.

In the sequel, we assume that $0 < t < C|\delta|^{1/4}$ for some constant C. Moreover, $-1 \le z \le 1$ in accordance with the interval of integration in (2.2). All estimates (O-symbols) will depend on this constant, but be uniform otherwise.

Then

$$(\delta + itz)^{-1/4} = \delta^{-1/4} + O(\delta^{-1})\,, \quad \frac{2}{3}(\delta + itz)^{3/2} = \frac{2}{3}\delta^{3/2} + itz\delta^{1/2} - \frac{z^2}{4}t^2\delta^{-1/2} + O(\delta^{-3/4})$$

and therefore, in the sector $|\arg\delta| < \pi$ (see (3.13))

$$\mathrm{Ai}(\delta + itz) = \frac{\delta^{-1/4}}{2\pi^{1/2}} \exp\left[-\frac{2}{3}\delta^{3/2}\right] \exp\left[-itz\delta^{1/2}\right] \exp\left[\frac{z^2}{4}t^2\delta^{-1/2}\right] \left(1 + O(\delta^{-3/4})\right)$$

Consequently, in this sector,

$$\begin{aligned}
&\exp\left[\frac{2}{3}\delta^{3/2}\right] \int_{-1}^{1} \mathrm{Ai}(\delta + itz) \exp[4\varepsilon t^3 z]\, dz = \\
&\quad = \frac{\delta^{-1/4}}{2\pi^{1/2}} \int_{-1}^{1} \exp\left[\left(4\varepsilon t^3 - it\delta^{1/2}\right) z\right] \exp\left[\frac{z^2}{4}t^2\delta^{-1/2}\right] \left(1 + O(\delta^{-3/4})\right)\, dz \\
&\quad = \frac{2\delta^{-1/4}}{\pi^{1/2}} \left\{ \frac{\sinh(a - iw)/2}{a - iw} + O\Big(C^2 (a - iw)_{\sqcap}^{-2}, (a - iw)_{\sqcap}^{-1}\delta^{-3/4}\Big) \cosh \mathrm{Re}((a - iw)/2) \right\}\,,
\end{aligned} \tag{4.74}$$

where we have taken advantage of the notations $a = 8\varepsilon t^3 = \sqrt{A}$, $w = 2t\delta^{1/2} = \sqrt{x}$ and $T_\sqcap := \min\{T, 1\}$. The $(a - iw)_\sqcap^{-1}$ terms have arisen from treating $O(z^2)$ in $\exp[z^2t^2\delta^{-1/2}/4] = 1 + O(z^2)$ by integration by parts.

Similar expansions are possible for the other integrals in (2.2); for the integrals involving Bi, say, with $|\arg z| < \pi/3$:

$$
\begin{aligned}
&\exp\Big[\frac{2}{3}\delta^{3/2}\Big]\int_{-1}^{1}\mathrm{Ai}(\delta+itz)\exp[-4\varepsilon t^3 z]\,dz = \\
&\quad= \frac{2\delta^{-1/4}}{\pi^{1/2}}\Big\{\frac{\sinh(a+iw)/2}{(a+iw)} + O\Big(C^2(a+iw)_{\sqcap}^{-2},(a+iw)_{\sqcap}^{-1}\delta^{-3/4}\Big)\cosh\mathrm{Re}((a+iw)/2)\Big\} \\
&\exp\Big[-\frac{2}{3}\delta^{3/2}\Big]\int_{-1}^{1}\mathrm{Bi}(\delta+itz)\exp[4\varepsilon t^3 z]\,dz = \\
&\quad= \frac{4\delta^{-1/4}}{\pi^{1/2}}\Big\{\frac{\sinh(a+iw)/2}{(a+iw)} + O\Big(C^2(a+iw)_{\sqcap}^{-2},(a+iw)_{\sqcap}^{-1}\delta^{-3/4}\Big)\cosh\mathrm{Re}((a+iw)/2)\Big\} \\
&\exp\Big[-\frac{2}{3}\delta^{3/2}\Big]\int_{-1}^{1}\mathrm{Bi}(\delta+itz)\exp[-4\varepsilon t^3 z]\,dz = \\
&\quad= \frac{4\delta^{-1/4}}{\pi^{1/2}}\Big\{\frac{\sinh(a-iw)/2}{(a-iw)} + O\Big(C^2(a-iw)_{\sqcap}^{-2},(a-iw)_{\sqcap}^{-1}\delta^{-3/4}\Big)\cosh\mathrm{Re}((a-iw)/2)\Big\}
\end{aligned}
\tag{4.75}
$$

Using (2.2), we can therefore conclude that in the sector $-\pi/3 < \arg\delta < \pi/3$, one has

$$
\begin{aligned}
\Delta(\delta,t,\varepsilon) &= \frac{4\delta^{-1/2}}{\pi}\Big[\frac{(a+iw)^2\sinh^2((a-iw)/2)-(a-iw)^2\sinh^2((a+iw)/2)}{(a^2+w^2)^2} + \\
&\qquad + O(C^2(a\pm iw)_{\sqcap}^{-3},(a\pm iw)_{\sqcap}^{-2}\delta^{-3/4})\cosh^2\mathrm{Re}((a\pm iw)/2)\Big] \\
&= \frac{4i\delta^{-1/2}}{\pi}\Big[\frac{2aw(\cosh a\cos w-1)-(a^2-w^2)\sinh a\sin w}{(a^2+w^2)^2} + \\
&\qquad + O(C^2(a\pm iw)_{\sqcap}^{-3},(a\pm iw)_{\sqcap}^{-2}\delta^{-3/4})\cosh^2\mathrm{Re}((a\pm iw)/2)\Big]\,.
\end{aligned}
\tag{4.76}
$$

Similar calculations are possible in the other sectors. Actually, the same calculation is immediately valid for the sectors $-\pi/3 < \arg\delta < \pi$ and $-\pi < \arg\delta < \pi/3$: just replace Bi by Bi $\pm$ Ai in (2.2) and use (3.19), (3.13) instead of (3.17). The missing neighbourhood of $\arg\delta = \pi$ can now be moved in the other sectors by employing formulas (3.7) and (3.8).

Therefore, (4.76) holds in all sectors. This is just the leading term $j = 0$ from Theorem 4.7. (Use $W = -1/\pi$). It is not clear from this approach that it is actually the leading term of a convergent series. On the other hand, it is much easier here to understand the behaviour as $w \to \infty$. But with Theorem 4.7 at hand, we can identify both approaches:

Carrying the expansions further will result in an asymptotic expansion in powers of $\delta^{-3/4}$ of, say, $\int_{-1}^{1}\mathrm{Ai}(\delta+itz)\exp[4\varepsilon t^3 z]\,dz$ for fixed a, w, with a corresponding series for Δ, the beginning of which is just (4.76). This series, originally calculated in the sector $-\pi/3 < \arg\delta < \pi/3$, is valid for arbitrary $\arg\delta$ by the very reasoning just given. As an asymptotic series in powers of $\delta^{-3/4}$ (for fixed a, w) representing the analytic function $\delta^{-3} \mapsto \Delta(\delta,t,\varepsilon)/(16Wi\varepsilon t^4)$, $\delta^{-3} \in \mathbb{C}$, it must coincide with the Taylor series of that function, namely (4.65); in particular, the coefficients of $\delta^{-3l/4}$ with non-integer $3l/4$ must vanish. We do not attempt here to describe the behaviour for varying w, a as $w^2 + a^2 \to 0$.

From (4.76), we obtain the following weak coercivity estimate right away:

Theorem 4.8 *For any $a \geq 0$, and any $0 < t < C|\delta|^{1/4}$ with C sufficiently small, independent of a, the entire function*

$$
w \mapsto \tilde{f}(w,a) := \frac{\Delta(w^2/4t^2,t,a/8t^3)(a^2+w^2)^2}{64Wi\varepsilon t^4} = \frac{\pi\Delta(w^2/4t^2,t,a/8t^3)(a^2+w^2)^2}{4iaw\delta^{-1/2}}
$$

has simple real zeros $\pm w_n$ $(n \geq 2)$, *where* $(n-1)\pi < w_n < w_{n+1} < (n+1)\pi$, $n = 2, 4, 6, \ldots$, *the trivial double zeros at* $w = \pm ia$, *and no other zeros. For* $a = 0$, *this means that the imaginary zeros coalesce at 0 with multiplicity four. The same holds without the smallness assumption on the fixed number* C, *except that finitely many of the* w_n *may fail to be real or to lie in the prescribed interval. An exception set can be chosen independent of* a, *only dependent on* C.

In either case, for any $L > 0$, *the connected components of the set* $\{w : |\tilde{f}(w,a)| < L\}$ *are each bounded ("weak coercivity"). The bound depends only on an upper bound for* a.

Given $\varepsilon \in [0, \infty[$ *and* $C > 0$ *sufficiently small (as in the first sentence), it holds*

$$\left\{(\delta, t) \in \mathbb{C} \times \mathbb{R}^+ \mid \Delta(\delta, t, \varepsilon) = 0\,,\ t < C|\delta|^{1/4}\right\} = \bigcup_{n=2}^{\infty} \{(\delta_n(t), t) \mid 0 < t < t_n\}\,,$$

where $\delta_n(\cdot)$ *are* <u>*real*</u> *functions,* $\lim_{t\to 0} 4\delta_n(t)t^2 = \omega_n^2$, *and* $\omega_n = n\pi$ *for even* n, *and the solution of* $w/2 = \tan w/2$ *in* $](n-1)\pi, n\pi[$ *for odd* n.

In the whole domain, it holds $2\sqrt{\delta_n(t)}\,t \in\ ](n-1)\pi, n\pi[$ *for* n *odd, and* $2\sqrt{\delta_n(t)}\,t \in\ ](n-1)\pi, (n+\frac{1}{2})\pi[$ *for* n *even.*

If C *is arbitrary, there is still* $n_0 = n_0(C)$ *(independent of* ε*), such that for* $n > n_0$, *the branches* $\delta_n(t)$ *in the small domain* $t < C_0|\delta|^{1/4}$ *remain real in the large domain* $t < C|\delta|^{1/4}$.

Note: This classification of solution for small t will be the basis of the following (global) classification of solutions. The arcs $(\delta_{2k}(t), t)$ will be denoted as R_{k0}, whereas the arcs $(\delta_{2k+1}(t), t)$ will be denoted as R_{k1} $(k = 1, 2, 3, \ldots)$. See in particular

- Figure 5.4 and Theorem 5.10 (with different asymptotics than here, because $\varepsilon = \infty$ there).
- Theorem 5.15 and Figure 5.5 for large ε
- Figure 6.1, Figure 6.6 and Theorem 5.15 for $\varepsilon = 0$.

Proof of Theorem 4.8:
Given any $t > 0$ in the specified interval and any fixed $\varepsilon \geq 0$, we apply Rouché's theorem to the function

$$\begin{aligned}\tilde{f}(w,a) &= f(w,a) + O(C^2(a \pm iw)_{\sqcap}^{-3}, (a \pm iw)_{\sqcap}^{-2}\delta^{-3/4}) \cosh \operatorname{Re}(a \pm iw) \\ &= f(w,a) + O(Cw^{-3}) \cosh a \cosh \operatorname{Im} w \qquad (\text{as } |w| \to \infty\,,\ C \text{ bounded})\end{aligned} \tag{4.77}$$

with

$$f(w,a) := 2(\cosh a \cos w - 1) - (a^2 - w^2)\frac{\sinh a}{a}\,\frac{\sin w}{w} \tag{4.78}$$

and the comparison function $g(w) := w \sin w\, (\sinh a)/a$ on certain rectangles

$$K_{NM} := \left\{w \;\middle|\; |\operatorname{Re} w| \leq \left(N + \frac{1}{2}\right)\pi\,,\ |\operatorname{Im} w| \leq M\right\}\,.$$

The $O(Cw^{-3})$-terms come from (4.76) and are valid, if $|w|$ is large enough (depending on a) or else if w is real (uniformly in a): In the former case, for $|w| \geq 2|a|$, we conclude $|(a \pm iw)_{\sqcap}^{-1}| \leq 2|w|^{-1}$, in the latter case $|(a \pm iw)_{\sqcap}^{-1}| \leq |w|^{-1}$. And in either case, we use $|\delta|^{-3/4} < 2C|w|^{-1}$.

Let $w = u + iv$, u, v real. We use $\sinh v \leq |\sin w|, |\cos w| \leq \cosh v$ and assume $|w| \geq a$. Then,

$$\begin{aligned}|f(w) - g(w)| &\leq \frac{a \sinh a\,|\sin w|}{|w|} + 2\cosh a\,|\cos w| + 2 \leq (2 + 2\cosh a + \sinh a)\cosh v \\ \text{and}\quad |g(w)| &\geq v \sinh v\, \frac{\sinh a}{a}\,.\end{aligned}$$

Moreover, we also have the estimate

$$|g(w)| \geq |w| \cosh v \, \frac{\sinh a}{a} \, , \qquad \text{if } u \equiv \pi/2 \bmod \pi \ \ (\Rightarrow |\sin w| = \cosh v \ ) \, .$$

This latter will be used on the vertical pieces of ∂K_{NM} (with N integer).

For Rouché's theorem to apply on the horizontal pieces of ∂K_{NM}, (where $v = \pm M$) we need

$$\frac{a}{\sinh a}\Big(2 + 2\cosh a + \sinh a + \cosh a \, O(CM^{-3})\Big) < M \, \tanh M \, .$$

This is clearly true for large M, and we even get $|\tilde{f} - g|/|g| \to 0$ on the horizontal piece of the boundary, as $M \to \infty$.

For Rouché's theorem to apply on the vertical pieces, we need

$$\frac{a}{\sinh a}\Big(2 + 2\cosh a + \sinh a + \cosh a \, O(CN^{-3})\Big) < (N + \tfrac{1}{2})\pi \, ,$$

which permits the very same conclusions.

Therefore, for large enough N (depending only on an upper bound for a), $\tilde{f}$ and g have the same number of zeros inside the strip $K_{N\infty}$, namely $2N + 2$. Four of these zeros of $\tilde{f}$ are immediate, namely the double zeros at $\pm ia$. We will try to locate the remaining zeros on the real axis.

But first note that $g|_{\partial K_{NN}} \to \infty$ and $(\tilde{f} - g)/g|_{\partial K_{NN}} \to 0$ as $N \to \infty$. This implies the weak coercivity property for $\tilde{f}$.

In the Rouché argument, the lower bounds for N, M had to depend on a. Now, for the location of the non-trivial zeros, we will use estimates that are uniform as $a \to \infty$. For odd n, we have $\tilde{f}(n\pi, a) \leq -2 - 2\cosh a + \cosh a \, O(Cn^{-3}) < 0$, if n is large enough, or else for all n, provided C is small enough.

For even $n > 0$ and $a \geq 1$, we have $\tilde{f}(n\pi, a) \geq -2 + 2\cosh a - \cosh a \, O(Cn^{-3}) > 0$, again if n is large enough or C is small enough. For $a \leq 1$ and even $n > 0$, we replace this estimate by $\tilde{f}((n + \frac{1}{2})\pi, a) \geq -2 + ((n + \frac{1}{2})^2\pi^2 - 1)/(n + \frac{1}{2})\pi - (\cosh 1) \, O(Cn^{-3}) > 0$, if n is large or C is small enough. These estimates are uniform in a, and therefore in ε.

This shows that there is some C such that for any ε and any $t < C|\delta|^{1/4}$, there are at least two real zeros of $w \mapsto \tilde{f}(w, a)$ in each of the intervals $]n\pi, (n+2)\pi[$ with n odd. They exhaust the total number of zeros in the strip $K_{N\infty}$ for N large enough, therefore there are no further zeros, and the ones found are simple. The estimates given above yield the locations of the curves as given in the theorem.

Without the smallness assumption on C, the same conclusion holds for sufficiently large n, i.e., all but finitely many solutions w are real and simple and lie in the claimed interval.

The zeros w_n of $\tilde{f}$ immediately give zeros $\delta_n(t)$ of Δ, namely $\delta_n(t) = w_n(a = 8\varepsilon t^2; t)^2/(4t^2)$. They continue to exist down to $t \searrow 0$ and are all real positive if C is small enough. The property $\lim_{t\to 0} f(w_n(t), 8\varepsilon t^3) = 0$, which follows from $\tilde{f} = 0$, and the estimate of w_n, imply that $\omega_n := \lim_{t=0} w_n(t)$ are the zeros of $f(\cdot, 0)$ and hence the numbers specified in the theorem.

The same reasoning applies for arbitrary C and $n > n_0(C)$. ∎

Chapter 5

The Infinite Viscosity or Short Wave Case

5.1 Why Reynolds Number Zero <u>is</u> Relevant

5.1.1 Definitions and Outline

The purpose of this chapter is to show that there are no changes in topology of the set of real zeros of $\Delta(\cdot,\cdot,\varepsilon)$ for sufficiently <u>large</u> $\varepsilon > \varepsilon_0$. The limit $\varepsilon \to \infty$, although it is the limit of high viscosity and seems to be of little interest in physics as compared to the limit $\varepsilon \to 0$, is mathematically much simpler, and it will serve as the starting point for a comparison of the spectrum in dependence of ε. It should be noted that in our formulation of the spectral condition, crucial scalings of the spectrum have already been made, such that the relevant variables have finite limits as $\varepsilon \to \infty$, and they also retain a fine structure in the limit $\varepsilon \to 0$, which leaves more information than just that the spectrum goes to the imaginary axis. The spectrum, expressed in the (δ, t)-variables undergoes only well-organized changes as ε varies between 0 and ∞, as is explained in more detail in Chapter 6.

When we consider

$$\Delta(\delta,t,\varepsilon) := \begin{vmatrix} \int_{-1}^{1} \mathrm{Ai}(\delta+itz)\sinh(4\varepsilon t^3 z)\,dz & \int_{-1}^{1} \mathrm{Ai}(\delta+itz)\cosh(4\varepsilon t^3 z)\,dz \\ \int_{-1}^{1} \mathrm{Bi}(\delta+itz)\sinh(4\varepsilon t^3 z)\,dz & \int_{-1}^{1} \mathrm{Bi}(\delta+itz)\cosh(4\varepsilon t^3 z)\,dz \end{vmatrix},$$

it is intuitively clear, that with appropriate scaling, the weight $\cosh 4\varepsilon t^3 z$ on $[-1,1]$ will converge to a Dirac weight concentrated on the end points of that interval. As long as the limit $\varepsilon \to \infty$ is carried out with (δ, t) restricted to a compact set, there is no difficulty. However, care must be taken to get more or less the same result even for (δ, t) unbounded.

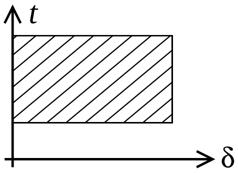
Obviously, the limit $\varepsilon \to \infty$ competes with the limit $t \to 0$. We have studied the solution curves for small t in section 4.3 and have seen that there is a balance between $4\delta t^2$ and $64\varepsilon^2 t^6$ as long as t remains bounded. This will make the limits $\delta \to \infty$ and $\varepsilon \to \infty$ compete.

Of course, care needs to be taken with $(\delta, t) \to \infty$ in any case, in order to control the error terms, but this can be done by careful technique, whereas the limit $\delta \to \infty$ with bounded t is a genuine difficulty caused by competing limits.

When we try to prove rigorously that the spectrum for large ε is homeomorphic to the one in the limiting case $\varepsilon = \infty$ (Section 5.3), it turns out, not too surprising when one looks at the formula for Δ, that the limit $t \to \infty$ for fixed $\varepsilon > 0$ has the same effects as the limit $\varepsilon \to \infty$ for fixed $t > 0$. As the combination $4\varepsilon t^3$ that enters is just (half) the wave number, we choose the double title "Infinite Viscosity" or "Short Wave" for this chapter.

Let us do the simplest case first, just to illustrate the principle. The detailed study of the limit $\varepsilon \to \infty$ will be done in Section 5.3.

Lemma 5.1 *As $\varepsilon \to \infty$, it holds*



$$\frac{\Delta(\delta,t,\varepsilon)\,(4\varepsilon t^3)^2}{2\sinh(4\varepsilon t^3)\cosh(4\varepsilon t^3)} = \mathrm{Ai}(\delta+it)\,\mathrm{Bi}(\delta-it) - \mathrm{Bi}(\delta+it)\,\mathrm{Ai}(\delta-it) + O(1/\varepsilon)$$

with the O-term uniform as long as (δ, t, t^{-1}) remain bounded.

Proof:
We integrate by parts

$$\int_{-1}^{1} \mathrm{Ai}(\delta+itz)\sinh(4\varepsilon t^3 z)\,dz = \left[\mathrm{Ai}(\delta+itz)\frac{\cosh(4\varepsilon t^3 z)}{4\varepsilon t^3}\right]_{-1}^{1} - it\int_{-1}^{1}\mathrm{Ai}'(\delta+itz)\frac{\cosh(4\varepsilon t^3 z)}{4\varepsilon t^3}\,dz\ .$$

Estimate the remaining integral using $\mathrm{Ai}'(\delta+itz) = O(1)$, and proceed similarly for the other terms in Δ. The calculation is routine. ■

Before continuing to study the limiting process $\varepsilon \to \infty$, we give a thorough discussion of the formal limiting case $\varepsilon = \infty$.

5.2 A Detailed Study of Reynolds Number Zero

It is the purpose of this section to study the set

$$\mathcal{Z}_\infty := \{(\delta,t) \in \mathbb{C}\times\mathbb{R} \mid \mathrm{Ai}(\delta+it)\,\mathrm{Bi}(\delta-it) - \mathrm{Ai}(\delta-it)\,\mathrm{Bi}(\delta+it) = 0\} \tag{5.1}$$

As a preparation, we collect some (more or less) well–known facts about Airy functions. As these facts are usually harvested from collections of formulas and tables of special functions, we do give full proofs in a self–contained way.

5.2.1 More on Airy Functions

Lemma 5.2 *The function $z \mapsto \mathrm{Ai}(z)$ (which is uniquely determined up to the normalization constant by $\mathrm{Ai}''(z) = z\,\mathrm{Ai}(z)$, $\lim_{z\to+\infty}\mathrm{Ai}(z) = 0$) has a sequence of negative real simple zeros $a_{-1} > a_{-2} > a_{-3} > \ldots \to -\infty$ and no other zeros. It holds $a_{-n} \sim -[\frac{3}{2}\pi(n-\frac{1}{4})]^{2/3}$ as $n \to \infty$.*

The function $y \mapsto |\mathrm{Ai}(x+iy)|^2$ is strictly convex for every x.

The function $z \mapsto \mathrm{Ai}'(z)$ has a sequence of negative real simple zeros $a'_0 > a'_{-1} > a'_{-2} > \ldots \to -\infty$ and no other zeros. It holds $a'_{-n+1} > a_{-n} > a'_{-n}$.

Within the domain $\mathrm{Re}\, z > a'_0$, $\mathrm{Im}\, z > 0$, all level lines of $|\mathrm{Ai}|$ are graphs of strictly increasing functions.

Proof:
Letting $G(x,y) := |\mathrm{Ai}(x+iy)|^2 = \mathrm{Ai}(x+iy)\,\mathrm{Ai}(x-iy)$ for real x, y, with $z = x+iy$, $\zeta = \frac{2}{3}(x+iy)^{3/2}$, it is easy to check that $G(x,0) \geq 0$, $G_y(x,0) = 0$, $G_{yy} = -2xG + 2|\mathrm{Ai}'|^2$. Fixing y

and letting $x \to +\infty$, the asymptotic expansion tells us that

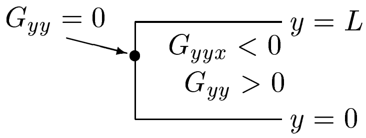
$$\begin{aligned}
\frac{1}{2}G_{yy}(x,y) &\sim \frac{|z|^{1/2}}{4\pi}\,|\exp(-2\zeta)|\left\{\left|1-\frac{d_1}{\zeta}+O(\zeta^{-2})\right|^2-\frac{x}{|z|}\left|1-\frac{c_1}{\zeta}+O(\zeta^{-2})\right|^2\right\}\\
&\sim \frac{x|\exp(-2\zeta)|}{4\pi|z|^{1/2}}\left\{\left(1-2d_1\operatorname{Re}\zeta^{-1}\right)\left(1+\frac{y^2}{2x^2}\right)-\left(1-2c_1\operatorname{Re}\zeta^{-1}\right)+O(\zeta^{-2},x^{-4})\right\}\\
&> \frac{x|\exp(-2\zeta)|}{4\pi|z|^{1/2}}\left\{\frac{y^2}{2x^2}+O(x^{-4})+2(c_1-d_1)\operatorname{Re}\zeta^{-1}+O(\zeta^{-2})\right\}>0\,.
\end{aligned}$$

(We have used $\operatorname{Re}\zeta^{-1}>0$, $d_1<0<c_1$.)

Moreover, it is straightforward to calculate $\frac{1}{2}G_{yyx}=-G-yG_y$.

This shows that, for any L, there exists an N_0, such that for $x \ge N_0$ and $y \in [0,L]$, we have $G_{yy}>0$, and therefore, for $y \in\,]0,L]$ also $G_y>0$, $G>0$, and $G_{yyx}<0$.

Now, for any given L, let $\check{x} := \inf\{N \mid G_{yy}>0 \text{ on } [N,\infty[\,\times\,]0,L]\}$. The inserted, self-explaining sketch shows a contradiction to the hypothesis $\check{x} > -\infty$. Therefore, $G_{yy}>0$ everywhere (except possibly $G_{yy}=0$ on the real axis), and $y \mapsto |\mathrm{Ai}(x+iy)|^2$ is strictly convex. In particular, Ai cannot have other than real zeros. In zeros of Ai′, level lines of G intersect orthogonally. This can happen only on the real axis, because otherwise, a contradiction with the vertical monotonicity of G would ensue.



Clearly, by the ODE, $x \mapsto \mathrm{Ai}(x)$ cannot have a positive maximum or a negative minimum for $x>0$, so the zeros of Ai and Ai′ are negative. They are simple by the uniqueness of solutions for the Airy ODE, and their interlacing property is then clear. There must be infinitely many zeros because of the ODE again.

For $x>a_0'$, the rightmost extremum of $G(\cdot,0)$, we have $G_x(x,0)<0$, but $G_{xy}(x,0)=0$, $G_{xyy}(x,y)<0$. This implies that $G_x(x,y)<0$ there. Using also $\operatorname{sign} G_y = \operatorname{sign} y$, we infer that, in the quadrant $x>a_0'$, $y>0$, all level lines of G are the graphs of strictly increasing functions.

■

Some level sets of the Airy function Ai are shown in Figure 5.1.

The following lemma is crucial:

Lemma 5.3 *It holds*

$$s,t_1,t_2\in\mathbb{R}\ \wedge\ \frac{\mathrm{Bi}(s+it_1)}{\mathrm{Ai}(s+it_1)}=\frac{\mathrm{Bi}(s+it_2)}{\mathrm{Ai}(s+it_2)}\ \wedge\ t_1<t_2 \quad\Longrightarrow\quad t_1<0<t_2\,,\quad s\ge 0$$

If the equation

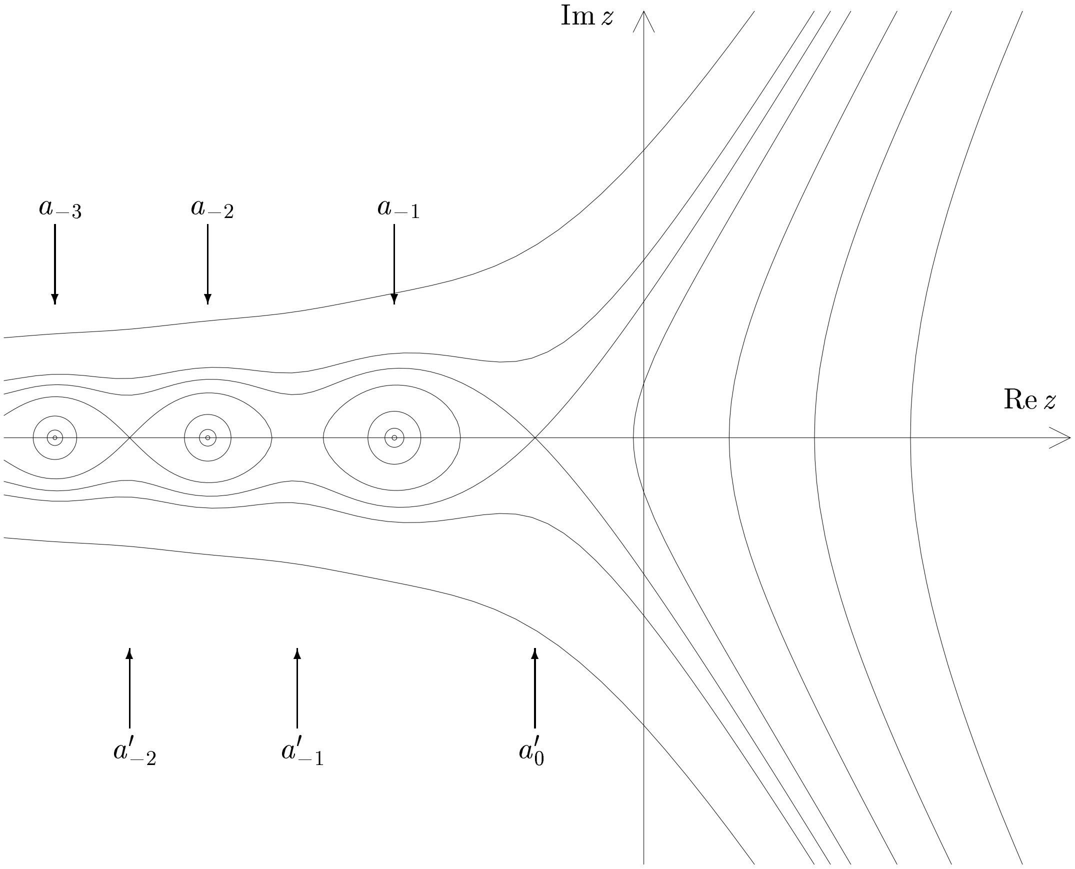
$$\frac{\mathrm{Bi}(\delta+it)}{\mathrm{Ai}(\delta+it)}=\frac{\mathrm{Bi}(\delta-it)}{\mathrm{Ai}(\delta-it)}$$

with $\delta\in\mathbb{C}$ and $0\ne t\in\mathbb{R}$ has a double solution (considered as an equation for δ), then δ is real positive, and $\mathrm{Ai}(\delta+it)=\pm\mathrm{Ai}(\delta-it)$, $\mathrm{Bi}(\delta+it)=\pm\mathrm{Bi}(\delta-it)$ are all either real or purely imaginary.

Remark: We shall see in the proof of Theorem 5.10 below that for double solutions, one has $\mathrm{Bi}(\delta+it)=\sqrt{3}\,\mathrm{Ai}(\delta+it)>0$.

PROOF:
Let the common value of Bi / Ai at the two points equal λ. Then, $\mathrm{Ci} := \mathrm{Bi}-\lambda\,\mathrm{Ai}$ satisfies the Airy ODE, which can be written as

$$u_{tt}=-su+tv\,,\quad v_{tt}=-sv-tu\,,\quad (\text{with } u=\operatorname{Re}\mathrm{Ci}\,,\ v=\operatorname{Im}\mathrm{Ci})\,.$$



Level lines of $|\mathrm{Ai}|$ at levels $\mathrm{Ai}(2.5) = 0.0157259$, $\mathrm{Ai}(1.6) = 0.0625369$, $\mathrm{Ai}(0.8) = 0.169846$, $\mathrm{Ai}(a'_{-2}) = 0.380406$, $\mathrm{Ai}(a'_0) = 0.535657$, 0.7, 1.8. The domain shown in the graph is $-6 < \operatorname{Re} z < 4$, $-4 < \operatorname{Im} z < 4$.

Figure 5.1: Level lines of $|\mathrm{Ai}|$

Then integrate the equality $(u_t v - u v_t)_t = t(u^2+v^2)$ from t_1 to t_2, and obtain $\int_{t_1}^{t_2} t(u^2+v^2)\,dt = 0$. This implies $0 \in]t_1, t_2[$.

Moreover,

$$\frac{1}{2}(u^2+v^2)_{tt} = u_t^2 + v_t^2 + u u_{tt} + v v_{tt} \geq -s(u^2+v^2) \geq 0 \quad \text{for} \quad s \leq 0\,.$$

Actually $(u^2+v^2)_{tt}(s+it_1)$ is strictly > 0, because $\mathrm{Ci}(s+it_1) = 0$ and hence $\mathrm{Ci}'(s+it_1) \neq 0$. This, together with $(u^2+v^2)(s+it_1) = 0$, $(u^2+v^2)_t(s+it_1) = 0$ implies that $u^2+v^2 > 0$ for $t \neq t_1$. This proves the first part of the lemma.

As far as the second part is concerned, we have $\operatorname{Re}\delta > 0$ from the first part even for simple solutions. Differentiating the equation and cancelling the Wronskian implies $\mathrm{Ai}(\delta+it) = \pm\,\mathrm{Ai}(\delta-it)$. The convexity and symmetry of $y \mapsto |\mathrm{Ai}(x+iy)|^2$ given in Lemma 5.2 shows that $\delta+it = \overline{\delta-it}$, i.e., δ is real. ∎

Lemma 5.4 *If* $\operatorname{Im} z > 0$, *then* $\operatorname{Im}(-\mathrm{Ai}'(z)/\mathrm{Ai}(z)) > 0$.

Proof:
Let $g(z) := -\mathrm{Ai}'(z)/\mathrm{Ai}(z)$. It is a meromorphic function with its only poles and zeros on the real axis, and these are simple (Lemma 5.3). For real x, it holds $g'(x) > 0$, wherever defined.

This is clear for $x \le 0$, because $g'(x) = g(x)^2 - x$. It is also true on some interval $]x_-, \infty[$, because we have $g(x) \sim \sqrt{x}(1 + x^{-3/2}/4 + O(x^{-3}))$ as $x \to +\infty$. Choose $x_- \ge 0$ as small as possible. If $x_- > 0$, then $g(x_-) = \sqrt{x}$ and $g'(x_-) = 0$. But then $g(x) < \sqrt{x}$ to the near right of x_-. This shows that $g(x) > \sqrt{x}$ and thus $g'(x) > 0$ for all $x \ge 0$.

By conformity, $g'(x) > 0$ implies $\operatorname{Im} g(z) > 0$ in the upper half of some neighbourhood of the real axis. In order to show that this property extends into the full upper half plane, we need to show that there are no points with $\operatorname{Im} g(z) = 0$ there.

Suppose that, to the contrary, $\operatorname{Im} g(z_0) = 0$, $\operatorname{Im} z_0 > 0$ for some z_0. Let L be the connected component of z_0 in the level set $\{\operatorname{Im} g(z) = 0\}$. L is a manifold, unless it contains a point where $g'(z) = 0$. But $g'(z) = g(z)^2 - z$, so g' can only vanish on L in a real point $z \ge 0$. We have seen that this is impossible.

This shows that L is a manifold. In particular, it cannot cross the real axis. By the classification of 1-dimensional connected manifolds, L is either a topological circle or a topological line.

It cannot be a topological circle, by the maximum principle for the harmonic function $\operatorname{Im} g$. So it must be a line and extend to complex infinity. In a sector $\arg z \in]0, \alpha]$ with $\alpha < \pi$, it holds $g(z) \sim z^{1/2} + O(z^{-1})$. Then $\operatorname{Im} g(z) = 0$ and $|\sqrt{z}|$ sufficiently large would imply $\operatorname{Im} \sqrt{z} = 0$, which is impossible on L. So L could only escape to infinity in some sector $\arg z \in]\pi - \varepsilon, \pi[$. In this sector, we use the asymptotics

$$g(z) \sim \sqrt{-z}\,\frac{(\cos\tilde\zeta)(1 + O(\tilde\zeta^{-1})) + (\sin\tilde\zeta)O(\tilde\zeta^{-1})}{(\sin\tilde\zeta)(1 + O(\tilde\zeta^{-1})) - (\cos\tilde\zeta)O(\tilde\zeta^{-1})} \qquad (\, \tilde\zeta = \frac{2}{3}(-z)^{3/2} + \pi/4 \,) \tag{5.2}$$

However, for $\tilde\zeta = n\pi + it$, the absolute value of the fraction can be calculated to be $(\tanh t + O(\tilde\zeta^{-1}))^{-1}$, so the factor $\sqrt{-z}$ guarantees that $|g(z)|$ is unbounded on both ends of L. Therefore $\operatorname{Re} g(z)$, which is monotonic on L, runs from $-\infty$ to $+\infty$ there. But then g must have a zero on L, which is a contradiction to Lemma 5.2. This proves the lemma. ■

Lemma 5.5 *The function $z \mapsto \operatorname{Bi}(z)$ has a sequence of real negative zeros $b_{-1} > b_{-2} > b_{-3} > \ldots \to -\infty$ and two sequences of complex zeros (conjugate to each other), $b_1, b_2, \ldots$ and $\bar b_1, \bar b_2, \ldots$, where*

$$b_n \sim e^{i\pi/3}\left[\frac{3}{2}\pi\left(n - \frac{1}{4}\right)\right]^{2/3} + O(n^{-1/3})\,, \quad b_{-n} \sim -\left[\frac{3}{2}\pi\left(n - \frac{3}{4}\right)\right]^{2/3} + O(n^{-1/3})$$

and no other zeros. There are no zeros in the closed sector $\arg z \in [-\pi/3, \pi/3]$.

PROOF:
First we show that $\operatorname{Bi}(z)\operatorname{Ai}(z)$ does not vanish in the closed sector $|\arg z| \le \pi/3$. Then, a fortiori, neither does $\operatorname{Bi}(z)$.

We use (3.10), namely

$$\operatorname{Bi}(z) - i\operatorname{Ai}(z) = 2e^{-i\pi/6}\operatorname{Ai}(e^{-2\pi i/3}z) \tag{5.3}$$

with $z = re^{i\pi/3}$ and multiply with $\operatorname{Ai}(re^{i\pi/3}) =: M$. We obtain

$$(\operatorname{Ai}\operatorname{Bi})(re^{i\pi/3}) = e^{-i\pi/6}(2|M|^2 - e^{-i\pi/3}M^2) \quad \text{with} \quad M = \operatorname{Ai}(re^{i\pi/3}) \tag{5.4}$$

Therefore, $\arg(\operatorname{Ai}\operatorname{Bi})(z) \in [-\pi/3, 0]$ on the ray $\arg z = \pi/3$. Similarly on the ray $\arg z = -\pi/3$.

By means of the asymptotic estimates (3.13) and (3.19), it holds

$$\operatorname{Ai}(z)\operatorname{Bi}(z) \sim \frac{z^{-1/2}}{2\pi}\left[1 + O(z^{-3/2}) + \frac{i}{2}\exp\left(-\frac{4}{3}z^{3/2}\right)\,(1 + O(z^{-3/2}))\right] \tag{5.5}$$

for $\arg z \in [0,\pi[$. For $\arg z \in [0,\pi/3]$ and $|z|$ large enough, the argument of this expression lies in a small neighbourhood of $[-\pi/3,0]$. Therefore, the argument of $\mathrm{Ai}(z)\,\mathrm{Bi}(z)$ changes at most $\pi/3+o(1)$ on the the arc $Re^{i\alpha}$, $\alpha\in[0,\pi/3]$, at most $\pi/3$ on the ray $e^{i\pi/3}[0,R]$, and not at all on $[0,R]$. The sector bounded by these curves can therefore not contain a zero of Ai Bi, and neither can the complex conjugate sector.

Next we show that there are exactly n zeros of Bi in the sector $\arg z\in[0,2\pi/3]$, $|z|\le R := (3\pi(n+\frac{1}{4})/2)^{2/3}$, for n large enough. Equation (3.8) implies $2\,\mathrm{Re}(e^{2\pi i/3}\,\mathrm{Bi}(re^{2\pi i/3})) = -\,\mathrm{Bi}(r)$, and therefore the argument of $\mathrm{Bi}(z)$ changes less than π along the ray $e^{2\pi i/3}[0,R]$. It does not change at all along $[0,R]$. We are left with calculating the change of argument along the arc:

$$\frac{1}{2\pi i}\int_{R}^{R\exp(2\pi i/3)}\!{}^{(\text{arc})}\qquad\frac{\mathrm{Bi}'(z)}{\mathrm{Bi}(z)}\,dz =$$

$$=\frac{1}{2\pi i}\int_{2R^{3/2}/3}^{2R^{3/2}\exp(i\pi)}\!{}^{(\text{arc})}\qquad\frac{2e^{\zeta}(1+d_1\zeta^{-1}+O(\zeta^{-2}))-ie^{-\zeta}(1-d_1\zeta^{-1}+O(\zeta^{-2}))}{2e^{\zeta}(1+c_1\zeta^{-1}+O(\zeta^{-2}))+ie^{-\zeta}(1-c_1\zeta^{-1}+O(\zeta^{-2}))}\,d\zeta$$

Lemma 5.6 *If $|w+i|\ge r$, then $\left|\frac{w-i}{w+i}\right|\le 1+\frac{2}{r}$, and $|w|+1\le|w+i|+|w-i|\le(2+\frac{2}{r})|w+i|$.*

PROOF:
The triangle inequality, in particular:
$2|w|+2 = |(w+i)+(w-i)|+|(w+i)-(w-i)|\le 2|w+i|+2|w-i|$. ■

The lemma can be applied with $w=2\exp(2\zeta)$ and $r<1$ on our arc for sufficiently large n: Indeed, $2\zeta=(2\pi n+\pi/2)e^{i\alpha}$, and if one had $|w+i|<r<1$, it would follow $0>\ln\frac{1+r}{2}>(2\pi n+\pi/2)\cos\alpha=\mathrm{Re}\,2\zeta>\ln\frac{1-r}{2}$, i.e., $0>\cos\alpha>-O(n^{-1})$ and $\pi/2<\alpha<\pi/2+O(n^{-1})$. But then $\sin\alpha>1-O(n^{-2})$ and $\mathrm{Im}\,2\zeta=(2\pi n+\pi/2)\sin\alpha=2\pi n+\pi/2+O(n^{-1})$ and consequently $\mathrm{Im}\,w=2\exp(\mathrm{Re}\,2\zeta)\,\sin(\mathrm{Im}\,2\zeta)>0$. This contradicts $|w+i|<1$.

Using the lemma with $w=2\exp 2\zeta$, we conclude

$$\left|\frac{2e^{\zeta}(1+d_1\zeta^{-1}+O(\zeta^{-2}))-ie^{-\zeta}(1-d_1\zeta^{-1}+O(\zeta^{-2}))}{2e^{\zeta}(1+c_1\zeta^{-1}+O(\zeta^{-2}))+ie^{-\zeta}(1-c_1\zeta^{-1}+O(\zeta^{-2}))}-\frac{2e^{\zeta}-ie^{-\zeta}}{2e^{\zeta}+ie^{-\zeta}}\right| =$$
$$=\left|\frac{w-i+d_1\zeta^{-1}(w+i)+O(\zeta^{-2})(|w|+1)}{w+i+c_1\zeta^{-1}(w-i)+O(\zeta^{-2})(|w|+1)}-\frac{w-i}{w+i}\right|\le$$
$$\le\frac{|w+i|^2[|d_1\zeta^{-1}|+|c_1\zeta^{-1}|(1+2/r)+O(\zeta^{-2})]}{|w+i|^2(1-|c_1\zeta^{-1}|(1+2/r)-O(\zeta^{-2}))}\le$$
$$\le[|d_1|+|c_1|(1+2/r)]|\zeta^{-1}|+O(\zeta^{-2})<|\zeta|^{-1}/3$$

(see (3.12) for c_1,d_1).

Therefore reducing the integrand to its leading term changes the integral by at most $\pi/3$. The leading term is

$$\mathrm{Re}\,\frac{1}{2\pi i}\int_{2R^{3/2}/3}^{2R^{3/2}\exp(i\pi)}\!{}^{(\text{arc})}\qquad\frac{2e^{\zeta}-ie^{-\zeta}}{2e^{\zeta}+ie^{-\zeta}}\,d\zeta =$$
$$=\frac{-1}{2\pi}\int_{-2R^{3/2}/3}^{2R^{3/2}/3}\frac{d}{d\zeta}\arg\{2e^{\zeta}+ie^{-\zeta}\}+\mathrm{Re}\sum_{k=1}^{n}\mathrm{res}\left\{\frac{2e^{\zeta}-ie^{-\zeta}}{2e^{\zeta}+ie^{-\zeta}}\,d\zeta;-\frac{\ln 2}{2}-i\pi/4+i\pi k\right\}=$$
$$=n-1/4+o(1)$$

as $R\to\infty$. Altogether, $1/2\pi$ times the change of argument along the boundary, i.e., the number of zeros included, differs from n by less than $1/4+1/2+1/6$, i.e., it necessarily equals n.

Similarly, we can determine the number of zeros of Bi in the sector $|\arg(-z)| < \pi/3$, using (3.21) and this time $R := (3\pi(n - \frac{1}{4})/2)^{2/3}$. The real zeros follow also by the same argument as for Ai (see Lemma 5.2), and they interlace with the zeros of Ai according to Sturm Liouville theory (i.e., by the fact that $\mathrm{Ai}' \mathrm{Bi} - \mathrm{Bi}' \mathrm{Ai}$ is constant). By a counting argument, there cannot be any zeros beyond the real ones in the sector $|\arg(-z)| \leq \pi/3$. This proves Lemma 5.5. ∎

5.2.2 The Solution for Infinite Viscosity

Lemma 5.7 *Let $D(z) := \mathrm{Bi}(z)/\mathrm{Ai}(z)$. The level sets $\Lambda(h) := |D|^{-1}(\{h\})$ of $|D|$ for $h > 0$ are manifolds (i.e., disjoint unions of simple closed curves and open curves diffeomorphic to $\mathbb{R}$).*

Let p_0 be the largest real number such that $D(p_0) = 1$. ($p_0 \approx -0.366$). Then, for any $x > p_0$, the connected component $\tilde{\Lambda}(x)$ of $\Lambda(D(x))$ containing x is an open curve running from $\infty \exp(-i\pi/3)$ to $\infty \exp(i\pi/3)$. The domain to the right of $\tilde{\Lambda}(x)$ is the union of all $\tilde{\Lambda}(x')$ with $x' > x$.

$|D| > 1$ everywhere in the sector $\arg z \in]-\pi/3, \pi/3[$ and also on the rays $\arg z \in \{\pm 2\pi/3\}$. The direction angle of $\nabla |D|$ in the sector $\arg z \in]0, \pi/3[$ lies in $]-\pi/3, 0[$.

PROOF:
The $\Lambda(h)$ are manifolds, because D' has no zeros. So the whole plane minus the zeros of Ai and Bi is fibered into connected components of level sets $\Lambda(h)$, which are each either topological circles, or else topological lines.

Based on the asymptotic formulas, we obtain, uniformly in $\arg z \in [0, \pi - \varepsilon]$, as $|z| \to \infty$,

$$D(z) \sim 2 \exp\left(\frac{4}{3} z^{3/2}\right) \left(1 + O(z^{-3/2})\right) + i \left(1 + O(z^{-3/2})\right) . \tag{5.6}$$

The behaviour in $\arg z \in [-\pi + \varepsilon, 0]$ is analogous (take the complex conjugate). The quadrant $0 \leq \arg z^{3/2} \leq \pi/2$ (i.e., $\arg z \in [0, \pi/3]$) contains (nonempty parts of) level sets $\Lambda(h)$ for each $h > 1$, and each such level set contains points of arbitrarily large absolute value. (Let $z^{3/2}$ run on the imaginary axis first, and then continue in the real direction). For h sufficiently large, a connected component of such a level set that hits the right half plane (for $z^{3/2}$), will never leave it. This holds, because D is bounded on the imaginary axis $z^{3/2} \in i\mathbb{R}$.

For any meromorphic function f, the normals onto the level lines of $|f|$ show in direction $\overline{f'/f}$, which, in the case of $f = D$, is the direction parallel to $\mathrm{Ai}(z)\,\mathrm{Bi}(z)$. We have seen in (5.5) that for $|z|$ large enough and z in the sector $\arg z \in [0, \pi/3]$, the argument of $\mathrm{Ai}(z)\,\mathrm{Bi}(z)$ lies in a small neighbourhood of $[-\pi/3, 0]$.

Now choose any $h > 1$ and any point p on $\Lambda(h)$ in the sector $\arg z \in [0, \pi/3]$ with sufficiently large imaginary part of z. The connected component L of p in $\Lambda(h)$ cannot be a closed curve, because then the normal in a presumed point of maximal imaginary part on this component would violate the direction estimate (5.5). Thus L has to be a curve diffeomorphic to $\mathbb{R}$ and can in particular not contain a subset homeomorphic to a circle.

As long as L remains inside the sector $\arg z \in [0, \pi - \varepsilon]$, the estimate (5.6) forces it to go to ∞ inside a strip with bounded $\mathrm{Re}\, z^{3/2}$ (and $\mathrm{Im}\, z \geq 0$). But both ends cannot go to ∞ in such a strip, because the normal vector would then have opposite orientations in points with large modulus, in contradiction to (5.5). Therefore, L has to leave that sector. We want to show that it leaves it through the real axis. Otherwise, the curve would have to escape into a sector where $\arg z \in [\pi - \varepsilon, \pi[$. This sector will however be dissected into slices due to the asymptotic estimate

$$D(z) \sim \frac{(\cos \tilde{\zeta})(1 + O(\tilde{\zeta}^{-1})) + (\sin \tilde{\zeta}) O(\tilde{\zeta}^{-1})}{(\sin \tilde{\zeta})(1 + O(\tilde{\zeta}^{-1})) - (\cos \tilde{\zeta}) O(\tilde{\zeta}^{-1})} , \quad (\tilde{\zeta} = 2/3(-z)^{3/2} + \pi/4) . \tag{5.7}$$

(This happens to be almost the same as (5.2), but some of the O-terms represent different expressions. The reasoning is just like after (5.2).)

On the lines $\tilde{\zeta} = (n+1/2)\pi + it$, we have $|D(z)| = \tanh t + O((\tilde{\zeta})^{-1}) < 1 + O(z^{-3/2})$. So they are obstacles for L. Therefore, L has to hit the real line, and this makes it identical to its complex conjugate. But it cannot hit $\mathbb{R}$ more than once, because then it would contain a topological circle.

We have therefore constructed, for any $h > 1$, a point $a + bi$ with b large enough (depending on h) such that $|D(a + bi)| = h$, and a mapping from the half line $[a, \infty[+ bi$ into the real axis. This mapping is defined by following the level set of $|D|$ through $x + bi$ until it hits the real axis. This mapping is injective, because the level set will never return to $[a, \infty[+ bi$, again by (5.5). It is continuous, because following the level set means to follow the gradient flow for $\arg D(z)$. The image of this mapping is therefore an interval $[x, \infty[$ with $D(x) = h$ (and x the largest real number with this property). The same fibration into level lines parametrized over $[x, \infty[$ would have been obtained with a larger b, so we may now let $h \searrow 1$ and take the union. This completes the construction of the curves $\tilde{\Lambda}(x)$. The confinement of $\operatorname{Re} z^{3/2}$ implied by (5.6) proves their asymptotic behaviour.

It is clear that for $x' > x$, $\tilde{\Lambda}(x')$ lies to the right of $\tilde{\Lambda}(x)$. No other level line but $\tilde{\Lambda}(x')$ can lie to the right of $\tilde{\Lambda}(x)$: Circles are impossible, because any compact set to the right of $\tilde{\Lambda}(x)$ is swept by the gradient flow for $\arg D$ between a segment $[a, \infty[+ bi$ (with b large enough and $a + bi \in \tilde{\Lambda}(x)$) and $[x, \infty[$. Topological lines different from the $\tilde{\Lambda}(x')$ are impossible, because both ends would have to go to infinity in such a way that $\operatorname{Im} z \to +\infty$, and the argument with opposite orientations of the normals contradicting (5.5) works again.

In order to finally show that $|D| > 1$ in the sector $\arg z \in]-\pi/3, \pi/3[$, we use the equation (3.10), i.e.,

$$\operatorname{Bi}(z) - i \operatorname{Ai}(z) = 2e^{-\pi i/6} \operatorname{Ai}(ze^{-2\pi i/3}) \ . \tag{5.8}$$

With $z = re^{i\pi/3}$, we find $D(re^{i\pi/3}) = i + 2e^{-i\pi/6} \frac{\operatorname{Ai}(re^{-i\pi/3})}{\operatorname{Ai}(re^{i\pi/3})}$. Clearly, for real r, the quotient runs on the unit circle. It is therefore clear that $|D(z)| \in [1, 3]$ for $z \in e^{i\pi/3}[0, \infty[$. The minimum is taken on when $D(e^{i\pi/3}r) = -i$, i.e., using (3.11) with the lower sign, exactly for $r = a_{-n}$. The maximum is taken on when $D(e^{i\pi/3}r) = 3i$, i.e., for $r = b_{-n}$. Here, (3.11) is used with $z = -r$, upper sign, and with $z = re^{i\pi/3}$, lower sign, and $\operatorname{Ai}(-r)$ is eliminated, to obtain $D(re^{i\pi/3}) = 3i + 2e^{-i\pi/3} \operatorname{Bi}(-r)/\operatorname{Ai}(re^{i\pi/3})$.

If we had $|D(z)| \leq 1$ at any point in the sector $|\arg z| < \pi/3$, then by the open mapping theorem, there would also be a point with $|D(z)| < 1$ in that sector. Its level line could not exit the sector, would be compact because of the estimate (5.6), and then, by the argument principle, its interior would have to contain a zero or a pole of D. But this is not the case, according to Lemmas 5.2 and 5.5.

Moreover, from (3.11), we have $2|\operatorname{Ai}(re^{\pm 2\pi i/3})| < |\operatorname{Bi}(r)|$. On the other hand, from (3.8), i.e.,

$$\operatorname{Bi}(z) + e^{2\pi i/3} \operatorname{Bi}(ze^{2\pi i/3}) + e^{-2\pi i/3} \operatorname{Bi}(ze^{-2\pi i/3}) = 0$$

we get $2|\operatorname{Bi}(re^{2\pi i/3})| > |2 \operatorname{Re}(e^{2\pi i/3} \operatorname{Bi}(ze^{2\pi i/3}))| = |\operatorname{Bi}(r)|$. This shows $|D| > 1$ on the rays $\arg z = \pm 2\pi/3$.

We have already seen after (5.4) that $\arg(\operatorname{Ai} \operatorname{Bi})(z) \in [-\pi/3, 0]$ on the ray $\arg z = \pi/3$ (and trivially on the ray $\arg z = 0$, too). arg is an open mapping (maximum principle for the harmonic function $z \mapsto \operatorname{Im} \log(\operatorname{Ai}(z) \operatorname{Bi}(z))$), so it maps the sector $\arg z \in]0, \pi/3[$ onto an open interval whose closure contains $-\pi/3$ and 0. If the image contained any point $\alpha \notin [-\pi/3, 0]$, the level line of constant $\arg \operatorname{Ai} \operatorname{Bi} = \alpha$, could not leave the sector through its boundary, so, being noncompact by the maximum principle, it would have to go to infinity in this sector. However, by (5.5), as $z \to \infty$, $\arg \operatorname{Ai}(z) \operatorname{Bi}(z)$ must accumulate in $[-\pi/3, 0]$. This contradiction proves the last claim in the lemma. ■

Lemma 5.8 *The set $\Lambda(1) := \{z \mid |D(z)| = 1\}$ is decomposed in the following connected components:*

$$\Lambda(1) = \bigcup_{n=0}^{\infty} S_{-n} \cup \bigcup_{n=1}^{\infty} S_n \cup \bigcup_{n=1}^{\infty} \bar{S}_n ,$$

which are all topological lines and different from each other. For $n \leq 0$, S_n crosses the negative real axis (and is invariant under complex conjugation), whereas for $n \geq 1$, S_n passes through $e^{i\pi/3}(-a_{-n})$ and lies in the upper half plane. $\bar{S}_n$ is its complex conjugate. Here, a_{-n} are the zeros of Ai *given in Lemma 5.2.*

All these lines are asymptotic to $\arg z = \pm 2\pi/3$ (i.e., $|\operatorname{Im} z^{3/2}|$ is bounded on them as $|z| \to \infty$).

Proof:
Using (3.11) again, one sees that the solutions of the equation $D(z) = -i$ are exactly the $z_n := e^{i\pi/3}(-a_{-n})$ mentioned in the lemma. The $\bar{z}_n$ are the solutions of $D(z) = i$.

There is a decreasing sequence $p_0 > p_{-1} > p_{-2} > \ldots \to -\infty$ of real numbers, where $D(z) = \pm 1$, namely $b_{-n-1} < p_{-2n} < a_{-n}$, $D(p_{-2n}) = 1$, and $a_{-n} < p_{-2n+1} < b_{-n}$, $D(p_{-2n+1}) = -1$.

For $n \geq 1$, we denote by S_n the connected component of $\Lambda(1)$ containing z_n, by $\bar{S}_n$ the component containing $\bar{z}_n$, and for $n \geq 0$ by S_{-n} the component of p_{-n}. (We do not know yet whether these are all different or whether there are still other components.)

S_0 cannot be a circle, because otherwise the gradient flow for $|D|$ would deform it into a circle within $\Lambda(h)$ for small $h > 1$ and intersecting $]p_0, \infty[$. But we have seen in the Lemma 5.7 that the level lines intersecting $]p_0, \infty[$ are not compact. Therefore S_0 is a topological line and cannot cross the real axis anywhere else but in p_0. As $D'(p_0) > 0$, conformity shows that $\operatorname{Im} D(z) > 0$ near p_0 on $S_0 \cap \{\operatorname{Im} z > 0\}$. S_0 is not connected with any of the S_n, $\bar{S}_n$ for $n > 0$, because the rays $\arg z = \pm 2\pi/3$ (on which $|D| > 1$) form a barrier. All S_n with $n < 0$ are to the left of S_0, the S_n, $\bar{S}_n$ with $n > 0$ are to its right.

If any component of $\Lambda(1)$ is a topological circle, D must take on both the values $+i$ and $-i$ on it. As no such values are taken on in the left piece, the S_n for $n < 0$ are topological lines. For $n > 0$, the S_n and $\bar{S}_n$ cannot intersect $S_0 \cup [p_0, \infty[$ and are therefore confined to the upper / lower half plane respectively. These S_n also have to be pairwise different topological lines, because otherwise they would have to contain points satisfying $D(z) = +i$; similarly for $\bar{S}_n$.

Before showing that there are no other components of $\Lambda(1)$, we determine the asymptotic directions of the S_n. Equation (3.11), used for $ze^{i\pi/3}$ instead of z, implies

$$D(ze^{i\pi/3}) - i = 2e^{-i\pi/6}\frac{\operatorname{Ai}(ze^{-i\pi/3})}{\operatorname{Ai}(ze^{i\pi/3})} \sim 2\exp\Big[\frac{4i}{3}z^{3/2}\Big]\left(1 + O(z^{-3/2})\right)$$

in the sector $0 \leq \arg z < 2\pi/3$, of which only $0 \leq \arg z < \pi/3$ is relevant for the S_n $(n > 0)$. If $\operatorname{Im} \frac{4i}{3} z^{3/2} \to \infty$, $\arg(D(z) - i)$ grows indefinitely, so $D(z)$ winds around i infinitely often. This cannot happen on a curve on which $|D(z)| \equiv 1$. Therefore, each S_n $(n > 0)$, as well as any other possible component of $\Lambda(1)$ inside the sector $\pi/3 < \arg z < 2\pi/3$, is asymptotic to the line $\arg z = 2\pi/3$.

Similarly, we obtain

$$\frac{2}{D(-ze^{-i\pi/3}) - i} = e^{i\pi/6}\frac{\operatorname{Ai}(-ze^{-i\pi/3})}{\operatorname{Ai}(z)} \sim \exp\Big[\frac{4}{3}z^{3/2}\Big]\left(1 + O(z^{-3/2})\right) + i\left(1 + O(z^{-3/2})\right)$$

in the sector $-\pi/3 < \arg z \leq \pi/3$, of which only $0 < \arg z \leq \pi/3$ is relevant for S_n $(n \leq 0)$. Rewriting this for the sector $\arg z \in]2\pi/3, \pi]$, we get $\frac{2}{D(z)-i} \sim \exp\Big[-\frac{4}{3}z^{3/2}\Big](1+O(z^{-3/2}))+i(1+$

$O(z^{-3/2})$) there. If $\operatorname{Im}\frac{4}{3}z^{3/2} \to \infty$ along S_n, then $\arg(-i+2/(D(z)-i))$ grows indefinitely (the absolute value of the exponential is always ≥ 1), so $2/(D(z)-i)$ winds around i infinitely often. This cannot happen on a curve on which $\operatorname{Im} 2/(D(z)-i) \equiv 1$ (i.e., $|D(z)| \equiv 1$). Therefore, each S_n ($n \leq 0$), as well as any other possible component of $\Lambda(1)$ inside the sector $2\pi/3 < \arg z < \pi$, is asymptotic to the line $\arg z = 2\pi/3$.

We may consider each of S_n ($n > 0$), $\bar{S}_n$ ($n > 0$), $S_n \cup S_{n-1}$ ($n \leq 0$) as closed curves on the Riemann sphere. For each of them, we define an "interior": For $n > 0$, the interior I_n of S_n is the connected component of its complement that does not contain the sector $|\arg z| < \pi/3$. $\bar{I}_n$ is defined analogously. For $n \leq 0$, the interior I_n of $S_n \cup S_{n-1}$ is defined as the connected component of its complement that contains the interval $]p_{n-1}, p_n[$. Neither I contains any of the curves S discussed so far.

Each of these interiors I contains either exactly one zero or exactly one pole $\hat{z}$ of D, by the argument principle. The argument is controlled on these lines by their behaviour near ∞ and the number of values $D(z) = \pm i$ lying on them. Closing ∂I near ∞ does not contribute to the change of argument along ∂I, because $D(z) \to \pm i$ uniformly as $z \to \infty$ inside $I \cap \{\operatorname{Im} z \gtrless 0\}$. For I_n, $\bar{I}_n$ with $n \geq 1$, there is no choice: They must contain the zeros b_n, $\bar{b}_n$ respectively, with matching subscripts.

Assume any of these I contains another component S^* of $\Lambda(1)$ different from the ones listed in the beginning. There would have to be a zero or pole of D in its interior by the argument principle again. It must be the same (unique) $\hat{z} \in I$. The exterior of S^* within I must contain compact level lines for some levels $h \neq 1$, because, by (5.6), $|D(z)| \to 1$ as $z \to \infty$ in I. These lines would have to contain another zero or pole. Contradiction! A similar argument shows that there cannot be another level line in the remaining domain, which is the connected component of $\mathbb{R}^+$ in the complement of $\Lambda(1)$. This proves the lemma. ■

Let us now collect the results to describe the level lines of $|D|$ in full detail:

Theorem 5.9 *For $D(z) := \operatorname{Bi}(z)/\operatorname{Ai}(z)$, the level sets $\Lambda(h)$ of $|D|$ are as follows:*

- *For $0 < h < 1$, $\Lambda(h) = \bigcup_{j=1}^{\infty} L_{-j}(h) \cup \bigcup_{j=1}^{\infty} L_j(h) \cup \bigcup_{j=1}^{\infty} \bar{L}_j(h)$, where $L_{-j}(h)$, $L_j(h)$, $\bar{L}_j(h)$ are topological circles containing b_{-j}, b_j and $\bar{b}_j$ respectively in their interior.*

- *For $h = 1$, $\Lambda(1) = \bigcup_{j=0}^{\infty} S_{-j} \cup \bigcup_{j=1}^{\infty} S_j \cup \bigcup_{j=1}^{\infty} \bar{S}_j$, where S_{-j} are topological lines passing through p_{-j} and S_j, $\bar{S}_j$ are topological lines in the upper / lower half plane passing through $e^{\pm i\pi/3}(-a_{-j})$ respectively.*

- *For $h > 1$, $\Lambda(h) = \bigcup_{j=1}^{\infty} L_{-j}(h) \cup L_1(h)$, where the $L_{-j}(h)$ are topological circles containing a_{-j} in their interior and $L_1(h)$ is a topological line intersecting the real axis between p_0 and $+\infty$.*

The following boundaries give the relative location:

$$\partial\Big(\bigcup_{h>1} L_1(h)\Big) = S_0 \cup \bigcup_{j=1}^{\infty} (S_j \cup \bar{S}_j) \qquad \partial\Big(\bigcup_{h>1} L_{-j}(h)\Big) = S_{-2j} \cup S_{-2j+1}$$

$$\partial\Big(\bigcup_{h<1} L_{-j}(h)\Big) = S_{-2j+1} \cup S_{-2j+2} \qquad \partial\Big(\bigcup_{h<1} L_j(h)\Big) = S_j$$

Moreover,

$$\bigcup_{h>1} L_1(h) \supset \{z \mid -\pi/3 < \arg z < \pi/3\} \supset \bigcup_{h>3} L_1(h)\ .$$

PROOF:
The claim about $\Lambda(1)$ is from Lemma 5.8. The $L_1(h)$ are the curves constructed in Lemma 5.7. Their union lies to the right of S_0, so $\partial(\bigcup L_1(h)) \subset S_0 \cup \bigcup_1^\infty (S_j \cup \bar{S}_j)$. This inclusion cannot be strict, because otherwise another component of $\Lambda(1)$ would have to exist.

There exist level sets $L_{-j}(h)$ for every $h > 1$ between S_{-2j} and S_{-2j+1}. Each such set crosses the real axis twice. They are all compact, because $D(z) \to \pm i$ as $z \to \infty$ between S_{-2j} and S_{-2j+1}. Therefore each $L_{-j}(h)$ must be a union of finitely many topological circles. But it can only be one circle each, because there is only one pole b_{-j} and no zero of D to fall in the interior of each circle. A similar argument applies to the $L_{-j}(h)$ for $h < 1$, as well as to the $L_j(h)$ and $\bar{L}_j(h)$ for $h < 1$.

These level lines fill the whole plane (except zeros and poles of D), hence there are no other level lines. The corresponding information about boundaries is immediate.

We have seen right after (5.8) that $|D| \leq 3$ on the rays $\arg z = \pm\pi/3$. This confines all lines $L_1(h)$ with $h > 3$ to the sector $|\arg z| < \pi/3$. On the other hand $\partial(\bigcup_{h>1} L_1(h))$ consists of curves in the complement of that sector: only the S_j, $\bar{S}_j$ meet its boundary at $e^{i\pi/3}(-a_{-j})$, $e^{-i\pi/3}(-a_{-j})$ respectively, and nowhere else, because these are the only points on that boundary where $|D| = 1$. Therefore $\bigcup_{h>1} L_1(h)$ covers the whole sector.

The level lines are shown in Figure 5.2. The corresponding image of the upper half plane under the mapping $z \mapsto D(z)$ is shown in Figure 5.3. ■

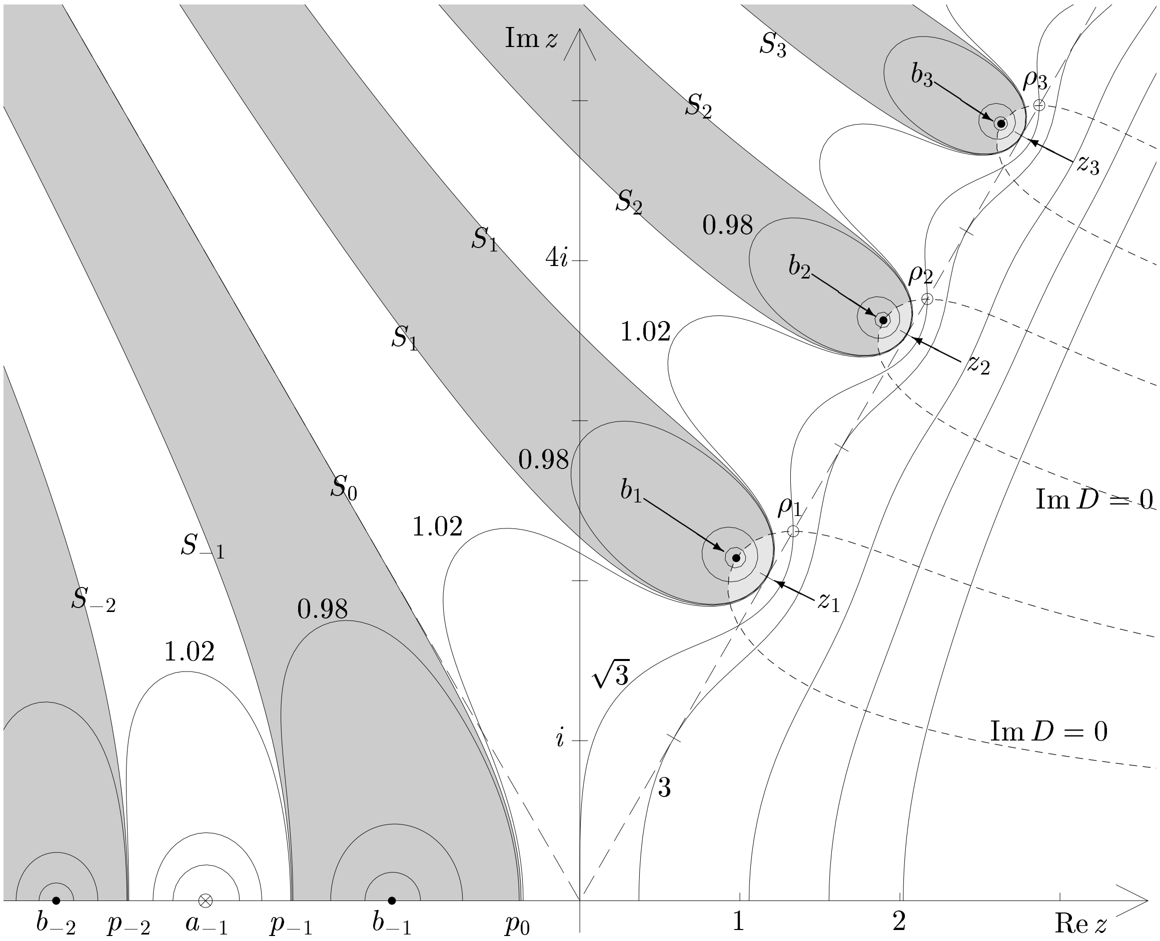
In order to find solutions $(\delta, t) \in \mathbb{C} \times \mathbb{R}^+$ to the equation $D(\delta + it) = D(\delta - it)$, we note that, from Lemma 5.3, a necessary condition is $\operatorname{Re}\delta > 0$ and $t > |\operatorname{Im}\delta|$. We distinguish two cases.

First let δ be real. Then the equation $D(\delta + it) = D(\delta - it)$ is equivalent to the condition that $D(\delta + it)$ is real. These solutions can readily be obtained from the level line portrait in Figure 5.2. Through each zero of D (they are simple) there passes exactly one curve $\operatorname{Im} D = 0$. Ignoring the obvious case of $z = \delta + it$ real, $t = 0$, we obtain the dotted curves $\tilde{R}_j$ passing through b_j $(j \geq 1)$. (The tilde in $\tilde{R}_j$ is merely chosen for conformity with future notation.)

As $\operatorname{Re} D(z)$ is monotonic on any curve $\operatorname{Im} D(z) \equiv 0$, such a curve (topologically a line, not a circle, by the maximum principle for the harmonic function $\operatorname{Im} D$) can cross $\Lambda(1)$ at most twice. Being also confined to $\{\operatorname{Re} z \geq 0\}$, it must go to infinity inside the sector $0 < \arg z < \pi/3$. Therefore, $\arg D(z) \to \infty$ and $\operatorname{Re} D(z)$ runs from $-\infty$ to $+\infty$ on such a curve. Therefore, each component of $\{\operatorname{Im} D(z) = 0\}$ must contain a zero of D, i.e., there are no other real solution curves beyond the ones dotted in Figure 5.2.

A curve of non-real solutions may bifurcate from $\tilde{R}_j$ only if $D'(\delta+it) = D'(\delta-it)$. Otherwise t is a local parameter for the solution continuum by the implicit function theorem. Possible bifurcation points (i.e., critical points) can <u>only</u> occur on the real solutions according to Lemma 5.3. We now show that there is indeed exactly one critical point on each of the $\tilde{R}_j$. It is at the point of $\tilde{R}_j$ where t is maximal.

The condition $D'(\delta + it) = D'(\delta - it)$ means that $\tilde{R}_j$ has a horizontal tangent and can locally be parametrized by δ: $t = t(\delta)$, $t'(\delta) = 0$. Then implicit differentiation (together with the condition $\operatorname{Ai}(\delta+it)^2 = \operatorname{Ai}(\delta-it)^2$ at a branch point) immediately yields $t''(\delta) = \operatorname{Im}(\operatorname{Ai}'(\delta+it)/\operatorname{Ai}(\delta+it))$. We have seen in Lemma 5.4 that this is always negative in the upper half plane. Therefore, each critical point on $\tilde{R}_j$ is a nondegenerate local maximum. This gives uniqueness. For existence, we show even more, namely that the two intersection points of $\tilde{R}_j$ with the critical line $\arg z = \pi/3$ satisfy $D(z) = \pm\sqrt{3}$ and one of them is a critical point.



Level lines of $|D|$ at levels 0.2, 0.5, 0.98; 1; 1.02, $\sqrt{3}$, 3, 10, 30, 100. The level lines at levels 1 and 3 are tangent to the critical line $\arg z = \pi/3$ (dashed). Among the orthogonal trajectories, we have dotted the lines $\operatorname{Im} D(z) = 0$. The slightly shaded area is the one, where $|D| < 1$. Compare Figure 5.3.

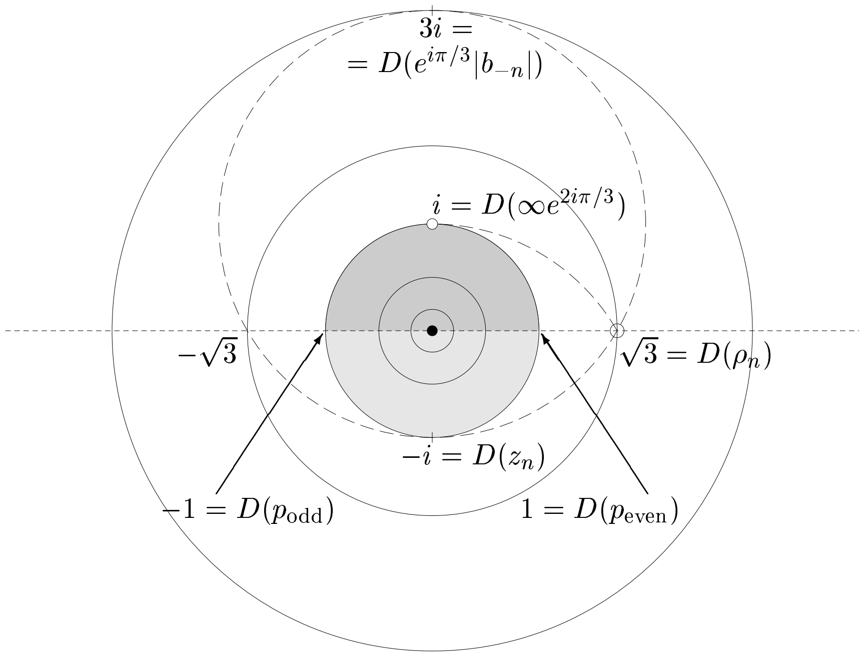
Figure 5.2: Level lines of $|D|$

Indeed, if for real r, $D(re^{i\pi/3}) = i + 2e^{-i\pi/6} \operatorname{Ai}(re^{-i\pi/3})/ \operatorname{Ai}(re^{i\pi/3}) \in \partial B_2(i)$ is real, then $D(re^{i\pi/3}) \in \{\sqrt{3}, -\sqrt{3}\}$ and $\arg \operatorname{Ai}(re^{i\pi/3}) = -\arg \operatorname{Ai}(re^{-i\pi/3}) \in \{0, -2\pi/3\}$. In the case $D = \sqrt{3}$, the tangential direction of $\tilde{R}_j$ (normal to $\Lambda(1)$) is, as near (5.5), $\operatorname{Ai}(re^{i\pi/3}) \operatorname{Bi}(re^{i\pi/3}) = \sqrt{3} \operatorname{Ai}(re^{i\pi/3})^2 > 0$. (In the case $D = -\sqrt{3}$, the direction is $e^{-i\pi/3}$.)

Analyticity guarantees that a bifurcation really occurs at the critical point. Near the critical point (δ_0, t_0), we have the equation $0 = D(\delta + it) - D(\delta - it) = a(t - t_0) + b(\delta - \delta_0)^2 + \text{hot}$, and the Weierstraß preparation theorem tells us that the local solution structure is diffeomorphic to the one of $a(t - t_0) + (\delta - \delta_0)^2 = 0$. More on this in Section 5.3.1.

Before discussing complex solution branches (i.e., δ non-real), we should clarify an ambiguity of notation: we have constructed the sets $\tilde{R}_j$ as subsets of the complex plane $\mathbb{C} \ni \delta + it$, but can (and should) rather consider them as subsets of $\mathbb{C} \times \mathbb{R} \ni (\delta, t)$. In Figure 5.4, the coordinate axes should be understood as coordinate axes of a complex plane, but as far as the $\tilde{R}_j$ are concerned, they may be viewed equally well as axes for $\operatorname{Re} \delta$ and t. For non-real branches, it is crucial to distinguish these two points of view, and from now on, we will understand $\tilde{R}_j \in \mathbb{C} \times \mathbb{R}$, whereas the corresponding set in $\mathbb{C}$ will be called $(\tilde{R}_j)_\pm$ for reasons that will become clear as soon as the distinction is actually used.

Any solution continuum $\hat{C} \subset \mathbb{C} \times \mathbb{R}^+$ to $D(\delta + it) = D(\delta - it)$ can, except for the bifurcation points, whose totality has just been exhibited, be locally parametrized by t. This parametrization is even



Compare Figure 5.2

Figure 5.3: The image under D of the upper half plane

global between the bifurcation points, as any failure of monotonicity of t along the continuum gives rise to a bifurcation point, due to the analytic dependence on $\delta \in \mathbb{C}$ and the Weierstraß preparation theorem. Let us denote a curve within the solution continuum, delimited by branch points, an arc. By the argument just given, any non-real arc C will therefore be non-compact, globally parametrized either over $]t_0, t_+[$ (if it bifurcates from a real branch at level t_0), or over $]t_-, t_+[$ (if it does not meet a real branch anywhere).

For any continuum $C \subset \{(\delta, t) \in \mathbb{C} \times \mathbb{R}^+ \mid D(\delta + it) = D(\delta - it)\}$, we define $C_+ := \{\delta + it \mid (\delta, t) \in C\} \subset \mathbb{C}$ and $C_- := \{\overline{\delta - it} \mid (\delta, t) \in C\} \subset \mathbb{C}$; both are in the upper half plane by Lemma 5.3. For the real branches $C = \tilde{R}_j$, $(\tilde{R}_j)_+$ and $(\tilde{R}_j)_-$ coincide into the single $(\tilde{R}_j)_\pm$, which has been announced before. For no arc C (unless it is part of some $\tilde{R}_j$) can either of the sets C_+ or C_- meet any $(\tilde{R}_j)_\pm$, except at the branch points, because to any $z_+ = \delta + it \in C_+$ there corresponds a $z_- = \bar{\delta} + it \in C_-$ such that $D(z_+) = D(\overline{z_-}) = \overline{D(z_-)}$ and $\operatorname{Re} z_+ = \operatorname{Re} z_-$. On the $(\tilde{R}_j)_\pm$, the first equation reduces to $D(z_+) = D(z_-)$, and Lemma 5.3 yields $z_+ = z_-$, hence δ is real, and thus C branches off from a real branch.

So the sets C_+ and C_- come in pairs, one of which lies in the domain $\{z \mid \operatorname{Im} D(z) > 0\}$ ("outside" the dotted curves in Figure 5.2), the other in $\{z \mid \operatorname{Im} D(z) < 0\}$ ("inside" one of these curves). With no loss of generality, we now assume $\operatorname{Im} \delta > 0$.

For an arc C, bifurcating from a real branch or not meeting it, we want to show that $t_+ = \infty$. If t_+ were finite, so would be $\operatorname{Im} \delta$ ($|\operatorname{Im} \delta| < t$ from Lemma 5.3). Therefore $\operatorname{Re} \delta \to \infty$ as $t \nearrow t_+$, by non-compactness of the arc. But then, $|D| \to \infty$ on C. This is however prohibited for non-real arcs by the fact that for $h > 3$ the $L_1(h)$ are graphs in the upper half plane (Lemma 5.7), so that $|D|(z_+) = |\overline{D(z_-)}| > 3$ implies $z_+ = z_-$. This contradiction shows $t_+ = \infty$.

Because we are considering the case $\operatorname{Im} \delta > 0$ now, we get $C_+ \subset \{z \mid \operatorname{Im} D(z) > 0\}$ and $C_- \subset \{z \mid \operatorname{Im} D(z) < 0\}$, as the opposite assumption would lead to a contradiction as $t \to \infty$. Consequently, $\operatorname{Im} \overline{\delta - it} = t - \operatorname{Im} \delta$ is bounded. If C does not meet a real branch, it is closed (but not compact), so it must go to infinity in $\mathbb{C} \times \mathbb{R}^+$ also if $t \searrow t_-$. Again a contradiction follows via $\operatorname{Re} \delta \to \infty$ and the graph property of the curves $L_1(h)$ for $h > 3$. This contradiction shows that there are no non-real branches except for the ones bifurcating from the real branches

$\tilde{R}_j$. We call these branches $\tilde{C}_j$. For them, too, $|D|$ is bounded on $(\tilde{C}_j)_-$, so $\operatorname{Re}\delta$ is bounded, whereas $t \to \infty$, and with it, $\operatorname{Im}\delta \to \infty$. We conclude $D(\delta + it) \to i$ and $D(\delta - it) \to -i$, hence $\operatorname{Re}\delta \to \frac{1}{2}|a_{-j}|$ and $t - \operatorname{Im}\delta \to \sqrt{3}\,|a_{-j}|/2$.

We collect these results in our following theorem and a corresponding figure:

Theorem 5.10 *Let $D(z) = \operatorname{Bi}(z)/\operatorname{Ai}(z)$. The set $\mathcal{Z}_\infty$ of solutions $(\delta, t) \in \mathbb{C} \times \mathbb{R}$ to $D(\delta + it) = D(\delta - it)$ consists of the following components (see Figure 5.4):*

- *The trivial solutions $t = 0$, δ arbitrary.*
- *A family of real solution branches $\tilde{R}_j$, $j \geq 1$ with $t > 0$, $\delta \in \mathbb{R}^+$. Each $\tilde{R}_j$ contains exactly one bifurcation point $\boldsymbol{r}_j = (\operatorname{Re}\rho_j, \operatorname{Im}\rho_j)$ and is cut by this bifurcation point in two pieces R_{j0} and R_{j1}, each parametrizable as $R_{j0|1} = \{(\delta_{j0|1}(t), t) \mid 0 < t < \operatorname{Im}\rho_j\}$. It holds $\lim_{t\searrow 0} \delta_{j0}(t)^{1/2} t = (j - 1/2)\pi$ and $\lim_{t\searrow 0} \delta_{j1}(t)^{1/2} t = j\pi$.*

  Add to this the reflected branches $\{(\delta, -t) \mid (\delta, t) \in \tilde{R}_j\}$.
- *A corresponding family of complex solution branches $\tilde{C}_j$, each coming in pairs quadruples (δ, t), $(\bar{\delta}, t)$. $(\bar{\delta}, -t)$. $\tilde{C}_j$ bifurcates from $\tilde{R}_j$ at $\boldsymbol{r}_j$ and can be parametrized as $\tilde{C}_j = \{(\delta_j(t), t), (\overline{\delta_j(t)}, t) \mid \operatorname{Im}\rho_j < t < \infty\}$. It holds $\lim_{t\to\infty} \delta_j(t) - it = e^{-i\pi/3}|a_{-j}|$ with $a_{-j} < 0$ the zeros of* Ai.

  Again, add to this the reflected branches $\{(\delta, -t) \mid (\delta, t) \in \tilde{C}_j\}$.

The complex numbers ρ_j defining the position of the branch point $\boldsymbol{r}_j$ are the solutions to the equation $D(z) = \sqrt{3}$ and lie on the ray $\arg\rho = \pi/3$.

5.3 The Large Viscosity Case

5.3.1 Overview, Notation

The purpose of this section is to study, in analogy with (5.1), the set

$$\mathcal{Z}_\varepsilon := \{(\delta, t) \in \mathbb{C} \times \mathbb{R} \mid \Delta(\delta, t, \varepsilon) = 0\} \tag{5.9}$$

It is already clear from (4.53) that, for small t, one has $\Delta(\delta, t, \varepsilon) = 16i\varepsilon t^4(W/3 + O(t^2))$. Therefore

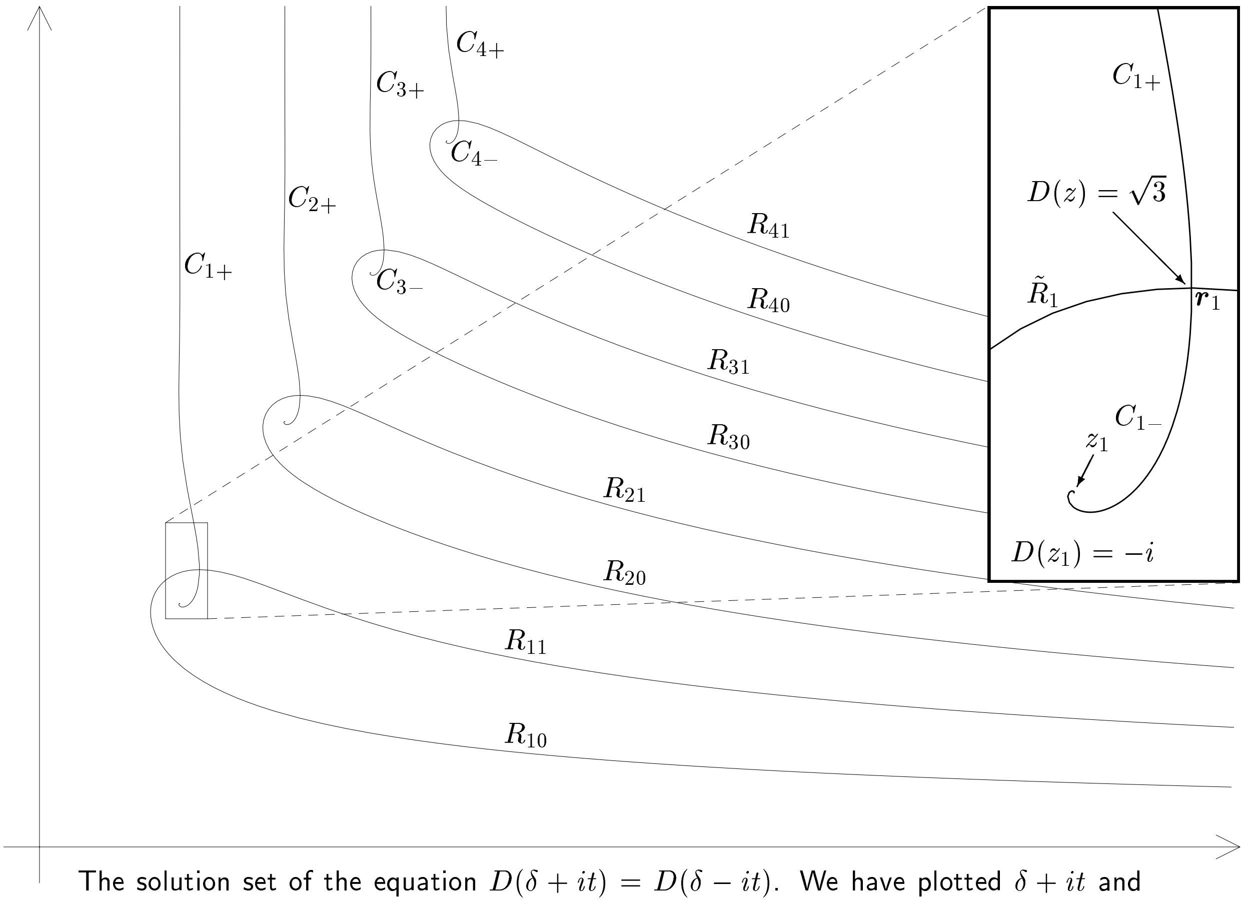
$$\mathcal{Z}_\varepsilon = (\mathbb{C} \times \{0\}) \cup \mathcal{Z}_\varepsilon^+ \cup \mathcal{Z}_\varepsilon^- \quad \text{with} \quad \mathcal{Z}_\varepsilon^\pm = \mathcal{Z}_\varepsilon \cap \{t \gtrless 0\}$$

consists of at least three connected components, the first of which, $\mathbb{C} \times \{0\}$, is trivial. Clearly, $\mathcal{Z}_\varepsilon^+$ and $\mathcal{Z}_\varepsilon^-$ are transformed into each other by $t \leftrightarrow -t$, see (2.1). Therefore, we need only discuss $\mathcal{Z}_\varepsilon^+$.

For any $(\delta_0, t_0) \in \mathcal{Z}_\varepsilon^+$, if $\partial_\delta \Delta(\delta_0, t_0, \varepsilon) \neq 0$, $\mathcal{Z}_\varepsilon^+$ is locally near (δ_0, t_0) a smooth curve parametrized by t. But even without this assumption, complex solution branches issue from the (higher order) zero $\delta = \delta_0$ of $\Delta(\cdot, t_0)$ for real t near t_0 (on either side of t_0). This is an immediate consequence of the Weierstraß preparation theorem, which permits us to write, unless $\Delta(\cdot, t_0) \equiv 0$,

$$\Delta(\delta, t) = \Big((\delta - \delta_0)^m + c_1(t)(\delta - \delta_0)^{m-1} + \ldots + c_m(t)\Big)\, g(\delta, t)$$

with g nonvanishing near (δ_0, t_0), $m \geq 0$ integer, c_i holomorphic functions, $c_i(t_0) = 0$. (The other possibility $\Delta(\cdot, t_0) \equiv 0$ will be excluded in a moment.) Here, m is the order to which $\Delta(\cdot, t_0)$ vanishes at δ_0. See any book on several complex variables for the Weierstraß preparation theorem, e.g., [3], [7].



The solution set of the equation $D(\delta + it) = D(\delta - it)$. We have plotted $\delta + it$ and $\overline{\delta - it}$ in the rectangle $\operatorname{Re} z \in [0, 10]$, $\operatorname{Im} z \in [0, 7]$ for the first four real and complex branches. This gives an undistorted picture of the real branches $\tilde{R}_j \ni (\delta, t)$. The shape of the complex branches $\tilde{C}_j \subset \mathbb{C} \times \mathbb{R}$, here represented by the curves $C_{j+} \cup C_{j-}$, is approximately like a hyperbola whose plane of reflection symmetry is the plotting plane here.

Figure 5.4: Solutions of $D(\delta + it) = D(\delta - it)$

Therefore $\mathcal{Z}_\varepsilon^+$ consists of smooth curves, which may meet at certain branch points. For each connected component of $\mathcal{Z}_\varepsilon^+$, its projection onto the t–axis is an open interval. When we follow such a connected component in the direction of decreasing t ($t \searrow t_* \geq 0$), making arbitrary choices at branch points, $|\delta|$ must tend to ∞, because $\mathcal{Z}_\varepsilon^+$ is also closed. Therefore, each connected component of $\mathcal{Z}_\varepsilon^+$ contains (at least) one of the solutions constructed in Theorem 4.8. Following the solution set along constant t, if that had been possible, would also have led into the domain covered by Theorem 4.8, where, however, such solutions have already been excluded.

Each connected component X_j of $\mathcal{Z}_\varepsilon^+$ can be further decomposed into arcs, i.e., the (bounded or unbounded) open simple curves that are the connected components of $X_j \setminus$ all branch points. By a branch of X_j, we denote a maximal smooth curve in X_j that is the closure of a union of some arcs. Having made a formal definition of the notion of a branch, we should point out that it applies to the technical sections following, but has still been used in an informal sense (now called arc) in Chapter 0.

In Theorem 4.8, we have constructed arcs that were characterized by their asymptotic behaviour $\delta_n(t) \sim \omega_n / 2\sqrt{t}$ ($n \geq 0$) as $t \searrow 0$. We call these arcs R_{k0} for $n = 2k$ and R_{k1} for $n = 2k + 1$, thus stressing their analogy with the case $\varepsilon = \infty$ covered in Theorem 5.10. Note however, that the correspondence between the cases $\varepsilon < \infty$ and $\varepsilon = \infty$ is given by the relative ordering of branches, not by their exact asymptotic behaviour: For $\varepsilon = \infty$, it holds $2\delta_{k0}(t)\sqrt{t} \to (2k-1)\pi \neq \omega_{2k} = 2k\pi$, and $2\delta_{k1}(t)\sqrt{t} \to (2k)\pi \neq \omega_{2k+1} = (2k+1)\pi - O(k^{-1})$.

This is, why in Lemma 5.11 below, a lower bound for t has to be assumed, and it is a consequence of the fact that in, say, $\cosh(4\varepsilon t^3)$ in (2.1), the limits $\varepsilon \to \infty$ and $t \to 0$ do not commute.

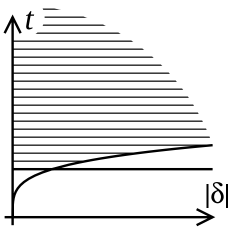
Given any arc S, we denote by $\tilde{S}$ the branch to which it belongs, and given any arc S or branch $\tilde{S}$, by $\hat{S}$ its connected component in $\mathcal{Z}_\varepsilon^+$. (The shape of the diacritics should vaguely symbolize the meaning: tilde for branches, hats for connected components.)

One possible worry may be disposed of from the beginning: the ad-hoc definition of a branch does not carry far in subtle situations of higher degeneracy of a branch point, but it does not need to. E.g., for the equation $\delta^2 - t^3 = 0$, the solution set would be classified into two branches, if "smooth" means C^1, but into four (diffeomorphic to $[0,\infty[$), if "smooth" means C^2 or better, whereas examples like $\delta^{2k} - t^2 = 0$ show that C^k is not sufficient to make branches unique. These worries are of no concern here, because our notion of a branch serves only as a label for such unions of arcs which "definitely belong together", in situations where $\partial_\delta^2 \Delta \neq 0 \neq \partial_t \Delta$, or else, $\partial_\delta \Delta = 0 = \partial_t \Delta$ with nondegenerate Hessian.

5.3.2 The Key Asymptotic Estimate

In order to describe the zero set outside the domain covered by Theorem 4.8, we prove the following lemma:

Lemma 5.11



Let $\delta \in \mathbb{C}$, $t > 0$ satisfy

$$t \geq C_1^{1/3} \quad \textit{and} \quad t \geq C^{1/4}|\delta|^{1/4} \tag{5.10}$$

for certain C, C_1. Assume $\varepsilon \geq c$ with $c := \sqrt{C_1^{-1} + C^{-1}}$. Then, in the specified domain, it holds, with a constant in the O-symbol that depends only on a lower bound for C, C_1:

$$\begin{aligned}\frac{2\Delta(\delta,t,\varepsilon)\,(4\varepsilon t^3)^2}{\exp(8\varepsilon t^3)} &= \mathrm{Ai}(\delta+it)\,\mathrm{Bi}(\delta-it) - \mathrm{Bi}(\delta+it)\,\mathrm{Ai}(\delta-it) \quad + \\ &+ \min\{1, |(\delta+it)(\delta-it)|^{-1/4}\}\, O\left(\frac{c}{\varepsilon}\right) \times \\ &\times \left\{\exp\frac{2}{3}\left[\left|\mathrm{Re}(\delta+it)^{3/2}\right| - \mathrm{Re}(\delta-it)^{3/2}\right] + \exp\frac{2}{3}\left[\left|\mathrm{Re}(\delta-it)^{3/2}\right| - \mathrm{Re}(\delta+it)^{3/2}\right]\right\}\end{aligned} \tag{5.11}$$

If both of $\arg(\delta \pm it)$ are restricted to the same sector $[-\pi/3 + a, \pi - a]$, or else to the same sector $[-\pi + a, \pi/3 - a]$, the absolute value signs may be dropped from the exponential error terms in the last line of (5.11).

The following formula, valid provided $\mathrm{Re}\,\delta \geq 0$, is more convenient for $\Delta(-\delta, t, \varepsilon)$:

$$\begin{aligned}&\frac{i\Delta(-\delta,t,\varepsilon)\,(4\varepsilon t^3)^2}{\exp(8\varepsilon t^3)} = \\ &= \mathrm{Ai}\Big((\delta+it)e^{i\pi/3}\Big)\,\mathrm{Ai}\Big((\delta-it)e^{-i\pi/3}\Big) - \mathrm{Ai}\Big((\delta+it)e^{-i\pi/3}\Big)\,\mathrm{Ai}\Big((\delta-it)e^{i\pi/3}\Big) \quad + \\ &+ \min\{1, |(\delta+it)(\delta-it)|^{-1/4}\}\, O\left(\frac{c}{\varepsilon}\right) \times \cosh\frac{2}{3}\,\mathrm{Im}\left\{(\delta+it)^{3/2} - (\delta-it)^{3/2}\right\}\end{aligned} \tag{5.12}$$

PROOF:
The assumptions (5.10) and the definition of c imply $(|\delta| + t)^{1/2} \leq ct^2$. Together with the

assumption $\varepsilon \geq c$, we obtain

$$4\varepsilon t^3 - t(|\delta| + t)^{1/2} \geq (4\varepsilon - c)\, t^3 \geq 3\varepsilon t^3 \; . \tag{5.13}$$

We use (2.2), i.e.,

$$\Delta(\delta, t, \varepsilon) := \frac{1}{2} \begin{vmatrix} \int_{-1}^{1} \mathrm{Ai}(\delta + itz)\exp(4\varepsilon t^3 z)\, dz & \int_{-1}^{1} \mathrm{Ai}(\delta + itz)\exp(-4\varepsilon t^3 z)\, dz \\ \int_{-1}^{1} \mathrm{Bi}(\delta + itz)\exp(4\varepsilon t^3 z)\, dz & \int_{-1}^{1} \mathrm{Bi}(\delta + itz)\exp(-4\varepsilon t^3 z)\, dz \end{vmatrix}$$

and obtain by means of integration by parts:

$$\begin{aligned} 4\varepsilon t^3 & \int_{-1}^{1} \mathrm{Bi}(\delta + itz)\exp(4\varepsilon t^3 z)\, dz = \\ & = \Big[\mathrm{Bi}(\delta + itz)\exp(4\varepsilon t^3 z)\Big]_{-1}^{1} - it \int_{-1}^{1} \mathrm{Bi}'(\delta + itz)\exp(4\varepsilon t^3 z)\, dz \end{aligned} \tag{5.14}$$

and therefore with the help of Lemma 3.1

$$\begin{aligned} 4\varepsilon t^3 & \int_{-1}^{1} \mathrm{Bi}(\delta + itz)\exp(4\varepsilon t^3 z)\, dz = \\ & = \mathrm{Bi}(\delta + it)\exp(4\varepsilon t^3) - \mathrm{Bi}(\delta - it)\exp(-4\varepsilon t^3) + \\ & \quad + O\left(t \max\{1, (|\delta| + t)^{1/4}\}\right) \int_{-1}^{1} \max\Big\{ \exp M_+(z), \exp M_-(z)\Big\}\, dz \end{aligned} \tag{5.15}$$

where

$$M_\pm(z) := 4\varepsilon t^3 z \pm \frac{2}{3}\,\mathrm{Re}(\delta + itz)^{3/2} \tag{5.16}$$

We estimate the exponents in the last integral of (5.15):

$$\frac{dM_\pm}{dz} = 4\varepsilon t^3 \pm \mathrm{Re}(it(\delta + itz)^{1/2}) = 4\varepsilon t^3 \mp t\,\mathrm{Im}(\delta + itz)^{1/2} \geq 4\varepsilon t^3 - t(|\delta| + t)^{1/2} \geq 3\varepsilon t^3 \tag{5.17}$$

Therefore

$$\int_{-1}^{1} \exp M_\pm(z)\, dz \leq \int_{-1}^{1} \exp[M_\pm(1) - (1 - z)\inf_z M_\pm']\, dz \leq \frac{\exp[4\varepsilon t^3 \pm \frac{2}{3}\,\mathrm{Re}(\delta + it)^{3/2}]}{3\varepsilon t^3} \tag{5.18}$$

For the term $\int_{-1}^{1} \mathrm{Bi}(\delta + itz)\exp(-4\varepsilon t^3 z)\, dz = \int_{-1}^{1} \mathrm{Bi}(\delta - itz)\exp(4\varepsilon t^3 z)\, dz$, we can use the same estimate with (formal) complex conjugation.

Equation (5.14) holds also for Ai instead of Bi. We estimate similarly as above:

$$\begin{aligned} 4\varepsilon t^3 & \int_{-1}^{1} \mathrm{Ai}(\delta + itz)\exp(4\varepsilon t^3 z)\, dz = \\ & = \mathrm{Ai}(\delta + it)\exp(4\varepsilon t^3) - \mathrm{Ai}(\delta - it)\exp(-4\varepsilon t^3) + \\ & \quad + O\left(t \max\{1, (|\delta| + t)^{1/4}\}\right) \int_{-1}^{1} \exp M_-(z)\, dz \end{aligned} \tag{5.19}$$

Here, we need to take care of the branch cut of $\delta + itz \mapsto (\delta + itz)^{3/2}$ when $\delta + itz < 0$, and we have to determine the sign according to Lemma 3.1. However, $\mathrm{Re}(\delta + itz)^{3/2}$ is still continuous at this cut, with a jump only in the first derivative, and the reasoning that led to (5.15) is not affected for (5.19).

Based on these estimates, we obtain:

$$\begin{aligned}
&\frac{2\Delta(\delta,t,\varepsilon)\,(4\varepsilon t^3)^2}{\exp(8\varepsilon t^3)} = \\
&\quad = \left\{\mathrm{Ai}(\delta+it) - \mathrm{Ai}(\delta-it)\exp(-8\varepsilon t^3) + O\left(\frac{\max\{1,(|\delta|+t)^{1/4}\}}{3\varepsilon t^2}\right)\exp\left[-\frac{2}{3}\mathrm{Re}(\delta+it)^{3/2}\right]\right\}\times \\
&\quad\times\left\{\mathrm{Bi}(\delta-it) - \mathrm{Bi}(\delta+it)\exp(-8\varepsilon t^3) + O\left(\frac{\max\{1,(|\delta|+t)^{1/4}\}}{3\varepsilon t^2}\right)\exp\left|\frac{2}{3}\mathrm{Re}(\delta-it)^{3/2}\right|\right\} \\
&\quad-\left\{\mathrm{Ai}(\delta-it) - \mathrm{Ai}(\delta+it)\exp(-8\varepsilon t^3) + O\left(\frac{\max\{1,(|\delta|+t)^{1/4}\}}{3\varepsilon t^2}\right)\exp\left[-\frac{2}{3}\mathrm{Re}(\delta-it)^{3/2}\right]\right\}\times \\
&\quad\times\left\{\mathrm{Bi}(\delta+it) - \mathrm{Bi}(\delta-it)\exp(-8\varepsilon t^3) + O\left(\frac{\max\{1,(|\delta|+t)^{1/4}\}}{3\varepsilon t^2}\right)\exp\left|\frac{2}{3}\mathrm{Re}(\delta+it)^{3/2}\right|\right\}
\end{aligned} \tag{5.20}$$

Under our conditions, those integrated terms containing $\exp(-8\varepsilon t^3)$ are also negligible: Indeed,

$$\begin{aligned}
&\exp\left[\left|\frac{2}{3}\mathrm{Re}(\delta-it)^{3/2}\right| - \left|\frac{2}{3}\mathrm{Re}(\delta+it)^{3/2}\right|\right] \\
&\qquad\le \exp\left[\frac{2}{3}\left|\mathrm{Re}\Big((\delta-it)^{3/2}-(\delta+it)^{3/2}\Big)\right|\right] \le \exp\left[\frac{2}{3}\left|(\delta-it)^{3/2}-(\delta+it)^{3/2}\right|\right] \\
&\qquad\le \exp\left[\frac{2}{3}\,2t\max_z\frac{3}{2}|(\delta+itz)^{1/2}|\right] \le \exp\left[2t(|\delta|+t)^{1/2}\right] \le \exp\left[2\,ct^3\right]\;.
\end{aligned} \tag{5.21}$$

We conclude that $|\mathrm{Bi}(\delta-it)|\exp(-8\varepsilon t^3) \le \exp[-6\varepsilon t^3]\exp|\frac{2}{3}\mathrm{Re}(\delta+it)^{3/2}|$, so this term can be absorbed in the O-term. The very same holds for the term $|\mathrm{Bi}(\delta+it)|\exp(-8\varepsilon t^3)$. For the corresponding Ai-terms, the absolute values are to be omitted (only) in the first line of estimate (5.21).

In order to bring the O-terms structurally closer to the main terms, we use

$$\begin{array}{rcl}
(|\delta|+t)^{1/4} \le & |\delta\pm it|^{-1/4}(|\delta|+t)^{1/2} & \le |\delta\pm it|^{-1/4}\,ct^2 \\
1 \le & |\delta\pm it|^{-1/4}\sqrt{c}\,t & \le |\delta\pm it|^{-1/4}\,ct^2/C_1^{1/12} \\
(|\delta|+t)^{1/4} \le & \sqrt{c}\,t & \le \qquad 1\cdot ct^2/C_1^{1/12} \\
1 \le & C_1^{-1/2}t^2/C_1^{1/6} & \le \qquad 1\cdot ct^2/C_1^{1/6}
\end{array}$$

Thus, $O\left(\max\{1,(|\delta|+t)^{1/4}\}/(3\varepsilon t^2)\right) = \min\{1,|\delta\pm it|^{-1/4}\}\,O(c/\varepsilon)$. With this processing, (5.20) yields formula (5.11) immediately.

The proofs of the modified (5.11) in the case $\arg(\delta\pm it)\in[-\pi/3+a,\pi-a]$ and of (5.12) are the same, with only a few modifications: For(5.12), one has to replace (2.2) by (2.3), thus selecting a different basis of Airy functions. Instead of (5.15), (5.19), we get the new Airy base functions and the error terms $\exp\tilde{M}_+$ and $\exp\tilde{M}_-$ respectively, where

$$\tilde{M}_\pm(z) := 4\varepsilon t^3 z \pm \frac{2}{3}\mathrm{Im}(\delta+itz)^{3/2}\;. \tag{5.22}$$

In (5.18) and (5.21), too, Re is replaced by Im, and the first line of (5.21) becomes obsolete. In (5.20), Re is replaced by $-\,\mathrm{Im}$ and the absolute value signs are omitted in the exponential error terms. This produces (5.12).

For the modified version of (5.11), one uses (2.4). ∎

5.3.3 Estimates for the Derivatives

We are going to need estimates like (5.11) also for the derivatives of Δ. The derivatives with respect to δ are rather simple, whereas the derivative with respect to t needs a bit of extra work.

Theorem 5.12 *Under the assumptions of Theorem 5.11 and with* $\mathbf{R}O(c/\varepsilon)$ *as an abbreviation for the complete error term in (5.11), it holds:*

$$\frac{2\partial_\delta^r \Delta(\delta,t,\varepsilon)\,(4\varepsilon t^3)^2}{\exp(8\varepsilon t^3)} = \partial_\delta^r\Big(\mathrm{Ai}_+\,\mathrm{Bi}_- - \mathrm{Bi}_+\,\mathrm{Ai}_-\Big) + (|\delta|+t)^{r/2}\mathbf{R}O\left(\frac{c}{\varepsilon}\right) \tag{5.23}$$

and

$$\frac{2(4\varepsilon t^3)^2\,\partial_\varepsilon\Delta(\delta,t,\varepsilon)}{\exp(8\varepsilon t^3)} = 2\Big(\mathrm{Ai}_+\,\mathrm{Bi}_- - \mathrm{Bi}_+\,\mathrm{Ai}_-\Big) + \mathbf{R}\,O\left(\frac{c}{\varepsilon}\right) \tag{5.24}$$

$$\frac{2(4\varepsilon t^3)^2}{\exp 8\varepsilon t^3}\,\partial_t\Delta(\delta,t,\varepsilon) = (6\varepsilon/t+\partial_t)(\mathrm{Ai}_+\,\mathrm{Bi}_- - \mathrm{Ai}_-\,\mathrm{Bi}_+) + \frac{(|\delta|+t)^{3/2}}{t}\mathbf{R}O(c/\varepsilon) \tag{5.25}$$

where $\mathrm{Ai}_\pm := \mathrm{Ai}(\delta \pm it)$

Before going into details of the proof, let us make a few general remarks:

Yes, you may take derivatives in an asymptotic series.

Of course, this claim is not generally true; but for asymptotic series arising from ordinary differential equations it is typically true. For example $\mathrm{Ai}'(z) = -\int_z^\infty \mathrm{Ai}(u)u\,du$, and with the asymptotic expansion for $\mathrm{Ai}(u)$ under the integral, this integral may be carried out term by term, and then $\int_\zeta^\infty y^{-k}\exp(-y)\,dy$ produces itself an asymptotic series. Rather than carrying this through, one can check that termwise differentiation of the asymptotic series in Section 3.3 does hold. Generally, when we talk about formal differentiation of an asymptotic series, we define the formal derivative $\frac{d}{d\zeta}O(\zeta^{-k}) := O(\zeta^{-k-1})$. With this definition, the formal derivative of asymptotic expansions of the Airy functions is the corresponding asymptotic expansion for the derivative of that Airy function. This holds for higher derivatives, too, because the ODE permits it.

When we want to do the same to asymptotic expansions of $\Delta(\delta,t,\varepsilon)$, the behaviour of derivatives with respect to t and ε seems not so easily controllable a-priori, but derivatives with respect to δ merely produce the corresponding derivatives of Airy functions; therefore the derivative of, say (5.11), with respect to δ can be taken easily, and arbitrarily often so. Unlike power series in increasing powers, it is not necessary to compute the asymptotic series to higher orders in order to take higher derivatives. To the contrary, in the leading order, it is only the exponentials that are affected by the differentiation, whereas the derivatives of the O-terms immediately vanish into the lower orders.

PROOF OF THEOREM 5.12:
Let us now go through the genesis of formula (5.11). (Formal) derivatives with respect to δ may be taken in (5.14)–(5.15), and all they produce is powers of $(\delta \pm it)^{1/2}$, whereas the O-symbols behave as if they were constants (in the leading order). We therefore conclude (5.23) with no difficulty at all.

We need to redo a bit more of the calculation, if we want the ε-derivative. Let us abbreviate $F := \int_{-1}^1 \mathrm{Bi}(\delta+itz)\exp(4\varepsilon t^3 z)\,dz$, the expression treated in (5.14), and $G := \int_{-1}^1 \mathrm{Bi}(\delta+itz)z\exp(4\varepsilon t^3 z)\,dz$. Moreover, we omit the argument $\delta+itz$ and use the abbreviations $\mathrm{Bi}_\pm := \mathrm{Bi}(\delta\pm it)$ for the integrated terms. We have seen that terms like $\mathrm{Bi}_\pm\exp(-4\varepsilon t^3)$ are negligible as exponentially small, and $\approx$ will denote equality up to such terms.

In analogy with (5.14) and (5.15), we get

$$\begin{aligned}
4\varepsilon t^3 G &:= 4\varepsilon t^3 \int_{-1}^{1} z \operatorname{Bi} \exp(4\varepsilon t^3 z)\, dz = \\
&= \Big[z \operatorname{Bi} \exp(4\varepsilon t^3 z)\Big]_{-1}^{1} - it \int_{-1}^{1} z \operatorname{Bi}' \exp(4\varepsilon t^3 z)\, dz - \int_{-1}^{1} \operatorname{Bi} \exp(4\varepsilon t^3 z)\, dz \approx \\
&\approx \Big[\operatorname{Bi} \exp(4\varepsilon t^3 z)\Big]_{-1}^{1} - it \int_{-1}^{1} \operatorname{Bi}' \exp(4\varepsilon t^3 z)\, dz - it \int_{-1}^{1} (z-1) \operatorname{Bi}' \exp(4\varepsilon t^3 z)\, dz - F = \\
&= (4\varepsilon t^3 - 1)F - \frac{i}{4\varepsilon t^2}\Big[(z-1)\operatorname{Bi}' \exp(4\varepsilon t^3 z)\Big]_{-1}^{1} - \\
&\qquad - \frac{1}{4\varepsilon t}\int_{-1}^{1} (z-1)\operatorname{Bi}'' \exp(4\varepsilon t^3 z)\, dz + \frac{i}{4\varepsilon t^2}\int_{-1}^{1} \operatorname{Bi}' \exp(4\varepsilon t^3 z)\, dz \approx \\
&\approx (4\varepsilon t^3 - 1)F + \frac{(|\delta|+t)^{3/4}}{4\varepsilon t}\, O\Big(\frac{\exp M_\pm(1)}{(3\varepsilon t^3)^2}\Big) + \frac{(|\delta|+t)^{1/4}}{4\varepsilon t^2}\, O\Big(\frac{\exp M_\pm(1)}{3\varepsilon t^3}\Big)\,.
\end{aligned} \tag{5.26}$$

In the last line, we have used an improvement upon (5.18), namely

$$\int_{-1}^{1} (1-z) \exp M_\pm(z)\, dz \le \frac{\exp[4\varepsilon t^3 \pm \frac{2}{3}\operatorname{Re}(\delta+it)^{3/2}]}{(3\varepsilon t^3)^2} \tag{5.27}$$

Had it been for the sole purpose of the ε derivative, we could have done without the second integration by parts (second last line in (5.26)) and somewhat weaker error terms. We will need this sharper version for the t-derivative.

Similar estimates are valid for the other expressions in (2.1), too. For our asymptotic estimate, we may pretend with sufficient accuracy that "$4\varepsilon t^3 G \sim 4\varepsilon t^3 F - F$", hence that "$\partial_\varepsilon \Delta = 2(1 - (4\varepsilon t^3)^{-1})\Delta$", and (5.24) follows.

Now some notation will be convenient: Let Ci be the vector valued Airy function $\begin{bmatrix}\mathrm{Ai}\\ \mathrm{Bi}\end{bmatrix}$ and $F(\delta,t,\varepsilon) := \int_{-1}^{1} \operatorname{Ci}(\delta+itz)\exp(4\varepsilon t^3 z)\, dz$, $\bar F(\delta,t,\varepsilon) := F(\delta,t,-\varepsilon) = \int_{-1}^{1} \operatorname{Ci}(\delta-itz)\exp(4\varepsilon t^3 z)\, dz$. The determinant of two 2-vectors will be denoted by $\wedge$.

In order to calculate $\partial_t\Delta$, we use from (7.2)

$$tF_t - 3\varepsilon F_\varepsilon = -F - \delta F_\delta + F_{\delta\delta\delta}$$

and the very same equation for $\bar F$. We get

$$t\partial_t \Delta = (3\varepsilon\partial_\varepsilon - 2 - \delta\partial_\delta + \partial_\delta^3)\Delta - 3\partial_\delta(F_\delta \wedge \bar F_\delta)\,.$$

All but the last term have already been calculated, and in complete analogy to (5.11) and (5.23), we obtain

$$\partial_\delta(F_\delta \wedge \bar F_\delta) = \partial_\delta(\mathrm{Ai}'_+\,\mathrm{Bi}'_- - \mathrm{Bi}'_+\,\mathrm{Ai}'_-) + (|\delta|+t)^{3/2} \times \mathbf{R}O(c/\varepsilon) \tag{5.28}$$

Carrying together the results produces (5.25)

$$\begin{aligned}
&\frac{2(4\varepsilon t^3)^2}{\exp 8\varepsilon t^3}\, t\partial_t\Delta = \\
&\quad = (6\varepsilon - 2 - \delta\partial_\delta + \partial_\delta^3)(\mathrm{Ai}_+\,\mathrm{Bi}_- - \mathrm{Ai}_-\,\mathrm{Bi}_+) - 3\partial_\delta(\mathrm{Ai}'_+\,\mathrm{Bi}'_- - \mathrm{Ai}'_-\,\mathrm{Bi}'_+) + (|\delta|+t)^{3/2}\mathbf{R}O(c/\varepsilon) = \\
&\quad = (6\varepsilon + t\partial_t)(\mathrm{Ai}_+\,\mathrm{Bi}_- - \mathrm{Ai}_-\,\mathrm{Bi}_+) + (|\delta|+t)^{3/2}\mathbf{R}O(c/\varepsilon)
\end{aligned} \tag{5.29}$$

This completes the proof of Theorem 5.12 ■

We are now in a position to give a rather precise description of the set $\mathcal{Z}_\varepsilon^+$ for large ε. Our procedure is a s follows:

The asymptotic information given in Lemma 5.11 permits to describe the set $\mathcal{Z}_\varepsilon^+$ near infinity, except for the region covered by Theorem 4.8. It is our next task here to solve the equation $\Delta = 0$ in the region covered by Lemma 5.11. All these arguments work for arbitrary $\varepsilon > 0$, but the compact set in (δ, t) space not covered by the reasoning will typically grow as $\varepsilon \to 0$. Finally, the behaviour in the compact set can be controlled by means of Lemma 5.1 for ε sufficiently large.

It is one thing to show existence of solutions in one domain and exclude solutions in another by means of the leading term of an asymptotic expansion, but quite another task to show uniqueness of the solutions found. This further task can be accomplished by giving an asymptotic expansion for the derivative as well. This appears to be a rather tedious enterprise, but can be avoided to a large extend. The complex–analytic nature of the equation prevents certain types of bifurcations (saddle node) from occurring, and local topological arguments control the (unsigned) number of solutions; essentially, the number of solutions is controlled everywhere by the number of solutions established for small t in Theorem 4.8. We will need the derivatives in the narrow domain where all the bifurcations occur, namely near the critical line $t = \sqrt{3}\,\delta$.

5.3.4 Solving the Spectral Equation Asymptotically

Let us first collect the essential facts of the transformation $w \mapsto \zeta = \frac{2}{3}z^{3/2}$, $r := \operatorname{Re} w > 0$ for simple reference:

Lemma 5.13 *If $w = r(1 + i\tan 2\alpha) = r + is$ with $r > 0$, $\alpha \in\,]-\pi/4, \pi/4[$, then $\zeta := \frac{2}{3}z^{3/2}$ satisfies:*

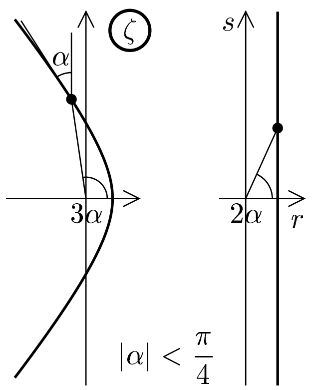
$$\zeta = \frac{2}{3}r^{3/2}(1 + i\tan 2\alpha)^{3/2} = \frac{2}{3}r^{3/2}\,\frac{\cos 3\alpha + i\sin 3\alpha}{\cos^{3/2} 2\alpha}$$

$$\frac{d\zeta}{d\alpha} = \frac{2r^{3/2}(-\sin\alpha + i\cos\alpha)}{\cos^{5/2} 2\alpha}$$

$$\frac{d\operatorname{Re}\zeta}{d\operatorname{Im}\zeta} = -\tan\alpha$$

$$\frac{d\operatorname{Re}\zeta}{ds} = \frac{-r^{1/2}\sin\alpha}{\cos^{1/2} 2\alpha} = \frac{-s}{\sqrt{2(\sqrt{s^2+r^2}+r)}}$$

$$\frac{d\operatorname{Im}\zeta}{ds} = \frac{r^{1/2}\cos\alpha}{\cos^{1/2} 2\alpha} = \sqrt{\frac{r}{1-\tan^2\alpha}} \geq r^{1/2}$$



$$\frac{d^2\operatorname{Re}\zeta}{(d\operatorname{Im}\zeta)^2} < 0\,, \qquad \frac{d^2\operatorname{Re}\zeta}{ds^2} < 0\,, \qquad \frac{d^2\operatorname{Im}\zeta}{ds^2} > 0$$

If $\operatorname{Re}\zeta \leq d \ll \frac{2}{3}r^{3/2}$, then $|\operatorname{Im} w| > \sqrt{3}\,r - \sqrt{2}\,d\,r^{-1/2} - O(d^2r^{-2})$.
If $\operatorname{Re}\zeta \geq -d$ with $d \ll \frac{2}{3}r^{3/2}$, then $|\operatorname{Im} w| < \sqrt{3}\,r + \sqrt{2}\,d\,r^{-1/2} + O(d^2r^{-2})$.

PROOF:
All calculations are straightforward. The second derivatives needn't be calculated explicitly, their signs follow from the monotonicity of the first derivatives as functions of α. ■

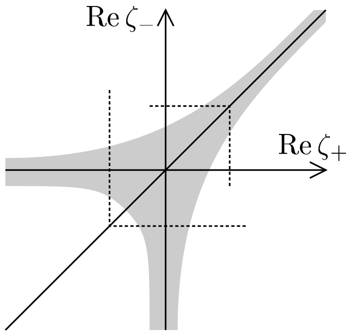
In the following, we generally use the abbreviations $\zeta_\pm := \frac{2}{3}(\delta \pm it)^{3/2}$.

In order to locate the zeros of Δ, asymptotically as $|\delta|^2 + t^2 \to \infty$, we first assume with no loss of generality that $t > 0$. Then solutions still come in pairs $\{(\delta, t), (\bar\delta, t)\}$ (unless δ is real), so it is no loss of generality to do the calculations under the assumption $\operatorname{Im}\delta \geq 0$, hence $\operatorname{Re}\zeta_+ \leq \operatorname{Re}\zeta_-$. The results can immediately be formulated without the assumption on $\operatorname{Im}\delta$.

Here are the results:

Theorem 5.14

1. *For every $a < 1$, there exists a compact set $K(a)$ such that for all $(\delta, t) \in \mathcal{Z}_\varepsilon^+ \setminus K(a)$, it holds:*
$$r := \operatorname{Re}\delta > 0\,,\; t > |\operatorname{Im}\delta|\,,\; (\operatorname{Re}\zeta_+, \operatorname{Re}\zeta_-) \in Y(a)$$
where



$$Y(a) = \Bigg\{(x,y)\;\Bigg|\; y \le \frac{1}{2}\ln\left(\frac{2+a}{2-a}\,e^{2x} + \frac{2+2a}{2-a}\right),\; x \le \frac{1}{2}\ln\left(\frac{2+a}{2-a}\,e^{2y} + \frac{2+2a}{2-a}\right),\; e^{2y} + e^{2x} \ge \frac{2-2a}{2+a}\Bigg\} \tag{5.30}$$

If b, $\tilde{b}$ satisfy the condition $b^2\tilde{b}^2 > 1 + \tilde{b}^2$, then there exists a such that every $(x,y) \in Y(a)$ satisfies at least one of the conditions $\max\{x,y\} \le \ln b$ ("wing condition") and $\min\{x,y\} \ge -\ln\tilde{b}$ ("body condition").

2. *For every b, solutions $(\delta,t) \in \mathcal{Z}_\varepsilon^+ \setminus K(a)$ satisfying the wing condition $\max\{x,y\} \le \ln b$ lie in the "wing domain"*
$$V(b) := \Bigg\{(\delta,t)\;\Bigg|\; -\sqrt{2}(\ln b)r^{-1/2} - O(r^{-2}) < t - \sqrt{3}\,r - |\operatorname{Im}\delta| < \frac{\sqrt{2}}{2}r^{-1/2}\ln\frac{2+a}{1-a} + O(r^{-2})\Bigg\} \tag{5.31}$$
and satisfy
$$|\sinh(\zeta_- - i\pi/4)| \le \frac{1}{2}\left[(1+a)b + a\sqrt{\tfrac{2+a}{1-a}}\right]\,. \tag{5.32}$$

3. *For every $\tilde{b}$, solutions $(\delta,t) \in \mathcal{Z}_\varepsilon^+ \setminus K(a)$ satisfying the body condition $\min\{x,y\} \ge -\ln\tilde{b}$ lie in the "body domain"*
$$I(\tilde{b}) := \Bigg\{(\delta,t)\;\Bigg|\; |\operatorname{Im}\delta| + t < \sqrt{3}\,r + \sqrt{2}\,(\ln\tilde{b})r^{-1/2} + O(r^{-2})\,,\; |\operatorname{Im}\delta| < O((\operatorname{Re}\delta/C)^{1/4})\Bigg\}\,. \tag{5.33}$$
and, provided $t > (C|\delta|)^{1/4}$, they satisfy
$$|\sinh(\zeta_- - \zeta_+)| \le \frac{1}{2}\left[\tilde{b}^2(1+a) + a(3+4\tilde{b}^2)^{1/2}\right]\,. \tag{5.34}$$

4. *For every $b < \sqrt{2}$, there exists $a > 0$ such that solutions $(\delta,t) \in \mathcal{Z}_\varepsilon^+ \setminus K(a)$ do not meet any one of the surfaces*
$$W(n,b) := \Bigg\{(\delta,t)\;\Bigg|\; \operatorname{Re}\delta = \left[\tfrac{3\pi\sqrt{2}}{8}\left(n+\tfrac{1}{4}\right)\right]^{2/3} =: r\,,\; t \ge \sqrt{3}\,r + |\operatorname{Im}\delta| - \sqrt{2}(\ln b)r^{-1/2} + O(r^{-2})\Bigg\}\,, \tag{5.35}$$
where $n \ge 0$ is an integer.

5. *For every $\tilde{b} < \sqrt{2}$ (and $\gamma := \frac{1}{2}\ln\tilde{b}$), there exists $a > 0$ such that solutions $(\delta, t) \in \mathcal{Z}_\varepsilon^+ \setminus K(a)$ satisfying $t > (C|\delta|)^{1/4}$ do not meet the surfaces*

$$T(n,\gamma) := \left\{ (\delta,t) \;\middle|\; \operatorname{Im}\frac{2}{3}(r+it)^{3/2} = \left(n+\frac{1}{4}\right)\pi\,,\ |\operatorname{Im}\delta| \le \hat{a}(r)\,,\ t < \sqrt{3}\,r + \gamma r^{-1/2} \right\} \tag{5.36}$$

where n is an integer or half an odd integer, and sufficiently large.

6. *For any $\gamma < \sqrt{2}(\ln 2)/2$, real solutions satisfying $t > (C\delta)^{1/4}$ do not meet the lines*

$$\begin{aligned} T_{\mathrm{r}}(n) &:= \{(\delta,t) \mid \delta \text{ real},\ \operatorname{Im}\zeta_+ = (n+\tfrac{1}{4})\pi\} \\ T_{\mathrm{r}}(n+\tfrac{1}{2}) &:= \{(\delta,t) \mid \delta \text{ real},\ \operatorname{Im}\zeta_+ = (n+\tfrac{3}{4})\pi\,,\ t < \sqrt{3}\,r + \gamma r^{-1/2}\}\,, \end{aligned}$$

provided n is a sufficiently large (dependent on γ) integer. Real solutions are restricted to the sector $t < \sqrt{3}\,r + \gamma r^{-1/2}$ for every $\gamma > \sqrt{2}(\ln 2)/2$, provided r is sufficiently large (dependent on γ).

On the other hand, inside the sector $t < \sqrt{3}\,r - \tilde{\gamma} r^{-1/2}$, but outside a compact set (depending on $\tilde{\gamma}$), there exist only real solutions, provided $\tilde{\gamma} > \pi\sqrt{6}/8 - \sqrt{2}(\ln 2)/2$.

Proof:
Our estimates hold for any $\varepsilon > 0$, but not for $\varepsilon = 0$, provided $\zeta_\pm$ is large enough (depending on ε). As $\zeta_\pm$ large enough implies t large, the c in $O(c/\varepsilon)$ from Lemma 5.11 goes to zero making all O-symbols small.

We are interested in solutions outside a compact set in (δ, t) space, hence we may assume that ζ_- or ζ_+ is large, but they need not necessarily be both large: It is conceivable that there are solutions (δ, t) to $\Delta(\delta, t, \varepsilon) = 0$, satisfying $\delta \to \infty$ and $t \to \infty$, but $\delta - it$ bounded. Such solutions actually do exist. We call this case the "non-asymptotic case". It is actually the easiest one, and we deal with it first:

The Non-Asymptotic Case: One of $\delta \pm it$ does not go to infinity

To be specific, we assume $\delta - it$ is bounded. We get a trivial bound on $\operatorname{Re}\delta$ by the boundedness assumption for $\delta - it$. As we follow $\mathcal{Z}_\varepsilon^+$ letting $t \to +\infty$, we clearly have $\operatorname{Im}\delta \ge t - \mathit{const} \to +\infty$, too. Employing Lemma 5.11 right away, we conclude from (5.11) that $\operatorname{Bi}(\delta - it) - \operatorname{Ai}(\delta - it)D(\delta + it) = O(c/\varepsilon) \to 0$ (because $c \to 0$ for $t \to \infty$). But as $D(\delta + it) \to i$ under this limit, we find that $\delta - it$ tends to the set $\{\bar{z}_j\} = D^{-1}(\{i\})$. (Clearly, on connected components of $\mathcal{Z}_\varepsilon^+$, this can only mean $\delta - it \to \bar{z}_j$ for some j.)

The first statement of the theorem is easily verified: Given a, we get upper bounds on ζ_- and thus $\operatorname{Re}\delta$ (otherwise, say, $O(\zeta_-^{-1}) < a/3$, and it's no longer the non-asymptotic case). Now as $t \to \infty$, $\delta - it \to \{z_j\}$, hence $\operatorname{Re}\zeta_- \to 0$, and $\operatorname{Re}\zeta_+ \to -\infty$. Therefore, outside a compact set, solutions do lie in $Y(a)$. As $\operatorname{Re}\bar{z}_j > 0$, we get $\operatorname{Re}\delta > 0$. As $\operatorname{Im}\bar{z}_j < 0$, we get $\operatorname{Im}\delta < t$. Moreover, $t - \sqrt{3}\,r - \operatorname{Im}\delta \to -\sqrt{3}\operatorname{Re}\bar{z}_j - \operatorname{Im}\bar{z}_j = 0$. This verifies (5.31). Using $\bar{z}_j = e^{-i\pi/3}|a_{-j}|$ with Lemma 5.2, we see that these solutions avoid the sets $W(n, b)$. The argument is valid even for small n, in which case the lower bound for t in (5.35) amounts to nothing but $O(1)$.

(To be very pedantic: The lemma tells us the asymptotic behaviour of the $\operatorname{Re}\bar{z}_j$; these numbers fall right between two of the r given for $W(j, b)$. We rely on the numerical fact, that also none of the $\operatorname{Re}\bar{z}_j$ for small j happens to fall exactly on the r specified in $W(j, b)$.)

The remaining statements of the theorem do not apply to the non-asymptotic case.

Having dealt with the non-asymptotic case, we now tacitly identify statements holding outside a compact set with statements following from asymptotic expansions as both ζ_- and ζ_+ going to infinity.

Now we show that there are no solutions (outside some compact set) satisfying $\operatorname{Re}\delta \le 0$.

EXCLUDING THE CASE $\operatorname{Re}\delta \le 0$

We treat this case by writing $-\delta$ instead of δ, with the new δ in the right half plane, and still with the definitions $\zeta_\pm = \frac{2}{3}(\delta \pm it)^{3/2}$. We have to avoid exponentially large cancellations by using an appropriate form of Δ.

By (3.7) and (3.9), we get:

$$\begin{aligned}&\operatorname{Ai}(-(\delta - it))\operatorname{Bi}(-(\delta + it)) - \operatorname{Bi}(-(\delta - it))\operatorname{Ai}(-(\delta + it)) = \\ &\quad = 2i\Big(\operatorname{Ai}((\delta + it)e^{-\pi i/3})\operatorname{Ai}((\delta - it)e^{\pi i/3}) - \operatorname{Ai}((\delta - it)e^{-\pi i/3})\operatorname{Ai}((\delta + it)e^{\pi i/3})\Big)\ ,\end{aligned} \tag{5.37}$$

and for all arguments of Ai occurring, we can apply (3.13) uniformly. This produces

$$\begin{aligned}&\operatorname{Ai}(-(\delta - it))\ \operatorname{Bi}(-(\delta + it)) - \operatorname{Bi}(-(\delta - it))\ \operatorname{Ai}(-(\delta + it)) = \frac{i(\delta^2 + t^2)^{-1/4}}{2\pi} \times \\ &\quad \times \Big\{\exp[i(\zeta_+ - \zeta_-)]\Big(1 + O(\zeta_+^{-1}) + O(\zeta_-^{-1})\Big) - \exp[i(\zeta_- - \zeta_+)]\Big(1 + O(\zeta_-^{-1}) + O(\zeta_+^{-1})\Big)\Big\}\end{aligned} \tag{5.38}$$

Inserting (5.38), (5.37) into (5.12), we obtain

$$\begin{aligned}\tilde\Delta(-\delta, t, \varepsilon) &:= \frac{4\pi\Delta(\delta, t, \varepsilon)\,(4\varepsilon t^3)^2}{\exp(8\varepsilon t^3)}\,(\delta^2 + t^2)^{1/4} = \\ &= -2\sin(\zeta_+ - \zeta_-) + \Big(O(\zeta_\pm^{-1}) + O(c/\varepsilon)\Big)\ \cosh\operatorname{Im}(\zeta_+ - \zeta_-)\end{aligned} \tag{5.39}$$

If this $\tilde\Delta(-\delta, t, \varepsilon)$ were 0, solutions with both ζ_+ and ζ_- large enough would be restricted to some strip $|\operatorname{Im}\zeta_+ - \operatorname{Im}\zeta_-| \le K$, which immediately implies a bound $2|t| \le K/\sqrt{r}$ contradicting $|t| \ge (C|\delta|)^{1/4}$. Therefore, no solutions arise from this case, except possibly inside some compact set.

Next we show that we may assume $\arg(\delta + it) \in\ ]-\pi/3, \pi[\ =: S_+$ and $\arg(\delta - it) \in\ ]-\pi, \pi/3[\ =: S_-$. We need only discuss the case $\operatorname{Im}\delta \ge 0$. Then, if our assumption fails, we have, say, $\pi/6 < \arg(\delta - it) < \arg(\delta + it) < \pi/2$.

EXCLUDING THE CASE $\pi/6 < \arg(\delta - it) < \arg(\delta + it) < \pi/2$

We proceed as before, but using the basis functions $\operatorname{Ai}(\delta)$, $\operatorname{Ai}(\delta e^{-2i\pi/3})$, and we obtain

$$\begin{aligned}&\operatorname{Ai}(\delta + it)\operatorname{Bi}(\delta - it) - \operatorname{Bi}(\delta + it)\operatorname{Ai}(\delta - it) = \frac{(\delta^2 + t^2)^{-1/4}}{2\pi} \times \\ &\quad \times \Big\{\exp[\zeta_- - \zeta_+](1 + O(\zeta_+^{-1}) + O(\zeta_-^{-1})) - \exp[\zeta_+ - \zeta_-](1 + O(\zeta_-^{-1}) + O(\zeta_+^{-1}))\Big\}\ .\end{aligned} \tag{5.40}$$

We use the modified version of (5.11) in Lemma 5.11. This time, for Δ to vanish, $\operatorname{Re}(\zeta_- - \zeta_+)$ must be bounded. With the same reasoning as explained below for the main case, this implies an estimate (5.43); hence $\arg\delta < \pi/6$ for r large enough. But this is incompatible with the assumption made in this case.

We are therefore left with the main case, namely

THE MAIN CASE $\arg(\delta \pm it) \in S_\pm$; $\operatorname{Re}\delta > 0$

BASIC ESTIMATES:

Using (3.13) and (3.19), we obtain

$$\begin{aligned}&\operatorname{Ai}(\delta + it)\operatorname{Bi}(\delta - it) - \operatorname{Bi}(\delta + it)\operatorname{Ai}(\delta - it) = \frac{(\delta^2 + t^2)^{-1/4}}{2\pi} \times \\ &\quad \times \Big\{\exp[\zeta_- - \zeta_+](1 + O(\zeta_+^{-1}) + O(\zeta_-^{-1})) - \exp[\zeta_+ - \zeta_-](1 + O(\zeta_-^{-1}) + O(\zeta_+^{-1})) \\ &\qquad - i\exp[-\zeta_+ - \zeta_-](1 + O(\zeta_+^{-1}) + O(\zeta_-^{-1}))\Big\}\ .\end{aligned} \tag{5.41}$$

The exponentials in the error term of (5.11) in Lemma 5.11 can be rewritten as

$$\begin{aligned}
&\exp\max\{|\mathrm{Re}\,\zeta_+| - \mathrm{Re}\,\zeta_-, |\mathrm{Re}\,\zeta_-| - \mathrm{Re}\,\zeta_+\} \leq \\
&\quad \leq \exp\max\{\mathrm{Re}\,\zeta_+ - \mathrm{Re}\,\zeta_-, -\mathrm{Re}\,\zeta_+ - \mathrm{Re}\,\zeta_-, \mathrm{Re}\,\zeta_- - \mathrm{Re}\,\zeta_+\} \\
&\quad \leq \exp\max\{|\mathrm{Re}(\zeta_+ - \zeta_-)|, -\mathrm{Re}\,\zeta_+ - \mathrm{Re}\,\zeta_-\}
\end{aligned}$$

therefore this error term matches the one we get from expansion (5.41). Inserting(5.41) into (5.11), we conclude that

$$\begin{aligned}
\tilde{\Delta}(\delta,t,\varepsilon) &:= \frac{4\pi\Delta(\delta,t,\varepsilon)\,(4\varepsilon t^3)^2}{\exp(8\varepsilon t^3)}\,(\delta^2+t^2)^{1/4} = \\
&= \exp(\zeta_- - \zeta_+) - \exp(\zeta_+ - \zeta_-) - i\exp(-\zeta_- - \zeta_+) + \\
&\quad + \Big(O(\zeta_\pm^{-1}) + O(c/\varepsilon)\Big)\Big(\cosh\mathrm{Re}(\zeta_- - \zeta_+) + \exp[-\mathrm{Re}(\zeta_- + \zeta_+)]\Big)
\end{aligned} \tag{5.42}$$

We want the error term in the above equation $O(\zeta_\pm^{-1}) + O(c/\varepsilon) < a$. These error terms can be made sufficiently small, provided $\zeta_\pm$ are large enough. As we are not dealing with the non-asymptotic case, we get a compact set $K(a)$ outside which the asymptotic statements are true.

Let us exploit $\tilde{\Delta} = 0$ by selecting one term A in (5.42), $A+B+C+D=0$, and concluding that $|A| \leq |B| + |C| + |D|$. Depending on which term in (5.42) we select, we get different necessary conditions for (δ, t) to be a solution of $\Delta = 0$, expressed in terms of $y := \mathrm{Re}\,\zeta_-$ and $x := \mathrm{Re}\,\zeta_+$. We again assume $y \geq x$ for definiteness. Selecting the first term in (5.42) yields

$$\exp(y-x) \leq \exp(x-y) + \exp(-x-y) + a[\exp(y-x) + \exp(x-y) + 2\exp(-y-x)]/2\,,$$

hence the first condition defining (5.30). The second condition is analogous for $y \leq x$, with the second term of (5.42) selected. The third condition arises from selecting the third term. We get the estimate

$$\exp(-x-y) \leq \exp(y-x) + \exp(x-y) + a[\exp(y-x) + \exp(x-y) + 2\exp(-x-y)]/2\,.$$

This completes the proof of (5.30). We immediately get the second claim in part 6, namely that real solutions are restricted to the sector $t < \sqrt{3}\,r + \gamma r^{-1/2}$. Clearly, the claim $t > |\mathrm{Im}\,\delta|$ follows for solutions satisfying (5.31). For real solutions, it is trivial. For solutions satisfying (5.33), it follows from the estimate in part 6, which tells us that non-real solutions occur only near the critical line $t = \sqrt{3}\,\delta$. This claim will be proved together with the following theorem (and not be used until then). Unlike in Lemma 5.3, where $t > |\mathrm{Im}\,\delta|$ was a-priori information for the case $\varepsilon = \infty$, we get the same information here a-posteriori.

In order to show that each solution satisfies either the wing or the body condition, by (5.30), it suffices to show that $\ln b > \frac{1}{2}\ln\left(\frac{2+a}{2-a}\tilde{b}^{-2} + \frac{2+2a}{2-a}\right)$ for a sufficiently small. This immediately yields the condition claimed in the statement of the theorem.

ESTIMATES FOR THE WING DOMAIN:

For the finer estimates, we distinguish the different "branches" of $Y(a)$. For the wing domain (again $y \geq x$), we select the the first and third term in (5.42) and calculate:

$$|2e^{i\pi/4}\sinh(\zeta_- - i\pi/4)| = |\exp(\zeta_-) - i\exp(-\zeta_-)| \leq \exp(2x-y) + a\Big(\frac{\exp y}{2} + \exp(-y) + \frac{\exp(2x-y)}{2}\Big)\,.$$

We already have automatically $\exp(-y) \leq (\frac{2+a}{1-a})^{1/2}$, and moreover, we have assumed $\exp(y) \leq b$, which also implies $\exp(2x-y) \leq b$. By Lemma 5.13, this implies (5.31). Inserting the estimates for $\exp(2x-y)$ etc. immediately implies (5.32).

In order to see that solutions in $V(b)$ don't meet the surface $W(n,b)$ given in (5.35), we make use of the a-priori bound on $|\operatorname{Re}\zeta_-|$ for solutions, which, together with Lemma 5.13, permits us to write

$$\delta - it = r(1-i\sqrt{3}) + i\sigma r^{-1/2}$$

with σ bounded. Inserting the value for r, we conclude

$$\begin{aligned}\zeta_- - i\pi/4 &= -i\pi(n+1/2) + \sqrt{2}e^{i\pi/3}\sigma + O(r^{-3/2})\\ |\sinh(\zeta_- - i\pi/4)| &= |\cosh(\sqrt{2}e^{i\pi/3}\sigma + O(n^{-1}))| \ .\end{aligned}$$

Now if $b < \sqrt{2}$ and a small enough, the right hand side of (5.32) is less than $\sqrt{2}/2$, Considering the last equation, we note that the set of z where $|\cosh z| < \sqrt{2}/2$ is contained in the union of disks of radius less than $\arcsin\sqrt{2}/2 = \pi/4$ centered at $\pm i\pi/2, \pm 3i\pi/2, \ldots$ The rays $z = \sqrt{2}\,e^{i\pi/3}\sigma$, σ real, do not meet these disks, however. This concludes the proof of statement 4 of our theorem. (We have argued asymptotically as $n\to\infty$; if we keep n bounded instead, we are in the non-asymptotic case, and the arguments given there apply.)

ESTIMATES FOR THE BODY DOMAIN:

Let us now consider the body part of $Y(a)$, where $\operatorname{Re}\zeta_\pm > -\ln\tilde{b}$. Then we have $e^{-(x+y)} < \tilde{b}^2$, and from (5.30) $e^{|x-y|} \le [(2+a+(2+2a)\tilde{b}^2)/(2-a)]^{1/2} < (3+4\tilde{b}^2)^{1/2} =: f(\tilde{b})$. From Lemma 5.13 (and repeating the basic assumption from Lemma 5.11) we get

$$(C|\delta|)^{1/4} < t \le |\operatorname{Im}\delta| + t < \sqrt{3}\,r + \sqrt{2}\,(\ln\tilde{b})r^{-1/2} + O(r^{-2}) \ ,$$

hence the first part of (5.33). Selecting the first two terms of (5.42), we obtain

$$|2\sinh(\zeta_- - \zeta_+)| \le \tilde{b}^2(1+a) + af(\tilde{b}) \ ,$$

i.e., (5.34). From $\exp|x-y| \le f(\tilde{b})$, we conclude

$$\begin{aligned}\ln f(\tilde{b}) \ge |\operatorname{Re}\zeta_+ - \operatorname{Re}\zeta_-| &= \int_{||\operatorname{Im}\delta|-t|}^{|\operatorname{Im}\delta|+t}\left|\frac{d\operatorname{Re}\zeta}{ds}\right| ds \ge \int_{||\operatorname{Im}\delta|-t|}^{|\operatorname{Im}\delta|+t}\frac{s\,ds}{2(s^2+r^2)^{1/4}} =\\ &= \frac{1}{3}\left[(r^2+s^2)^{3/4}\right]_{||\operatorname{Im}\delta|-t|}^{|\operatorname{Im}\delta|+t} \ge \frac{(|\operatorname{Im}\delta|+t)^2 - ||\operatorname{Im}\delta|-t|^2}{4(r^2+(|\operatorname{Im}\delta|+t)^2)^{1/4}} \ge \frac{t|\operatorname{Im}\delta|}{(2(|\delta|^2+t^2))^{1/4}} \ .\end{aligned}$$

Together with $t > (C|\delta|)^{1/4}$, this bounds $|\operatorname{Im}\delta|$, namely

$$\begin{aligned}&|\operatorname{Im}\delta| < 2^{1/4}\ln f(\tilde{b})\left(|\delta|/C + (C|\delta|)^{-1/2}\right)^{1/4}\\ \Longrightarrow\ &|\operatorname{Im}\delta| < \hat{a}(\operatorname{Re}\delta) = O((\operatorname{Re}\delta/C)^{1/4}) \ ,\end{aligned} \tag{5.43}$$

and therefore solutions in the diagonal region must be nearly real. This proves part 3 of the theorem.

For δ real, we have $\zeta_+ = \bar{\zeta}_-$, and (5.42) together with (5.30) implies

$$|2\sin(2\operatorname{Im}\zeta_+) + \exp(-2\operatorname{Re}\zeta_+)| \le a(1+\exp(-2\operatorname{Re}\zeta_+)) < \frac{3a}{1-a} \ . \tag{5.44}$$

Clearly, if $\operatorname{Im}\zeta_+ = (n+\frac{1}{4})\pi$, (5.44) cannot be satisfied at all, whereas, if $\operatorname{Im}\zeta_+ = (n+\frac{3}{4})\pi$, it cannot be satisfied for $\operatorname{Re}\zeta_+ > \frac{1}{2}\ln\frac{1+a}{2-a}$, i.e., for $t < \sqrt{3}\,r + (\sqrt{2}/2)\ln\frac{2-a}{1+a}$. This latter domain contains $\{t < \sqrt{3}\,r + \gamma r^{-1/2}\}$. This establishes the first statement of part 6.

Therefore, the lines $\operatorname{Im}\zeta = (n/2+1/4)\pi$ separate real solutions. In the (δ,t)-plane, they correspond to curves $t = t_n(\delta)$ disconnecting the sector $t \le \sqrt{3}\,\delta$; namely $t_n(\delta) \sim \delta^{-1/2}(n/2+1/4)\pi$ as $\delta\to\infty$. By the intermediate value theorem (applied to the real valued function $\delta \mapsto i\tilde{\Delta}(\delta,t,\varepsilon)$), there is indeed at least one real solution between each of these lines, at least up to the maximal value of t where $T_{\mathrm{r}}(n+\frac{1}{2})$ ends.

For n an integer or half an odd integer, we extend the curves $T_{\mathrm{r}}(n)$ to the surfaces $T(n,\gamma)$. ($T(n,\gamma)$, unlike $T_{\mathrm{r}}(n)$, depends on γ also for integers n.) We want to show that these surfaces are still disjoint from $\mathcal{Z}_\varepsilon^+$ for n large enough. To this end, it suffices to show for all $\sigma := \operatorname{Im}\delta \in [-\hat{a}(r), \hat{a}(r)]$ and all r, t satisfying $t \le \sqrt{3}\,r + \gamma r^{-1/2}$, it holds

$$\frac{2}{3}\operatorname{Im}\left\{(r+i\sigma+it)^{3/2} - (r+i\sigma-it)^{3/2}\right\} \in \left]\beta-\pi/4, \beta+\pi/4\right[\tag{5.45}$$

with

$$\beta = (2n+\frac{1}{2})\pi = \frac{2}{3}\operatorname{Im}\left\{(r+it)^{3/2} - (r-it)^{3/2}\right\}\ .$$

To this end, we note that for fixed (r,t), $\operatorname{Im}(\zeta_+ - \zeta_-)$ is a convex function of σ; indeed,

$$\frac{d}{d\sigma}\frac{2}{3}\left(\operatorname{Im}(r+i\sigma+it)^{3/2} - \operatorname{Im}(r+i\sigma-it)^{3/2}\right) = \frac{2}{3}\int_{-\sigma}^{\sigma}\left(\frac{d^2}{d\tau^2}\operatorname{Im}(r+it+i\tau)^{3/2}\right)d\tau\ ,$$

which is $\gtrless 0$ for $\sigma \gtrless 0$ and monotonically increasing in σ (by Lemma 5.13). Therefore it is sufficient to carry out the estimate for $\sigma = \hat{a}(r)$ only. Now,

$$\begin{aligned}&\frac{2}{3}\operatorname{Im}\left((r+it+i\hat{a}(r))^{3/2} + (r+it-i\hat{a}(r))^{3/2}\right) = \\ &\qquad = \frac{4}{3}\operatorname{Im}(r+it)^{3/2} - \frac{1}{2}\hat{a}(r)^2\operatorname{Im}\{(r+it)^{-1/2}\} + \ldots = \beta + O(C^{-1/2}) + O(r^{-3/2}C^{-1})\end{aligned} \tag{5.46}$$

Having thus shown (5.45), we still get a contradiction to (5.34), with $\tilde{b} < \sqrt{2}$, provided r (and with it, C) is large enough. This proves part 5 of the theorem.

We have still left over the very last claim, and as announced after equation (5.42), we will prove it together with the next theorem. ■

5.3.5 Structure of the Solution Set $\mathcal{Z}_\varepsilon$

Theorem 5.15

1. *For all real solutions (i.e., δ real) to the equation $\Delta(\delta,t,\varepsilon) = 0$ with $\varepsilon > 0$ and $t > 0$, it holds $\delta > -O(1)$ and $t < \sqrt{3}\delta + O(\delta^{-1/2})$.*

2. *For any $\varepsilon > 0$ and large enough k (depending on ε), the arcs R_{k0} and R_{k1} (defined in Theorem 5.10) belong to one branch, $\tilde{R}_k$. In the whole domain $t < \sqrt{3}\operatorname{Re}\delta - O((\operatorname{Re}\delta)^{-1/2})$ (except possibly a compact set) there are no other solutions beyond the real ones arising from the branches $\tilde{R}_k$.*

   There is exactly one point $\boldsymbol{r}_k$ on $\tilde{R}_k$ from which non-real solutions bifurcate. t has a nondegenerate maximum on $\tilde{R}_k$ at $\boldsymbol{r}_k$.

3. *The same as in 2 is true for every k, provided ε is large enough.*

4. *For every k and every $\varepsilon > 0$, there exists a branch $\tilde{C}_k$ of solutions, on which $t \to \infty$ and $\delta \sim \bar{z}_k + it + o(1)$ or $\delta \sim z_k - it + o(1)$ respectively as $t \to \infty$. Here $z_k = |a_{-k}|e^{i\pi/3}$ with $\operatorname{Ai}(a_{-k}) = 0$ as in Theorem 5.10.*

   There is exactly one point $\boldsymbol{c}_k$ on $\tilde{C}_k$ from which other solutions bifurcate (real ones, in fact). t has a nondegenerate minimum on $\tilde{C}_k$ at $\boldsymbol{c}_k$. In all of the domain $t > \sqrt{3}\operatorname{Re}\delta + O((\operatorname{Re}\delta)^{-1/2})$, there are no other solutions except those belonging to the branches $\tilde{C}_k$.

5. *For k sufficiently large (or else for all k, and ε sufficiently large), $\tilde{C}_k$ and $\tilde{R}_k$ belong to the same connected component $\hat{K}_k$ of $\mathcal{Z}_\varepsilon^+$, and the $\hat{K}_k$ are different for different k.*

6. *For k sufficiently large (or else for all k, and ε sufficiently large), $\hat{K}_k$ consists only of $\tilde{C}_k$ and $\tilde{R}_k$; in particular, $\boldsymbol{c}_k = \boldsymbol{r}_k$.*

Proof:
We have just seen the claim about the location restrictions for real solutions.

Now select C_0 small enough such that for the "lower domain" $t < C_0|\delta|^{1/4}$, all solutions are classified according to Theorem 4.8; follow each one of the arcs R_{k0} and R_{k1} along increasing t. They all have to enter the "upper domain" $t > (C|\delta|)^{1/4}$, with $C^{1/4}$ chosen (slightly) smaller than C_0, because otherwise a branch point (and consequently non-real solutions) would occur in the lower domain. The ordering of the arcs is preserved when they enter the upper domain. There exists some $C_1 > 0$ such that $t > C_1$ at all points where the arcs enter, so we are really in the situation of Lemma 5.11, and Theorem 5.14 prevails. Its part 6 confines all real branches each to a compact set, as long as they do not return to the lower domain. See Figure 5.5.

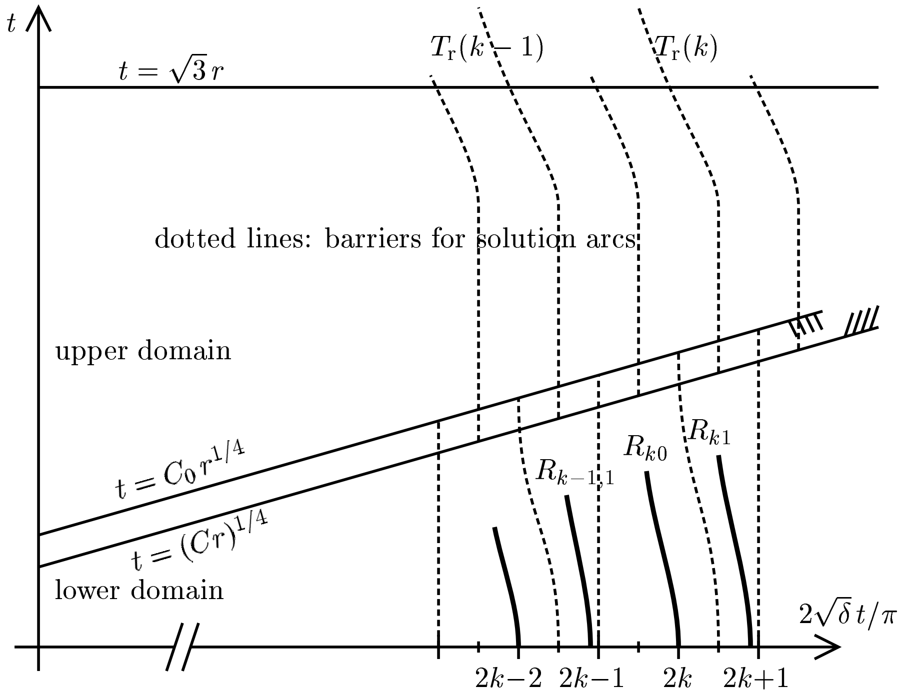


Figure 5.5: Schematic sketch of solutions and barriers

Consider the arc R_{k1}. According to Theorem 4.8, this arc satisfies $2\sqrt{\delta}\,t \in\,]2k\pi, (2k+1)\pi[$ in the lower domain. It can therefore enter the upper domain either in the interval $2\sqrt{\delta}\,t \in\,](2k+\frac{1}{2})\pi, (2k+\frac{3}{2})\pi[$ or in the interval $2\sqrt{\delta}\,t \in\,](2k-\frac{1}{2})\pi, (2k+\frac{1}{2})\pi[$. If k is large enough for the asymptotic estimates in the upper domain to hold, the former possibility is ruled out, because it would leave an odd number of arcs in the domain $2\sqrt{\delta}\,t < (2k+\frac{1}{2})\pi$. But these arcs have to assemble to branches, so there must be an even number of them.

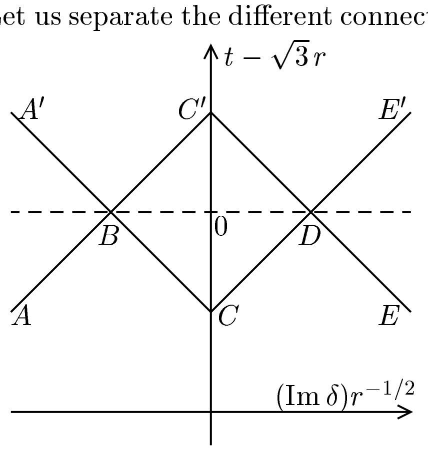
If k is large enough for the same reasoning to apply to $R_{k-1,1}$ as well, then R_{k0} is forced to enter the upper domain in the interval $2\sqrt{\delta}\,t \in\,](2k-\frac{3}{2})\pi, (2k-\frac{1}{2})\pi[$, because otherwise there would not be a solution left for this interval. (The argument applies to the small intersection zone between the upper and lower domains.) This forces the arcs R_{k0} and R_{k1} to belong to the same branch, $\tilde{R}_k$. In order to meet each other, the arcs have to extend beyond the sector $t \le \sqrt{3}\,\delta$, because they are separated by the line $2\sqrt{\delta}\,t = (2k-\frac{1}{2})\pi$ inside this sector. In particular, for large k, the arcs R_{k0} and R_{k1} can be followed all the way through the upper domain.

In order to see that there are no solutions except the real ones in the domain $t < \sqrt{3}\operatorname{Re}\delta - O((\operatorname{Re}\delta)^{-1/2})$, outside some compact set, we now choose $\tilde{C}$ and r large enough for (5.46) to imply (5.45) and k large enough for the domain $t < (\tilde{C}|\delta|)^{1/4}$ to be covered by Theorem 4.8. In the remaining part of the body domain, no solution branches can enter through the boundary of the tubes into which that domain is cut by the surfaces $W(n,b)$. A degree (or winding number) argument for the function $\delta \mapsto \tilde{\Delta}(\delta, t, \varepsilon)$ in each tube, with t as a homotopy parameter, shows that the number of solutions δ to $\tilde{\Delta} = 0$ within each tube is constant as a function of t, namely it is one, as can be seen in the intersection between the lower and upper domain. No solutions can therefore branch off inside the tube.

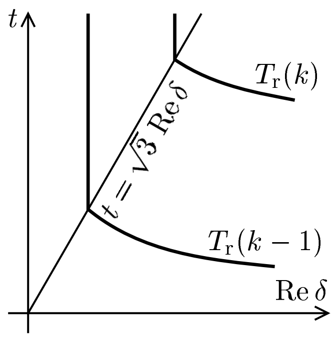
Considering t as a function on $\tilde{R}_k$, its lowest critical point must be a local (nondegenerate) maximum, because otherwise, non-real branches would bifurcate there in the direction of smaller t, but they must have returned to the real branch inside the tube; so we would get a contradiction to the minimality of t. Therefore, there is only one critical point on $\tilde{R}_k$.

As we have a lower estimate for t at that branch point, given by the curve $T_{\mathrm{r}}(k - \frac{1}{2})$, and as all bifurcating solutions are confined to a neighbourhood of $\tilde{C}_k$ and $\tilde{R}_k$ (in a way to be detailed out in a minute), we get a lower bound of the type $t > \sqrt{3}\operatorname{Re}\delta - O((\operatorname{Re}\delta)^{-1/2})$, below which no other solutions but real ones can occur. This argument can be done quantitatively:

Let us separate the different connected components. In Theorem 5.14, we can choose $b = \tilde{b} < \sqrt{2}$, but still satisfying the condition $b^2\tilde{b}^2 > 1 + \tilde{b}^2$. Look at the sketch. The surface $W(k,b)$ reaches down to the line $A'CE'$, but may safely be extended to smaller t until the line $ABCDE$, the upper boundary of the body region, without hitting any solutions. The surface $T(k,\gamma)$ in turn reaches up to the line $AC'E$, but may be extended to larger t until the lower boundary of the wing domain, i.e., up to the line $A'BC'DE'$. Both surfaces join along the straight line BD, i.e., $t = \sqrt{3}\,r$, with the same $r = (3\pi\sqrt{2}(k+1/4)/8)^{2/3}$ and thus disconnect $\mathcal{Z}_\varepsilon^+$. A cross section along constant $\operatorname{Im}\delta$ is shown in the inset sketch. We have shown part 5 of the theorem for the case of large k and arbitrary $\varepsilon > 0$.



We can now do the estimate from part 2 quantitatively, i.e., fill in the remaining gap from Theorem 5.14,6. The branch point $\boldsymbol{r}_k = (r_k, t_k)$ satisfies $t_k > \sqrt{3}\,\rho_0 + \gamma\rho_0^{-1/2}$ with $\rho_0 = [3\pi\sqrt{2}(k - \frac{1}{4})/8]^{2/3}$, and on the whole bifurcating branch (plus those branches arising through possible further bifurcations), there will also be $t \geq t_k$. But $\operatorname{Re}\delta$ will remain smaller than $\rho_1 = [3\pi\sqrt{2}(k+\frac{1}{4})/8]^{2/3}$, because the branch cannot cross $T(k,\gamma)$. Therefore we get $t > \sqrt{3}\operatorname{Re}\delta - \tilde{\gamma}(\operatorname{Re}\delta)^{-1/2}$ on that branch, provided $-\tilde{\gamma} < \lim_{k\to\infty}(\sqrt{3}\rho_0 + \gamma\rho_0^{-1/2} - \sqrt{3}\rho_1)\rho_1^{1/2}$. This gives the quantitative limit $\tilde{\gamma} > \pi\sqrt{6}/8 - \sqrt{2}(\ln 2)/2$ claimed in Theorem 5.14.



For given integer $k \geq 1$ and sufficiently large t (depending on k according to Theorem 5.14,4), equation (5.11) permits us to apply Rouché's theorem to the functions

$$\begin{aligned} &\tilde{\Delta} : \delta \mapsto 2\Delta(\delta, t, \varepsilon)(4\varepsilon t^3)^2/\exp(8\varepsilon t^3) \qquad\qquad \text{and} \\ &\tilde{D} : \delta \mapsto \operatorname{Ai}(\delta + it)\operatorname{Bi}(\delta - it) - \operatorname{Bi}(\delta + it)\operatorname{Ai}(\delta - it) \end{aligned}$$

in any domain $(3\pi\sqrt{2}(k - \frac{3}{4})/8)^{2/3} < \operatorname{Re}\delta(3\pi\sqrt{2}(k - \frac{3}{4})/8)^{2/3}$, $|\operatorname{Im}\delta| < A$ with $A > t$ and conclude that $\tilde{\Delta}$ has exactly two zeros there, because $\tilde{D}$ has, by Theorem 5.10.

These solutions cannot be real, so they must be complex conjugates to each other. Following them for growing t, there cannot occur any bifurcations, because the number of solutions is limited to exactly 2, and the two solutions cannot become real. Following the for decreasing t, must eventually lead to a real bifurcation point $\boldsymbol{c}_k$. This establishes the existence of a branch $\tilde{C}_k$ as claimed in part 4 for each k. The asymptotic behaviour as $t \to \infty$ follows from the estimates for the non-asymptotic case in the proof of Theorem 5.14.

As for part 3 and the variant of part 5 concerning small k, but large ε, we merely need to deal with the remaining compact set. First note that there is a neighbourhood of $(\delta, t) = (0, 0)$ in $\mathbb{C} \times \mathbb{R}^+$, independent of ε, containing no solutions at all. This follows from Theorem 4.8. One may take the cylinder $|\delta| < \delta_*$, $0 < t < t_*$, where (δ_*, t_*) is the intersection of the curves $2\sqrt{\delta}\, t = \pi$ and $t = C\delta^{1/4}$ (with sufficiently small C). Following any presumable solution in this cylinder along decreasing t, one could not enter into the domain $t < C|\delta|^{1/4}$, hence would have to go to $(0, 0)$. This is however impossible, because $\Delta(\delta, t, \varepsilon) \sim -16i\varepsilon t^4/3\pi$ as $(\delta, t) \to (0, 0)$, due to (4.53).

But then, Theorem 4.8 controls the totality of solutions satisfying $0 < t < t_*$, and the remaining compact set is of the type to which Lemma 5.1 applies. If we choose now ε sufficiently large, then the barriers constructed from $T(k, \gamma)$ and $W(k, b)$, originally for large k, are also barriers for the finitely many remaining (small) k, and the above reasoning carries over. All but part 6 has been proved by now.

5.3.6 Nonexistence of Real Compact Branches; Proof of Theorem 5.15,6

Note that the topological arguments are not sufficient to conclude $\boldsymbol{r}_k = \boldsymbol{c}_k$, even for k and ε large. The branch bifurcating from $\tilde{R}_k$ at $\boldsymbol{r}_k$ cannot undergo further bifurcations, unless it returns to the real axis (counting solutions again), and if so, it has to return soon, because real solutions are impossible for large t. But a finite chain of branches, each diffeomorphic to a circle, alternately consisting of real and complex solutions could well connect $\boldsymbol{r}_k$ and $\boldsymbol{c}_k$ in a way entirely compatible with the constraints established so far.

In order to rule out this possibility, one has to show that the zero set of the equation $\Delta(\delta, t, \varepsilon)$ for real δ does not contain closed curves. One conceivable approach would be to use the partial differential equations established for Δ in Section 7.2, possibly together with homotopy methods with respect to ε and some kind of maximum principle. Enticing as the approach may seem, it has eluded the author so far. Given the strong location restrictions $\sqrt{3}\,\delta - \tilde{\gamma}\delta^{-1/2} < t < \sqrt{3}\,\delta + \gamma\delta^{-1/2}$ that such a branch has to satisfy (Theorem 5.14,6; this theorem, part 2), an asymptotic approach is again feasible, provided it also controls the derivatives of Δ as in Section 5.3.3.

We assume here that $\delta > 0$ real, and $t = \sqrt{3}\,\delta + a/\sqrt{\delta}$ with an a-priori bound on a. Then we obtain

$$\begin{aligned}
\zeta_+ &= \tfrac{4i}{3}\sqrt{2}\,\delta^{3/2} - a\sqrt{2}e^{-i\pi/3} + O(\delta^{-3/2}) \\
\zeta_- &= -\tfrac{4i}{3}\sqrt{2}\,\delta^{3/2} - a\sqrt{2}e^{i\pi/3} + O(\delta^{-3/2}) \\
(\delta - it)^{1/2} - (\delta + it)^{1/2} &= -i(2\delta)^{1/2} - ia\sqrt{6}/(4\delta) + O(\delta^{-5/2}) \\
(\delta - it)^{1/2} + (\delta + it)^{1/2} &= (6\delta)^{1/2} + a\sqrt{2}/(4\delta) + O(\delta^{-5/2}) \\
(\delta^2 + t^2)^{1/4} &= (2\delta)^{1/2} + \sqrt{6}\,a/(8\delta) + O(\delta^{-5/2}) \\
(\delta^2 + t^2)^{-1/4} &= (2\delta)^{-1/2} - \sqrt{6}\,a/(16\delta) + O(\delta^{-5/2})
\end{aligned} \tag{5.47}$$

and by (3.13), (3.19),

$$2\pi i(\mathrm{Ai}_+\,\mathrm{Bi}_- - \mathrm{Bi}_+\,\mathrm{Ai}_-) = (2\delta)^{-1/2}\left(2\sin\Big(\frac{8\sqrt{2}}{3}\delta^{3/2} + a\sqrt{6}\Big) + \exp(a\sqrt{2}) + O(\delta^{-3/2})\right) \tag{5.48}$$

Note that the $O(\delta^{-3/2})$ is actually multiplied by angular and exponential functions like the ones preceding it. We are right to ignore them, because they are bounded in the very narrow neighbourhood we are considering, but these exponentials do have an effect when we take the derivatives; in this equation we have to pretend $\partial_\delta O(\delta^{-k}) = O(\delta^{-k+1/2})$.

If $g(\delta,t) = \tilde g(\delta,a)$ with $t = \sqrt{3}\,\delta + a\sqrt{\delta}$, then

$$\begin{aligned}\frac{\partial g}{\partial \delta} &= \frac{\partial \tilde g}{\partial \delta} - \left(\sqrt{3} - a/(2\delta^{3/2})\right)\sqrt{\delta}\,\frac{\partial \tilde g}{\partial a}\\ \frac{\partial g}{\partial t} &= \sqrt{\delta}\,\frac{\partial \tilde g}{\partial a}\,,\end{aligned}$$

of which the term $a/(2\delta^{3/2})$ may be neglected right away. We conclude

$$2\pi i\partial_\delta(\mathrm{Ai}_+\,\mathrm{Bi}_- - \mathrm{Bi}_+\,\mathrm{Ai}_-) = \left(2\cos\Bigl(\frac{8\sqrt{2}}{3}\delta^{3/2} + a\sqrt{6}\Bigr) - \sqrt{3}\exp(a\sqrt{2}) + O(\delta^{-3/2})\right)\,, \tag{5.49}$$

$$2\pi i\partial_t(\mathrm{Ai}_+\,\mathrm{Bi}_- - \mathrm{Bi}_+\,\mathrm{Ai}_-) = \left(2\sqrt{3}\cos\Bigl(\frac{8\sqrt{2}}{3}\delta^{3/2} + a\sqrt{6}\Bigr) + \exp(a\sqrt{2}) + O(\delta^{-3/2})\right)\,. \tag{5.50}$$

From (5.10), we conclude $c = O(\delta^{-3/2})$, and, using (5.11) and (5.48), we obtain

$$\begin{aligned}\frac{\pi i(8\varepsilon t^3)^2}{\exp(8\varepsilon t^3)}\Delta(\delta,t,\varepsilon) &= 2\pi i(\mathrm{Ai}_+\,\mathrm{Bi}_- - \mathrm{Bi}_+\,\mathrm{Ai}_-) + O(\delta^{-2}\varepsilon^{-1})\\ &= (2\delta)^{-1/2}\left(2\sin\Bigl(\tfrac{8\sqrt{2}}{3}\delta^{3/2} + a\sqrt{6}\Bigr) + \exp(a\sqrt{2})\right) + O(\delta^{-2})\end{aligned} \tag{5.51}$$

Similarly, using (5.23), (5.25) and (5.49), (5.50), we obtain

$$\begin{aligned}\frac{\pi i(8\varepsilon t^3)^2}{\exp(8\varepsilon t^3)}(\partial_t - \sqrt{3}\partial_\delta)\Delta(\delta,t,\varepsilon) &= 2\pi i(6\varepsilon t^{-1} + \partial_t - \sqrt{3}\partial_\delta)(\mathrm{Ai}_+\,\mathrm{Bi}_- - \mathrm{Bi}_+\,\mathrm{Ai}_-) + O(\delta^{-3/2}\varepsilon^{-1})\\ &= 4\exp(a\sqrt{2}) + O(\varepsilon\delta^{-3/2})\end{aligned} \tag{5.52}$$

For given $\varepsilon > 0$ and δ sufficiently large, the whole domain where real solutions are possible is covered by the vector field $\partial_t - \sqrt{3}\partial_\delta$, along which Δ is monotonic (second line of (5.52). So we get no closed curves of real solutions, and $\boldsymbol{r}_j = \boldsymbol{c}_j$. In the remaining compact set K (which can be kept constant as we increase ε), we use the results from Theorem 5.10. The solutions to $\mathrm{Ai}_+\,\mathrm{Bi}_- - \mathrm{Ai}_-\,\mathrm{Bi}_+ = 0$ are simple smooth curves, along which $\nabla(\mathrm{Ai}_+\,\mathrm{Bi}_- - \mathrm{Ai}_-\,\mathrm{Bi}_+) \neq 0$. Fix a compact tubular neighbourhood of this set such that normal coordinates are defined there and the normal derivative of $\mathrm{Ai}_+\,\mathrm{Bi}_- - \mathrm{Ai}_-\,\mathrm{Bi}_+$ doesn't vanish. Outside this neighbourhood (still inside the compact set K), there is a positive lower bound of $|\mathrm{Ai}_+\,\mathrm{Bi}_- - \mathrm{Ai}_-\,\mathrm{Bi}_+|$, hence there are no zeros of Δ there, provided ε is large enough. Inside the tubular neighbourhood, the gradient does not vanish, so there can be no closed curves there on which $\Delta = 0$ could hold. This proves the last part of Theorem 5.15. ■

Combining parts 3,5,6 of 5.15 and Theorem 5.10, we immediately have

Corollary 5.16 *For ε sufficiently large, $\mathcal{Z}_\varepsilon$ is homeomorphic to $\mathcal{Z}_\infty$.*

Chapter 6

The Zero Viscosity (Euler) Case

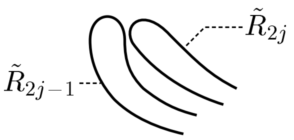
6.1 An Overview

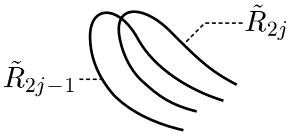
We have seen that for large ε, the zero set $\mathcal{Z}_\varepsilon$ is homeomorphic to $\mathcal{Z}_\infty$ and thus known quite accurately. Starting from there, we view ε as a homotopy parameter approaching 0. It would be too bold to expect that no changes in the topology of $\mathcal{Z}_\varepsilon$ occur as $\varepsilon \searrow 0$. However, numerical experiments suggest that the changes that do occur are very orderly ones. Significant parts of these observations can be confirmed analytically.

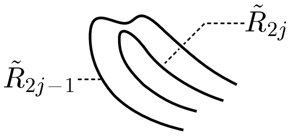
6.1.1 Numerical Observations

As $\varepsilon \to 0$, there is a countably infinite sequence of changes in the topology of $\mathcal{Z}_\varepsilon$ as follows: For a sequence $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \cdots \to 0$, the branches $\tilde{R}_{2j-1}$ and $\tilde{R}_{2j}$ (which have not undergone any topological change for $\varepsilon > \varepsilon_j$) meet at a single point (δ_j, t_j), more precisely the arcs $R_{2j-1,1}$ and $R_{2j,0}$ meet there. We have $\Delta(\delta_j, t_j, \varepsilon_j) = 0 = \partial_\delta \Delta(\delta_j, t_j, \varepsilon_j) = \partial_t \Delta(\delta_j, t_j, \varepsilon_j)$. The arcs combine into new branches as follows:

$$
\begin{array}{lrl}
\text{For } \varepsilon > \varepsilon_j & R_{2j-1,0} \cup R_{2j-1,1} & = \tilde{R}_{2j-1} \\
 & R_{2j,0} \cup R_{2j,1} & = \tilde{R}_{2j} \\
\text{For } \varepsilon = \varepsilon_j & R_{2j-1,0} \cup R_{2j,0} & =: \tilde{R}_{2j-1} \\
 & R_{2j-1,1} \cup R_{2j,1} & = \tilde{R}_{2j} \\
\text{For } \varepsilon < \varepsilon_j & R_{2j-1,0} \cup R_{2j,1} & \subset \tilde{R}_{2j-1} \\
 & R_{2j-1,1} \cup R_{2j,0} & =: \tilde{R}_{2j}
\end{array}
\qquad (6.1)
$$







These bifurcations take place near the critical line $t = \sqrt{3}\,\delta$. Whereas for large ε, two arcs meet near that line and form a branch, for small ε, four such arcs can be continued into the vicinity of that line, and be seen to belong together there. This "period doubling" (with all due apologies for the non-dynamic use of this expression) or "mode crossing" was already observed, but not explained, by Gallagher [6], and is partially, but apparently not completely understood in [22]. See the remarks in the introduction page 9. The real solutions for a sequence of different values of ε are displayed in Figure 6.1. As a sample, we also plot real and complex solutions in the case $\varepsilon = 0.001218$, which is interesting because different combinations of branches coexist there:

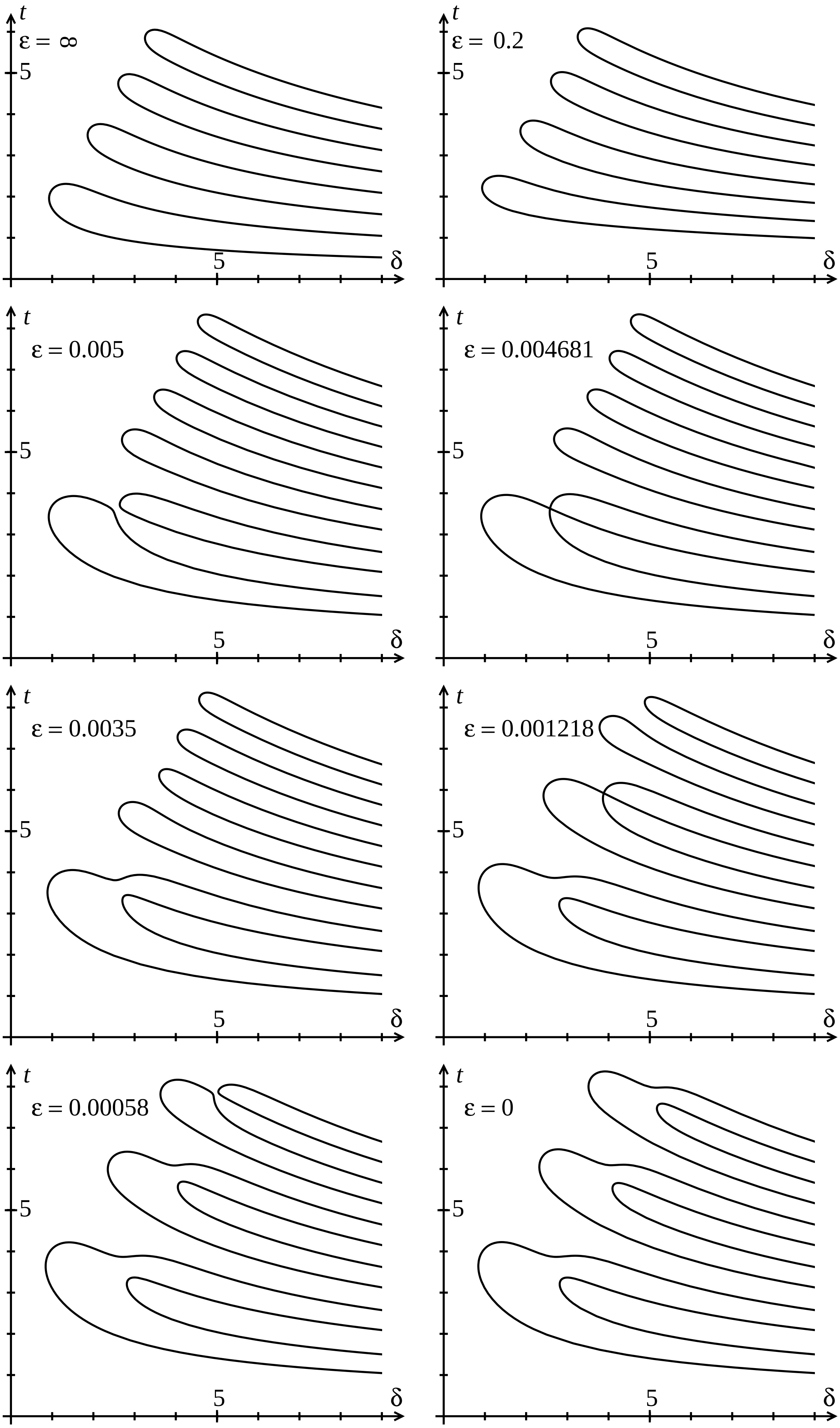


Figure 6.1: The real solutions for different values of ε

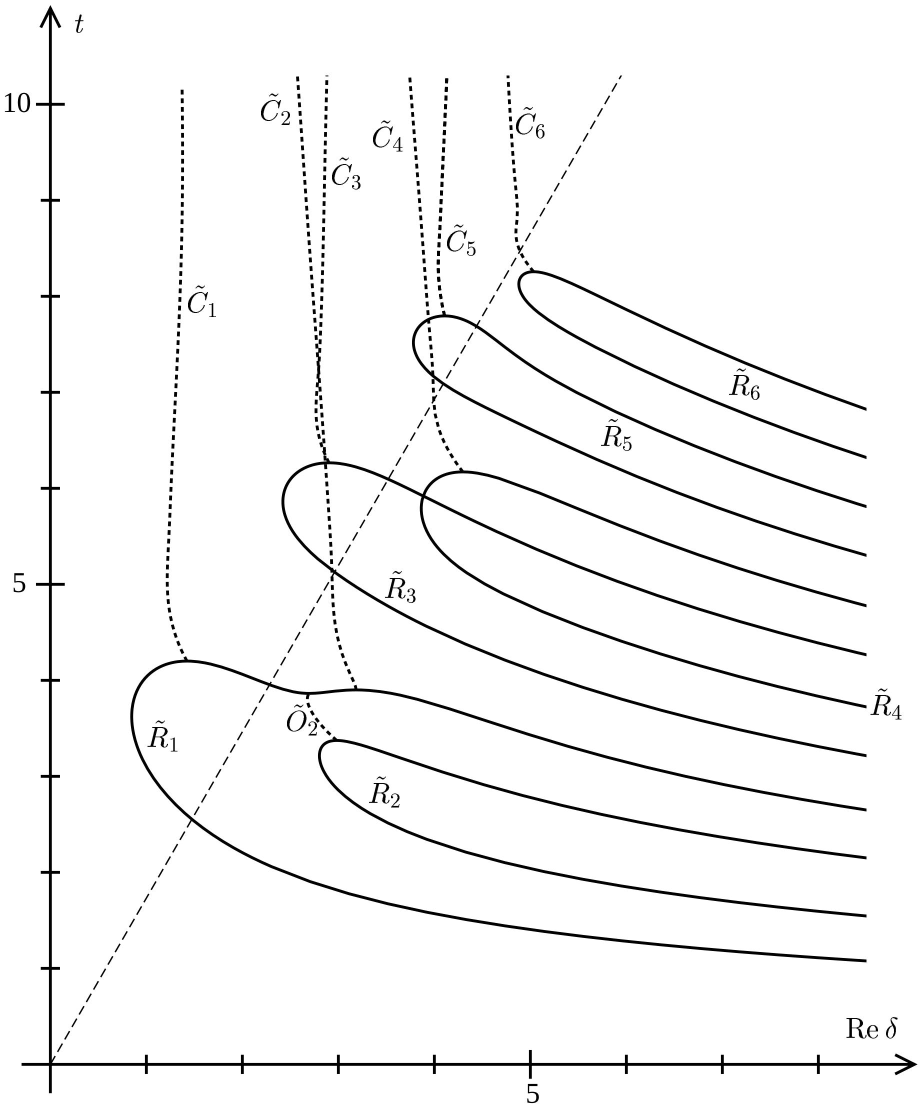


Figure 6.2: The first six real and complex branches for $\varepsilon = 0.001218$

See Figures 6.2 and 6.3. Once and for all, we help with the interpretation of the transformation (1.12) by graphing the corresponding curves in the complex λ plane as well (Figure 6.4).

The branch points $\boldsymbol{r}_n = \boldsymbol{c}_n$ defined in Theorem 5.15 are <u>not</u> affected by these bifurcations, and the complex branches $\tilde{C}_n$ bifurcating at these points do not undergo any changes of topology. For $\varepsilon < \varepsilon_j$, the new branch $\tilde{R}_{2j-1}$ has a local minimum $\boldsymbol{k}_{2j-1}$ between the local maxima $\boldsymbol{r}_{2j-1}$ and $\boldsymbol{r}_{2j}$, and this minimum is connected to the unique local maximum $\boldsymbol{k}_{2j}$ on the new branch $\tilde{R}_{2j}$ by a branch $\tilde{O}_{2j}$ of complex solutions, which is a topological circle. We give a table of the numerical values of the first branch points:

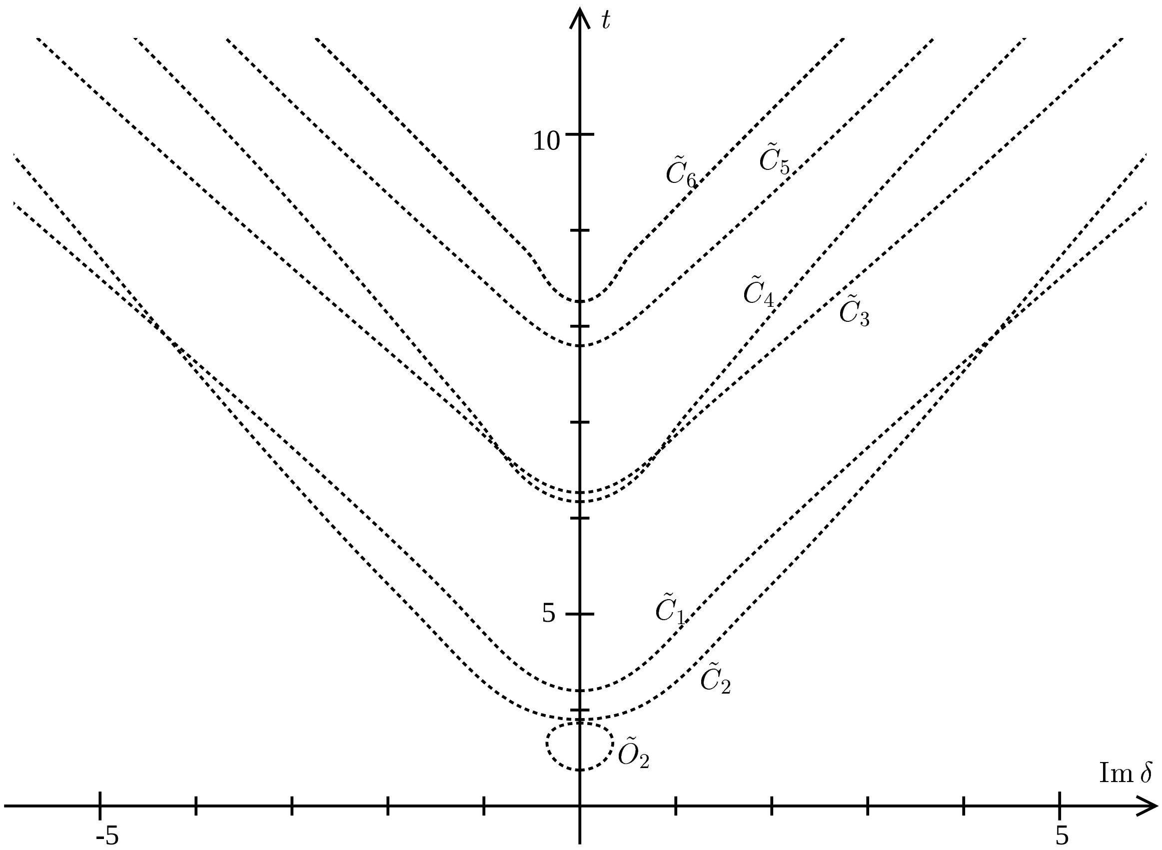
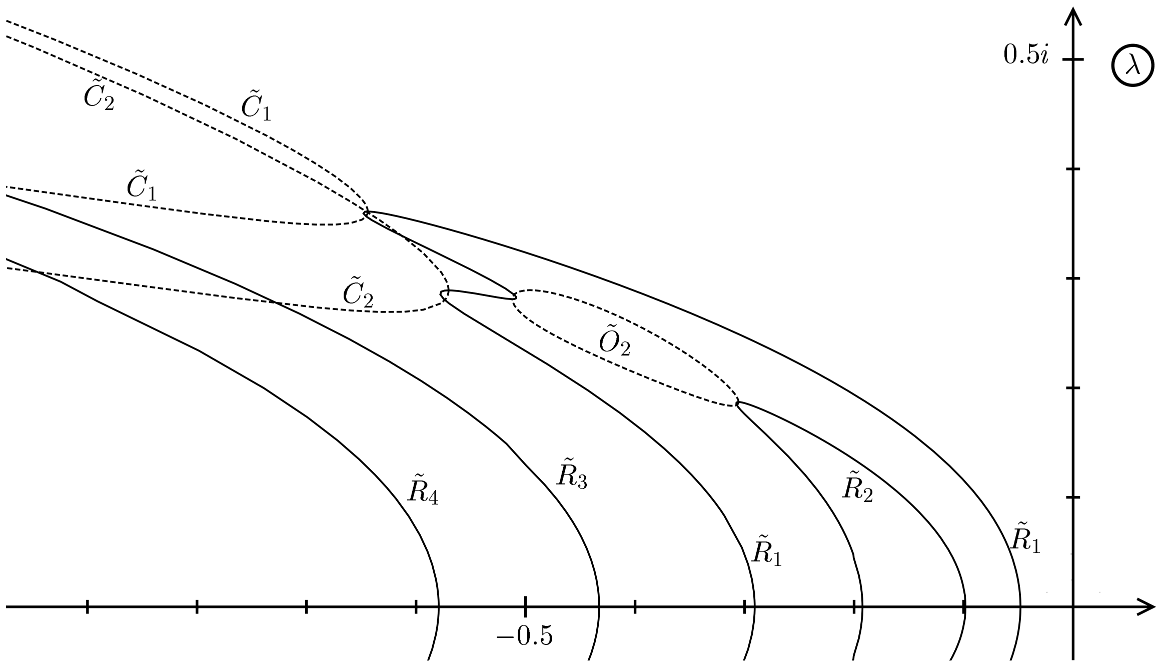


Figure 6.3: The first six complex branches for $\varepsilon = 0.001218$



Note that for given periodic boundary conditions of the original problem in the directions of indefinite extension of the plates, only discrete points are sampled from these curves. Therefore, the crossings of these curves do not imply that there are generically eigenvalues of higher multiplicity.

Figure 6.4: The first branches for $\varepsilon = 0.001218$ in the original complex λ plane.

j	$10^3\varepsilon_j$	δ_j	t_j	$\partial_\varepsilon\tilde{\Delta}$	$10^2\partial_\delta^2\tilde{\Delta}$
1	4.681007	2.581125	3.628174	−1.263092	5.169139
2	1.217666	3.884814	5.919795	−1.221998	1.928315
3	0.551212	4.981716	7.831294	−1.144286	1.052417
4	0.313355	5.960433	9.535014	−1.077669	0.677787
5	0.201936	6.859595	11.099995	−1.022832	0.479396
6	0.140919	7.700104	12.562858	−0.977198	0.360230
7	0.103908	8.494832	13.946013	−0.938561	0.282404

(6.2)

Here, $\tilde{\Delta} := \Delta/(4i\varepsilon t^3)$. As $j \to \infty$, $\varepsilon_j \to 0$.

No other bifurcations seem to occur.

The zero set $\mathcal{Z}_0^+$ consists of

- Real branches $\tilde{R}_n$ ($n \geq 1$), where $\tilde{R}_{2j}$ lies "inside" $\tilde{R}_{2j-1}$.
- Complex branches $\tilde{C}_n$ ($n \geq 1$), where both $\tilde{C}_{2j-1}$ and $\tilde{C}_{2j}$ bifurcate from $\tilde{R}_{2j-1}$ at points $\boldsymbol{r}_{2j-1}$ and $\boldsymbol{r}_{2j}$ respectively. ($j \geq 1$)
- Complex compact branches $\tilde{O}_{2j}$ that connect the unique maximum $\boldsymbol{k}_{2j}$ of $\tilde{R}_{2j}$ with the unique local minimum $\boldsymbol{k}_{2j-1}$ of $\tilde{R}_{2j-1}$. ($j \geq 1$)

Let us now discuss which of these numerical observations can be proved rigorously and how.

6.1.2 Statement of Analytic Results

Clearly it holds $\Delta(\delta, t, 0) \equiv 0$; therefore let us put $\Delta'(\delta,t) = \lim_{\varepsilon\to 0}\Delta(\delta,t,\varepsilon)/\varepsilon$. From (2.1), we obtain

$$\begin{aligned}\Delta'(\delta,t) &= 4t^3 \begin{vmatrix} \int_{-1}^{1} z\,\mathrm{Ai}(\delta+itz)\,dz & \int_{-1}^{1} \mathrm{Ai}(\delta+itz)\,dz \\ \int_{-1}^{1} z\,\mathrm{Bi}(\delta+itz)\,dz & \int_{-1}^{1} \mathrm{Bi}(\delta+itz)\,dz \end{vmatrix} \\ &= -4t \begin{vmatrix} \mathrm{Ai}'(\delta+it)-\mathrm{Ai}'(\delta-it) & \int_{-1}^{1} \mathrm{Ai}(\delta+itz)\,dz \\ \mathrm{Bi}'(\delta+it)-\mathrm{Bi}'(\delta-it) & \int_{-1}^{1} \mathrm{Bi}(\delta+itz)\,dz \end{vmatrix} .\end{aligned} \tag{6.3}$$

Similar as in (5.1) and (5.9), we define

$$\mathcal{Z}_0 := \{(\delta,t) \in \mathbb{C}\times\mathbb{R} \mid \Delta'(\delta,t) = 0\} \tag{6.4}$$

and $\mathcal{Z}_0^+ := \mathcal{Z}_0 \cap \{t > 0\}$. Δ' has a remarkable connection to

$$\underset{\infty}{\Delta} := 4\lim_{\varepsilon\to\infty}(4\varepsilon t^3)^2\exp(-8\varepsilon t^3)\Delta(\delta,t,\varepsilon) = 2(\mathrm{Ai}_+\,\mathrm{Bi}_- - \mathrm{Bi}_+\,\mathrm{Ai}_-)\ ,$$

namely

$$(\partial_\delta^2 + \partial_t^2)\Delta' = 16t\underset{\infty}{\Delta}\ . \tag{6.5}$$

This can be checked in a straightforward way from the definition (6.3), but is also a special case of a more general PDE satisfied by Δ, cf. Section 7.

Formula (6.5) is remarkable in itself. It seems enticing to use the maximum principle to make conclusions for the real zero set of Δ' from the much simpler zero set for $\underset{\infty}{\Delta}$. Alas, all attempts along these lines fail: Each time, the sign is just opposite to what would be needed for the maximum principle.

A different argument based on differential relations gives however the following result:

Theorem 6.1 *For $\varepsilon = 0$, real solutions to $\Delta'(\delta, t) = 0$, $t \neq 0$ cannot cross the line $\delta = 0$.*

This result is less than a linearized stability result, of course; only if $\mathrm{Re}\,\delta > 0$ for all solutions and all ε, one gets $\mathrm{Re}\,\lambda < 0$ from the transformation (1.12) given in the introduction. On the other hand, Romanov's stability result $\mathrm{Re}\,\lambda < 0$ for all ε does imply $\mathrm{Re}\,\delta \geq 0$ as $\varepsilon \to 0$. So the result in itself is no genuine progress. But the author believes it is noteworthy for its method of proof.

What can be said about the changes in the topology of the set $\mathcal{Z}_\varepsilon$ as ε varies?

A careful study of the set $\mathcal{Z}_0$ shows (by means of asymptotic methods again) that all (except possibly finitely many) of the real arcs coming from $t < O(|\delta|^{1/4})$ are arranged in the way shown in Figure 6.1 and predicted by the bifurcation pattern in (6.1). Therefore all (except possibly finitely many) of the real branches $\tilde{R}_{2j}$ and $\tilde{R}_{2j-1}$ have to undergo a topological change, whereas, on the other hand, all such changes can only occur for ε below some fixed number. Moreover, given $\varepsilon^* > 0$, we have seen in Sections 5.3.5 and 5.3.6 how to obtain a compact set K such that branches in $\mathcal{Z}_\varepsilon \setminus K$ can be identified with corresponding branches in $\mathcal{Z}_\infty \setminus K'$, say, for all $\varepsilon \geq \frac{1}{2}\varepsilon^*$. Therefore, if we let ε_j equal the largest ε for which $\tilde{R}_{2j}$ undergoes a change of topology, $\varepsilon_j \to 0$.

By itself, the asymptotic method cannot prove that even finitely many exceptions of bifurcating branches $\tilde{R}_n$ do not occur. Presumably, the asymptotic estimates could be made quantitative, and supported by interval arithmetics and quantitative error terms for the asymptotic series involved, one could enhance the numerical evidence for the first branches by a rigorous proof. We have not undertaken such a program. The numerical calculations based on equation (6.3) and carried out with the help of Mathematica clearly display the claimed pattern for small j.

By the implicit function theorem (IFT), a bifurcation at $(\delta_j, t_j, \varepsilon_j)$ can only happen, if $\Delta(\delta_j, t_j, \varepsilon_j) = 0 = \partial_\delta \Delta(\delta_j, t_j, \varepsilon_j) = \partial_t \Delta(\delta_j, t_j, \varepsilon_j)$. This is the condition from which the branch points tabled in (6.2) were calculated.

The asymptotic expansion in the critical region for $\varepsilon = 0$ is already pretty complicated, and the question how it perturbs for $\varepsilon > 0$ small (while at the same time (δ, t) may be large) is still under investigation. It would give more detailed information about the branch points and could possibly show rigorously that the single series of such points is already the whole story.

At the same time, there is the question to decide how the branches rearrange under bifurcation and to show in particular that each real branch undergoes only one bifurcation. Presently, the evidence for this is a merely a calculation of the first few branches for a sequence of values of ε. However, if we can control the direction of each of the single bifurcations and can prove that each bifurcation creates a pair of extrema rather than annihilate one, as ε decreases, then the direct study of the case $\varepsilon = 0$ bounds the number of bifurcations that have occurred on the way. A first step in this direction is the following simple lemma, which gives the condition in a nutshell:

Lemma 6.2 *Assume $\Delta = 0 = \partial_\delta \Delta = \partial_t \Delta$ for a real valued function Δ at some real point $(\delta_j, t_j, \varepsilon_j)$. And let $H[\Delta]$ be its Hessian determinant $\partial_\delta^2 \Delta \partial_t^2 \Delta - (\partial_\delta \partial_t \Delta)^2$ in the first two variables. If $\partial_\varepsilon \Delta \partial_\delta^2 \Delta / H[\Delta] < 0$, then a pair consisting of a local maximum and a local minimum is created as ε decreases through ε_j; if however $\partial_\varepsilon \Delta \partial_\delta^2 \Delta / H[\Delta] > 0$, then a pair consisting of a local maximum and a local minimum is annihilated as ε decreases through ε_j. If $H[\Delta] > 0$, this pair of extrema is created from nothing as part of small topological circles, if $H[\Delta] < 0$, $\Delta(\cdot, \cdot, \varepsilon_j) = 0$ consists locally of two crossing arcs.*

The proof is a routine calculation using the leading order of Taylor's expansion.

6.2 Proof of Theorem 6.1

For real δ, Δ is purely imaginary. Instead of Δ, we will therefore use $\Delta/i =: \Delta$ and similarly $\underset{\infty}{\Delta}/i =: \underset{\infty}{\Delta}$, $\Delta'/i =: \Delta'$.

We want to show $\Delta'(0,t) < 0$ for $t > 0$. For that purpose, we use the series expansion: It holds, from (4.53) and (4.25)

$$\Delta'(\delta,t) = -\frac{16}{\pi}\sum_{j=0}^{\infty}\sum_{m=3j}^{\infty}\frac{(-1)^m H_{0,m,j}}{(2m+1)!}\,\delta^{m-3j}t^{2m+4} \tag{6.6}$$

and therefore

$$\Delta'(0,t) = -\frac{16}{\pi}\sum_{j=0} s_j t^{6j+4} \quad \text{with} \quad s_j = \frac{4^{3j+1}R_j(3j,0)(4j+2)!}{(6j+1)!(6j+4)!} \tag{6.7}$$

We claim that $s_j > 0$ for all j (which is sufficient for $\Delta'(0,t) < 0$). Alas this is not obvious from any of the formulas (4.46)–(4.50) for R_j, and also not from a direct calculation based on immediate termwise integration of the series (3.2) inserted into (6.3). Merely for reference, we give this formula nevertheless:

$$s_j = (-1)^j\sum_{k=0}^{j}\left(\frac{(6k+2)!!!(6l-2)!!!}{(6k+3)!\,(6l+1)!} - \frac{(6k+2)!!!(6l-2)!!!}{(6k+5)!\,(6l-1)!}\right)\Bigg|_{l=j-k}. \tag{6.8}$$

Instead we use Lemma 7.4. This ODE for $t \mapsto \Delta'(0,t)$ implies a recursion formula for its Taylor coefficients, namely

$$\begin{bmatrix} -39216(6j+10)^{\downarrow 6}(6j+1)^{\downarrow 3} \\ -24(6j+1)^{\downarrow 3}A(j) \\ 12B(j) \\ 15000(j-1) \end{bmatrix}^T \begin{bmatrix} s_{j+1} \\ s_j \\ s_{j-1} \\ s_{j-2} \end{bmatrix} = 0 \qquad \text{with} \tag{6.9}$$
$$A(j) = 1215000j^6 - 3037500j^5 + 1370250j^4 + 1698975j^3 - 1466382j^2 - 460821j + 86918$$
$$B(j) = 1215000j^5 - 5062500j^4 + 8250750j^3 - 6567075j^2 + 2646977j - 350000$$

The substitution $S_j := 12j(s_{j-1} + (6j-1)(6j+1)(6j+2)(6j+4)s_j)$ simplifies this recursion to

$$\begin{bmatrix} -19608(6j+9)(6j+1)^{\downarrow 3} \\ -3(-659 - 300975j + 594750j^2 - 472500j^3 + 135000j^4) \\ 1250 \end{bmatrix}^T \begin{bmatrix} S_{j+1} \\ S_j \\ S_{j-1} \end{bmatrix} = 0 \tag{6.10}$$

Together with the initial values $s_0 = 1/3$, $s_1 = 1/14175$, $s_2 = 2/127702575$ obtained from either (6.7), (4.26ff) or (6.8), hence $S_1 = 172/27$, $S_2 = 584/42525$, the solution of this recursion is

$$S_j = \frac{2^{j+1}(30j+13)}{3^{j+1}(2j-1)(2j+1)\,(j-1)!\,(6j-5)!!}, \tag{6.11}$$

as can be easily checked by straightforward calculation. Then we get, after a small calculation

$$s_j = \frac{3^{2j}(2j+1)!}{(6j+4)!}\left\{(-1)^j 8 + \sum_{k=1}^{j}(-1)^{j-k}\frac{2^{4k-1}(30k+13)\,(3k)!}{3^{3k}(2k+1)!\,k!}\right\} \tag{6.12}$$

The braces contain a sum of terms with alternating sign, but with absolute value strictly increasing. Therefore the sign of the sum is determined by its last term. This proves Theorem 6.1.

6.3 Asymptotic Estimates

6.3.1 Basic Estimates for Δ'

In this section, we give asymptotic expansions for $\Delta'(\delta,t)$ as $|\delta|^2+t^2\to\infty$. The essential new objects are the integrals $\int_{-1}^{1}(\mathrm{Ai}\,|\,\mathrm{Bi})(\delta+itz)\,dz$. As before, we use $\zeta_\pm := \frac{2}{3}(\delta\pm it)^{3/2}$. For reasons that become clear later, we calculate two terms of the asymptotic expansions.

Lemma 6.3 *As* $|\delta|^2+t^2\to\infty$ *with* $\arg(\delta+it)\in\,]-\pi/3,\pi[$ *and* $\arg(\delta-it)\in\,]-\pi,\pi/3[$, *we have*

$$\begin{aligned}
E_1 &:= \int_{-1}^{1}\mathrm{Ai}(\delta+itz)\,dz = \\
&\frac{-i}{\sqrt{6\pi}\,t}\Big[e^{-\zeta_-}\Big(\zeta_-^{-1/2}-\tfrac{41}{72}\zeta_-^{-3/2}+O(\zeta_-^{-5/2})\Big)-e^{-\zeta_+}\Big(\zeta_+^{-1/2}-\tfrac{41}{72}\zeta_+^{-3/2}+O(\zeta_+^{-5/2})\Big)\Big]
\end{aligned}\tag{6.13}$$

$$\begin{aligned}
E_2 &:= \int_{-1}^{1}\mathrm{Bi}(\delta+itz)\,dz = \frac{\boldsymbol{2}}{\boldsymbol{t}}+ \\
&+\frac{2i}{\sqrt{6\pi}\,t}\Big[e^{\zeta_-}\Big(\zeta_-^{-1/2}+\tfrac{41}{72}\zeta_-^{-3/2}+O(\zeta_-^{-5/2})\Big)-e^{\zeta_+}\Big(\zeta_+^{-1/2}+\tfrac{41}{72}\zeta_+^{-3/2}+O(\zeta_+^{-5/2})\Big)\Big] \\
&-\frac{1}{\sqrt{6\pi}\,t}\Big[e^{-\zeta_-}\Big(\zeta_-^{-1/2}-\tfrac{41}{72}\zeta_-^{-3/2}+O(\zeta_-^{-5/2})\Big)+e^{-\zeta_+}\Big(\zeta_+^{-1/2}-\tfrac{41}{72}\zeta_+^{-3/2}+O(\zeta_+^{-5/2})\Big)\Big]
\end{aligned}\tag{6.14}$$

$$\begin{aligned}
E_3 &:= \mathrm{Ai}'_+-\mathrm{Ai}'_- = \\
&-\frac{(3/2)^{1/6}}{2\sqrt{\pi}}\Big[\zeta_+^{1/6}e^{-\zeta_+}\Big(1+\tfrac{7}{72}\zeta_+^{-1}+O(\zeta_+^{-2})\Big)-\zeta_-^{1/6}e^{-\zeta_-}\Big(1+\tfrac{7}{72}\zeta_-^{-1}+O(\zeta_-^{-2})\Big)\Big]
\end{aligned}\tag{6.15}$$

$$\begin{aligned}
E_4 &:= \mathrm{Bi}'_+-\mathrm{Bi}'_- = \\
&\frac{(3/2)^{1/6}}{\sqrt{\pi}}\Big[\zeta_+^{1/6}e^{\zeta_+}\Big(1-\tfrac{7}{72}\zeta_+^{-1}+O(\zeta_+^{-2})\Big)-\zeta_-^{1/6}e^{\zeta_-}\Big(1-\tfrac{7}{72}\zeta_-^{-1}+O(\zeta_-^{-2})\Big)\Big] \\
&-\frac{i(3/2)^{1/6}}{2\sqrt{\pi}}\Big[\zeta_+^{1/6}e^{-\zeta_+}\Big(1+\tfrac{7}{72}\zeta_+^{-1}+O(\zeta_+^{-1})\Big)+\zeta_-^{1/6}e^{-\zeta_-}\Big(1+\tfrac{7}{72}\zeta_-^{-1}+O(\zeta_-^{-1})\Big)\Big]
\end{aligned}\tag{6.16}$$

Already here, let us point at the term $2/t$ in (6.14). This term will be responsible for the mode crossing displayed in (6.1). We are going to trace it in subsequent formulas by typesetting it in boldface.

PROOF:
(6.15), (6.16) are just repetitions from Section 3.3. With the purpose of later integrating the asymptotic series from points at infinity where the exponentials decay, we write

$$\int_{-1}^{1}\mathrm{Ai}(\delta+itz)\,dz=\frac{1}{it}\left(\int_{\delta-it}^{\infty}\mathrm{Ai}(y)\,dy-\int_{\delta+it}^{\infty}\mathrm{Ai}(y)\,dy\right)\,.$$

For Bi, a careful decomposition is necessary. With the shorthand notations $\omega_k := \infty e^{ki\pi/3}$ and $\delta_\pm := \delta \pm it$, we obtain

$$\begin{aligned}
it\int_{-1}^{1} \mathrm{Bi}(\delta+itz)\,dz &= \Big(\int_{\delta_-}^{\omega_{-2}} - \int_{\delta_+}^{\omega_2} + \int_{\omega_{-2}}^{\omega_2}\Big)\,\mathrm{Bi}(y)\,dy = \\
&= \Big(-i\int_{\delta_-}^{\omega_{-2}} - i\int_{\delta_+}^{\omega_2}\Big)\,\mathrm{Ai}(y)\,dy \\
&\quad + 2e^{i\pi/6}\int_{\delta_-}^{\omega_{-2}} \mathrm{Ai}(ye^{2\pi i/3})\,dy - 2e^{-i\pi/6}\int_{\delta_+}^{\omega_2} \mathrm{Ai}(ye^{-2\pi i/3})\,dy + \int_{\omega_{-2}}^{\omega_2} \mathrm{Bi}(y)\,dy = \\
&= \Big(-i\int_{\delta_-}^{\omega_0} - i\int_{\delta_+}^{\omega_0}\Big)\,\mathrm{Ai}(y)\,dy + i\Big(\int_{\omega_{-2}}^{\omega_0} + \int_{\omega_2}^{\omega_0}\Big)\,\mathrm{Ai}(y)\,dy \\
&\quad + 2e^{i\pi/6}\int_{\delta_-}^{\omega_{-2}} \mathrm{Ai}(ye^{2\pi i/3})\,dy - 2e^{-i\pi/6}\int_{\delta_+}^{\omega_2} \mathrm{Ai}(ye^{-2\pi i/3})\,dy + \int_{\omega_{-2}}^{\omega_2} \mathrm{Bi}(y)\,dy \\
&\text{“="} \Big(\int_{\delta_-}^{\omega_{-1}} - \int_{\delta_+}^{\omega_1}\Big)\,\mathrm{Bi}(y)\,dy + 6i\int_0^\infty \mathrm{Ai}(r)\,dr
\end{aligned} \tag{6.17}$$

The last equality sign is in quotes: it is completely legitimate on the level of integrals, but less so, when asymptotic series are to be integrated termwise. The paths of integration have been shifted to the extreme argument possible, producing (only) conditionally convergent integrals, but permitting to combine two Ai-integrals into one Bi-integral. The usage of (3.19) under this integral is technically dubious, but legitimized by using (3.13) in the the previous line, which amounts to the very same series. In effect, we have just argued that it is permissible to integrate formally (3.19) as specified in the last line and not bother about contributions from infinity, and the quotes in “=” are a reminder of just that subtlety.

From (3.6), we know already that $\int_0^\infty \mathrm{Ai}(r)\,dr = 1/3$. This already accounts for the $2/t$ term. The asymptotic expansion (3.13) is uniform on the paths of integration chosen in the case of Ai, and in the case of Bi, we may pretend that (3.19) were uniform on the path of integration, as has just been justified. The integrals $\int^\infty \exp[-\zeta]\zeta^{-1/2}d\zeta$ and $\int^{-\infty} \exp[\zeta]\zeta^{-1/2}d\zeta$ can now be treated by integration by parts. Only the value of the integrand at the lower limit of integration affects the leading order. The lemma follows immediately. ■

With sufficient perseverance, one obtains:

Theorem 6.4 *Under the assumptions of Lemma 6.3, namely* $\arg(\delta+it) \in]-\pi/3, \pi[$ *and* $\arg(\delta-it) \in]-\pi, \pi/3[$*, the following asymptotic expansion holds as* $\delta \pm it \to \infty$*:*

$$\begin{aligned}
&-\frac{\pi}{4}\Big(\frac{3}{2}\Big)^{1/3}\Delta'(\delta,t) = \\
&= -\frac{\sqrt{6\pi}}{2}\Big[e^{-\zeta_+}\zeta_+^{1/6} - e^{-\zeta_-}\zeta_-^{1/6}\Big] && + e^{-\zeta_\pm}\,O(\zeta_\pm^{-5/6}) \\
&\quad - i\sinh(\zeta_+ - \zeta_-)\,(\zeta_+\zeta_-)^{1/6}\,(\zeta_+^{-2/3} - \zeta_-^{-2/3}) && + \cosh\mathrm{Re}(\zeta_+ - \zeta_-)\,O(\zeta_*^{-4/3}) \\
&\quad + \frac{2i}{3}(\zeta_+^{-4/3} + \zeta_-^{-4/3}) && + O(\zeta_\pm^{-7/3}) \\
&\quad - \frac{1}{2}e^{-(\zeta_+ + \zeta_-)}(\zeta_+\zeta_-)^{1/6}\,(\zeta_+^{-2/3} - \zeta_-^{-2/3}) && + \exp[-\mathrm{Re}(\zeta_+ + \zeta_-)]\,O(\zeta_*^{-4/3})
\end{aligned} \tag{6.18}$$

where we have used the shorthand $O(\zeta_*^{-4/3})$ *for* $O(\zeta_\pm^{-1/2}\zeta_\mp^{-5/6}) + O(\zeta_\pm^{-3/2}\zeta_\mp^{1/6})$.

In these formulas, each of the O–symbols containing $\pm$ *refers to two error terms, one with a consistent choice of the upper signs, one with the lower signs.*

Similar formulas hold for the derivatives (under the same assumptions):

$$
\begin{aligned}
&-\frac{\pi}{4}\,\partial_\delta \Delta'(\delta,t) = && (6.19)\\
&= +\frac{\sqrt{\boldsymbol{6\pi}}}{\mathbf{2}}\Big[e^{-\zeta_+}\zeta_+^{1/2} - e^{-\zeta_-}\zeta_-^{1/2}\Big] && + e^{-\zeta_\pm}\, O(\zeta_\pm^{-1/2})\\
&\quad - i\cosh(\zeta_+ - \zeta_-)\,(\zeta_+^{1/3} - \zeta_-^{1/3})\,(\zeta_+\zeta_-)^{1/6}\,(\zeta_+^{-2/3} - \zeta_-^{-2/3}) && + \cosh\mathrm{Re}(\zeta_+ - \zeta_-)\, O(\zeta_*^{-1})\\
& && + O(\zeta_\pm^{-2})\\
&\quad + \frac{1}{2}e^{-(\zeta_+ + \zeta_-)}(\zeta_+^{1/3} + \zeta_-^{1/3})\,(\zeta_+\zeta_-)^{1/6}\,(\zeta_+^{-2/3} - \zeta_-^{-2/3}) && + \exp[-\mathrm{Re}(\zeta_+ + \zeta_-)]\, O(\zeta_*^{-1})
\end{aligned}
$$

and

$$
\begin{aligned}
&-\frac{\pi}{4}\,(-i)\partial_t \Delta'(\delta,t) = && (6.20)\\
&= +\frac{\sqrt{\boldsymbol{6\pi}}}{\mathbf{2}}\Big[e^{-\zeta_+}\zeta_+^{1/2} + e^{-\zeta_-}\zeta_-^{1/2}\Big] && + e^{-\zeta_\pm}\, O(\zeta_\pm^{-1/2})\\
&\quad - i\cosh(\zeta_+ - \zeta_-)\,(\zeta_+^{1/3} + \zeta_-^{1/3})\,(\zeta_+\zeta_-)^{1/6}\,(\zeta_+^{-2/3} - \zeta_-^{-2/3}) && + \cosh\mathrm{Re}(\zeta_+ - \zeta_-)\, O(\zeta_*^{-1})\\
& && + O(\zeta_\pm^{-2})\\
&\quad + \frac{1}{2}e^{-(\zeta_+ + \zeta_-)}(\zeta_+^{1/3} - \zeta_-^{1/3})\,(\zeta_+\zeta_-)^{1/6}\,(\zeta_+^{-2/3} - \zeta_-^{-2/3}) && + \exp[-\mathrm{Re}(\zeta_+ + \zeta_-)]\, O(\zeta_*^{-1})
\end{aligned}
$$

where we have used the shorthand $O(\zeta_*^1)$ *for* $O(\zeta_\pm^{-1/2}\zeta_\mp^{-1/2}) + O(\zeta_\pm^{-3/2}\zeta_\mp^{1/2})$.

PROOF:
Just a clumsy calculation: The strategy is that in the end we want to have left one non-vanishing term with each type of exponential that comes up.

$$
\begin{aligned}
&-\tfrac{\pi}{4}\,(\tfrac{3}{2})^{1/3}\Delta'(\delta,t) = t\pi(3/2)^{1/3}(E_2E_3 - E_1E_4) =\\
&= -\frac{\sqrt{\boldsymbol{6\pi}}}{\mathbf{2}} \times \Big\{e^{-\zeta_+}\Big(\zeta_+^{1/6} + O(\zeta_+^{-5/6})\Big) - e^{-\zeta_-}\Big(\zeta_-^{1/6} + O(\zeta_-^{-5/6})\Big)\Big\}\\
&\quad + \Big\{\frac{i}{2}\Big[e^{\zeta_+}\Big(\zeta_+^{-1/2} + \tfrac{41}{72}\zeta_+^{-3/2} + O(\zeta_+^{-5/2})\Big) - e^{\zeta_-}\Big(\zeta_-^{-1/2} + \tfrac{41}{72}\zeta_-^{-3/2} + O(\zeta_-^{-5/2})\Big)\Big]\\
&\qquad + \frac{1}{4}\Big[e^{-\zeta_+}\Big(\zeta_+^{-1/2} - \tfrac{41}{72}\zeta_+^{-3/2} + O(\zeta_+^{-5/2})\Big) + e^{-\zeta_-}\Big(\zeta_-^{-1/2} - \tfrac{41}{72}\zeta_-^{-3/2} + O(\zeta_-^{-5/2})\Big)\Big]\Big\}\\
&\quad \times \Big\{e^{-\zeta_+}\Big(\zeta_+^{1/6} + \tfrac{7}{72}\zeta_+^{-5/6} + O(\zeta_+^{-11/6})\Big) - e^{-\zeta_-}\Big(\zeta_-^{1/6} + \tfrac{7}{72}\zeta_-^{-5/6} + O(\zeta_-^{-11/6})\Big)\Big\}\\
&\quad + \frac{1}{2i}\Big\{e^{-\zeta_+}\Big(\zeta_+^{-1/2} - \tfrac{41}{72}\zeta_+^{-3/2} + O(\zeta_+^{-5/2})\Big) - e^{-\zeta_-}\Big(\zeta_-^{-1/2} - \tfrac{41}{72}\zeta_-^{-3/2} + O(\zeta_-^{-5/2})\Big)\Big\}\\
&\quad \times \Big\{\ \Big[e^{\zeta_+}\Big(\zeta_+^{1/6} - \tfrac{7}{72}\zeta_+^{-5/6} + O(\zeta_+^{-11/6})\Big) - e^{\zeta_-}\Big(\zeta_-^{1/6} - \tfrac{7}{72}\zeta_-^{-5/6} + O(\zeta_-^{-11/6})\Big)\Big]\\
&\qquad - \frac{i}{2}\Big[e^{-\zeta_+}\Big(\zeta_+^{1/6} + \tfrac{7}{72}\zeta_+^{-5/6} + O(\zeta_+^{-11/6})\Big) + e^{-\zeta_-}\Big(\zeta_-^{1/6} + \tfrac{7}{72}\zeta_-^{-5/6} + O(\zeta_-^{-11/6})\Big)\Big]\Big\} =
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{\sqrt{6\pi}}{2}\Big[e^{-\zeta_+}\zeta_+^{1/6} - e^{-\zeta_-}\zeta_-^{1/6}\Big] + O(e^{-\zeta_\pm}\zeta_\pm^{-5/6})\\
&\quad + \frac{i}{2}\Big[\zeta_+^{-1/3} + \tfrac{2}{3}\zeta_+^{-4/3} + \zeta_-^{-1/3} + \tfrac{2}{3}\zeta_-^{-4/3} - e^{\zeta_+-\zeta_-}\zeta_+^{-1/2}\zeta_-^{1/6} - e^{\zeta_--\zeta_+}\zeta_-^{-1/2}\zeta_+^{1/6}\Big]\\
&\quad + O(\zeta_\pm^{-7/3}) + \cosh\operatorname{Re}(\zeta_+-\zeta_-)\,O(\zeta_*^{-4/3})\\
&\quad + \frac{1}{4}\Big[e^{-2\zeta_+}\zeta_+^{-1/3} - \tfrac{34}{72}e^{-2\zeta_+}\zeta_+^{-4/3} + e^{-2\zeta_-}\zeta_-^{-1/3} - \tfrac{34}{72}e^{-2\zeta_-}\zeta_-^{-4/3}\\
&\qquad\qquad + e^{-(\zeta_++\zeta_-)}(\zeta_-^{-1/2}\zeta_+^{1/6} - \zeta_+^{-1/2}\zeta_-^{1/6})\Big]\\
&\quad + e^{-\operatorname{Re}(\zeta_++\zeta_-)}\,O(\zeta_*^{-4/3}) + e^{-2\operatorname{Re}\zeta_\pm}\,O(\zeta_\pm^{-7/3})\\
&\quad - \frac{i}{2}\Big[\zeta_+^{-1/3} - \tfrac{2}{3}\zeta_+^{-4/3} + \zeta_-^{-1/3} - \tfrac{2}{3}\zeta_-^{-4/3} - e^{\zeta_+-\zeta_-}\zeta_-^{-1/2}\zeta_+^{1/6} - e^{\zeta_--\zeta_+}\zeta_+^{-1/2}\zeta_-^{1/6}\Big]\\
&\quad - \frac{1}{4}\Big[e^{-2\zeta_+}\zeta_+^{-1/3} - \tfrac{34}{72}e^{-2\zeta_+}\zeta_+^{-4/3} + e^{-2\zeta_-}\zeta_-^{-1/3} - \tfrac{34}{72}e^{-2\zeta_-}\zeta_-^{-4/3}\\
&\qquad\qquad - e^{-(\zeta_++\zeta_-)}(\zeta_-^{-1/2}\zeta_+^{1/6} - \zeta_+^{-1/2}\zeta_-^{1/6})\Big] =
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{\sqrt{6\pi}}{2}\Big[e^{-\zeta_+}\zeta_+^{1/6} - e^{-\zeta_-}\zeta_-^{1/6}\Big] && + e^{-\zeta_\pm}\,O(\zeta_\pm^{-5/6})\\
&\quad - i\sinh(\zeta_+-\zeta_-)\,(\zeta_+\zeta_-)^{1/6}\,(\zeta_+^{-2/3} - \zeta_-^{-2/3}) && + \cosh\operatorname{Re}(\zeta_+-\zeta_-)\,O(\zeta_*^{-4/3})\\
&\quad + \frac{2i}{3}(\zeta_+^{-4/3} + \zeta_-^{-4/3}) && + O(\zeta_\pm^{-7/3})\\
&\quad - \frac{1}{2}e^{-(\zeta_++\zeta_-)}(\zeta_+\zeta_-)^{1/6}\,(\zeta_+^{-2/3} - \zeta_-^{-2/3}) && + e^{-\operatorname{Re}(\zeta_++\zeta_-)}\,O(\zeta_*^{-4/3})\\
& && + e^{-2\operatorname{Re}\zeta_\pm}\,O(\zeta_\pm^{-7/3})
\end{aligned}
$$

In these formulas, we have used the abbreviation $O(\zeta_*^{-4/3})$ defined in the theorem, and the O–symbols containing $\pm$ refer to the worst choice of signs, but not crosswise, as explained in the theorem. The final formula is exactly the claimed one, except that we have dropped the $\exp[-2\operatorname{Re}\zeta_\pm]\,O(\zeta_\pm^{-7/3})$ term in the very last line, which, of course, needs justification. Not only do all the coefficients in the asymptotic series with factor $e^{-2\zeta_\pm}$ vanish, but the whole contribution vanishes. This can be seen when we look at the $\int_{\delta_\pm}^{\infty}\operatorname{Ai}(y)\,dy$ term before the quoted equality sign in (6.17), and similarly leave the terms from $\operatorname{Bi}'_+ - \operatorname{Bi}'_-$ in a form containing only Airy functions Ai, according to (3.10), of which (3.19) is a corollary. Then the terms $\int_{\delta_+}^{\infty}\operatorname{Ai}(y)\,dy \times \operatorname{Ai}'_+$ cancel exactly, and so do the corresponding ones with δ_-.

Let us now turn to the derivatives. Basically, the asymptotic series in Lemma 6.3 may be derived termwise, for the reason discussed already in Section 5.3.3, and this fact can also be checked directly by glancing at Section 3.3.

Now note $(\frac{3}{2})^{-1/3}\partial_\delta\zeta_\pm = \zeta_\pm^{1/3}$ and $(\frac{3}{2})^{-1/3}(-i)\partial_t\zeta_\pm = \pm\zeta_\pm^{1/3}$. The $O(\cdot)$-terms involved receive a factor $\zeta_\pm^{1/3}$ from the derivative of the exponentials, whereas the powers of $\zeta_\pm$ improve under differentiation such as to vanish into the $O(\cdot)$-terms. All derivatives carry through the huge calculation until we get the corresponding derivatives of (6.18). ■

These calculations have assumed that $\arg(\delta+it) \in\,]-\pi/3,\pi[$ and $\arg(\delta-it) \in\,]-\pi,\pi/3[$. If however both of these are in $]-\pi/3,\pi[$, we replace the basis function $\operatorname{Bi}(x)$ by $\operatorname{Bi}(x) - i\operatorname{Ai}(x) = 2e^{-i\pi/6}\operatorname{Ai}(xe^{-2\pi i/3})$, with the effect that the $\exp[-\zeta]$ term in (3.19) can be dropped. Moreover, the simple decomposition of the path of integration is possible, and no $2/t$ term arises. We get therefore only the middle term of (6.18). Then the formula is similar to (5.40), and as in the case of large ε, we conclude that this case does not contribute zeros of Δ'.

On the other hand, if either of $\delta \pm it$ approaches the negative real axis, the above estimates lose their uniformity. In that case, we assume $\operatorname{Re}\delta \geq 0$ and calculate $\Delta'(-\delta, t)$, using the basis functions $\operatorname{Ai}(xe^{\pm 2\pi i/3})$.

$$\frac{\Delta'(\delta,t)}{8it} = \begin{vmatrix} e^{-i\pi/6}\Big[\operatorname{Ai}'(\delta_+ e^{-i\pi/3}) - \operatorname{Ai}'(\delta_- e^{-i\pi/3})\Big] & -e^{i\pi/6}\displaystyle\int_{-1}^{1}\operatorname{Ai}\Big((\delta+itz)e^{-i\pi/3}\Big)\,dz \\ e^{i\pi/6}\Big[\operatorname{Ai}'(\delta_+ e^{i\pi/3}) - \operatorname{Ai}'(\delta_- e^{i\pi/3})\Big] & e^{-i\pi/6}\displaystyle\int_{-1}^{1}\operatorname{Ai}\Big((\delta+itz)e^{i\pi/3}\Big)\,dz \end{vmatrix} \tag{6.21}$$

with $\delta_\pm := \delta \pm it$.

Here, too, we can employ (6.13), the leading terms suffice, and finally we obtain

$$\begin{aligned}\Delta'(-\delta,t) = {} & \frac{4i}{\pi}\left(\frac{(\delta-it)^{1/4}}{(\delta+it)^{3/4}} - \frac{(\delta+it)^{1/4}}{(\delta-it)^{3/4}}\right)\sin(\zeta_+ - \zeta_-) \\ & + O(\zeta_\pm^{-4/3}) + \cosh\operatorname{Im}(\zeta_+ - \zeta_-)\Big(O(\zeta_\mp^{1/6}\zeta_\pm^{-3/2}) + O(\zeta_\mp^{-1/2}\zeta_\pm^{-5/6})\Big)\end{aligned} \tag{6.22}$$

The same reasoning as was applied for the case of large ε with equation (5.39) shows that there are no non-trivial solutions to $\Delta' = 0$ in this domain, except possibly inside some compact set. (Theorem 4.8 is again used to deal with $t \ll |\delta|$.) We also conclude that the sign of $\Delta'(-\delta,t)/i$ is negative for $t > 0$ and real $\delta \geq 0$. This fits well to the result in Theorem 6.1.

Let us collect these observations in

Lemma 6.5 *Except possibly for solutions restricted to some compact set, only the case* $\operatorname{Re}\delta > 0$, $\arg(\delta+it) \in \,]-\pi/3, \pi[$, $\arg(\delta-it) \in \,]-\pi, \pi/3[$ *contributes to solutions of the equation* $\Delta'(\delta,t) = 0$.

PROOF:
For the asymptotic cases, we have just seen this. Like for $\varepsilon > 0$, there is a non-asymptotic case to be considered for $\varepsilon = 0$, too. It does contribute solutions, and is dealt with in the following theorem. ∎

The non-asymptotic case for $\varepsilon = 0$ is qualitatively like the one for $\varepsilon > 0$, but differs quantitatively. The term $\cosh 4\varepsilon t^3$ and $\sinh 4\varepsilon t^3$ that play the crucial role in the spectral condition account for this jump: Like the competition of the limits $\varepsilon \to \infty$ and $t \to 0$ produced the jump discontinuity between $\varepsilon < \infty$ and $\varepsilon = \infty$ in the asymptotic behaviour for the branches with small t (nicely visible in Figure 6.1 above), the competition of the limits $\varepsilon \to 0$ and $t \to \infty$ produces a similar jump discontinuity between $\varepsilon > 0$ and $\varepsilon = 0$ for the branches with large t:

Theorem 6.6 *Suppose that on the set $\mathcal{Z}_0$, t tends to infinity, but $\delta - it$ remains bounded. Then $\delta - it$ converges to some number $A_k e^{2\pi i/3}$, where A_k is a zero of the function* $\mathbf{A} : z \mapsto \int_z^\infty \operatorname{Ai}(x)\,dx$.
All zeros of this function lie in the sector $|\arg(-z)| < \pi/6$ and come in pairs of complex conjugates $A_n^\pm$, $n \geq 1$ (no real zeros), and obey the asymptotic law

$$A_n^\pm \sim -\left[3\pi\Big(n - \tfrac{1}{8}\Big) \mp \tfrac{3}{2}i \operatorname{arcosh}\pi\sqrt{3\Big(n-\tfrac{1}{8}\Big)}\right]^{2/3} \tag{6.23}$$

PROOF:
Using the basis functions $\operatorname{Ai}(x)$ and $\operatorname{Ai}(xe^{-2\pi i/3})$, we write the condition $\Delta'(\delta,t) = 0$ in the form

$$\begin{aligned}&\int_{\delta-it}^{\delta+it}\operatorname{Ai}(ye^{-2\pi i/3})\,dy = \\ &= \frac{\int_{\delta-it}^{\delta+it}\operatorname{Ai}(y)\,dy}{\operatorname{Ai}'(\delta+it) - \operatorname{Ai}'(\delta-it)}\left[\operatorname{Ai}'\Big((\delta+it)e^{-2\pi i/3}\Big) - \operatorname{Ai}'\Big((\delta-it)e^{-2\pi i/3}\Big)\right].\end{aligned} \tag{6.24}$$

As $t \to \infty$ with $\delta - it$ bounded, the quotient on the right hand side tends to zero (like $O(|\delta + it|^{-1})$), and also $\mathrm{Ai}'\big((\delta + it)e^{-2\pi i/3}\big) \to 0$. This shows that $(\delta - it)e^{-2\pi i/3}$ must be a zero of $z \mapsto \int_z^{\infty \exp(-i\pi/6)} \mathrm{Ai}(x)\,dx$.

A count of the total number of zeros of $\mathbf{A}$ shows that the numbering in (6.23) with n starting at 1 takes account of all zeros. We shall carry this out in a minute.

The proof that no zeros lie outside the given sector follows from the argument principle (using some numerical work) and is carried out in Wasow [31] section 3. The asymptotic series (3.24) in connection with (3.6) shows that zeros $-z$ of large modulus are close to solutions of $\cos(\zeta + \frac{\pi}{4}) = (3\pi\zeta/2)^{1/2}$ (with $\zeta = \frac{2}{3}z^{3/2}$). This gives a two-sided logarithmic bound of $\operatorname{Im}\zeta$ in terms of $|\zeta|$, which then implies the estimates for $\operatorname{Re}\zeta$ and $\operatorname{Im}\zeta$ that lead to the given formula.

The proof that no zeros of $\mathbf{A}$ are real is also given in [31], but we repeat it for its brevity and elegance: Multiply the differential equation $\mathbf{A}'''(x) + x\mathbf{A}'(x) = 0$ by $2\mathbf{A}(x)$, obtain $(2\mathbf{A}''\mathbf{A} - (\mathbf{A}')^2)' + x(\mathbf{A}^2)' = 0$, integrate from $-\infty$ to a presumptive real zero a, and get the contradiction $\mathbf{A}'(a)^2 + \int_{-\infty}^a \mathbf{A}^2 = 0$.

In order to get the total number of zeros, we want to evaluate $\int \mathbf{A}'(z)/\mathbf{A}(z)\,dz$ over a circle of radius r, where $r = (3\pi(n - \frac{1}{8}) + \frac{3\pi}{2})^{2/3}$. The part of this circle where $\arg z \in [-2\pi/3, 2\pi/3]$ — let us call it $C_1(r)$ —, is easy:

$$\int_{C_1(r)} \frac{\mathbf{A}'(z)}{\mathbf{A}(z)}\,dz = -ir^{3/2}\int_{-2\pi/3}^{2\pi/3} e^{3i\varphi/2}\frac{1 - c_1/\zeta + O(\zeta^{-2})}{1 - g_1/\zeta + O(\zeta^{-2})}\,d\varphi =$$
$$= -ir^{3/2}\int_{-2\pi/3}^{2\pi/3} e^{3i\varphi/2}\left(1 + \frac{3}{4}r^{-3/2}e^{-3i\varphi/2} + O(r^{-3})\right)d\varphi = -\frac{3i}{4}\frac{4\pi}{3} + O(r^{-3/2})$$

For the second part, we transform the variable z into $-z$ and denote by $C_2(r)$ the arc of radius r with $\arg z \in [-\pi/3, \pi/3]$:

$$-\int_{C_2(r)} \frac{\mathbf{A}'(-z)}{\mathbf{A}(-z)}\,dz = ir\int_{-\pi/3}^{\pi/3}$$
$$\frac{r^{-1/4}e^{-i\varphi/4}/2\sqrt{\pi}\left(e^{i\zeta - i\pi/4}(1 + c_1/\zeta + O(\zeta^{-2})) + e^{-i\zeta + i\pi/4}(1 - c_1/\zeta + O(\zeta^{-2}))\right)}{1 - r^{-3/4}e^{-3i\varphi/4}/2\sqrt{\pi}\left(e^{i\zeta + i\pi/4}(1 + g_1/\zeta + O(\zeta^{-2})) + e^{-i\zeta - i\pi/4}(1 - g_1/\zeta + O(\zeta^{-2}))\right)}\,e^{i\varphi}\,d\varphi$$

This integral is $2\pi i$ times the change of the change of the argument in the denominator along the path of integration. With $\zeta = (\pi(2n+1) - \frac{\pi}{4})e^{3i\varphi/2}$ and $\varphi \geq 0$, the leading term in the parenthesis in the denominator, i.e., $e^{-i\zeta - i\pi/4}$, is $-\exp\left[\pi(2n + \frac{3}{4})\sin\frac{3\varphi}{2}\right]\exp\left[i\pi(2n + \frac{3}{4})(1 - \cos\frac{3\varphi}{2})\right]$. As long as φ remains bounded by, say, $O(\ln n/n)$, this term is nearly real negative and does not contribute a change in the argument. Beyond this bound, the 1 in the denominator is already negligible, and so is the competitor $e^{i\zeta + i\pi/4}$. Therefore, the change in the argument from $\varphi = 0$ to $\varphi = \pi/3$ equals $-\frac{\pi}{4} + \pi(2n + \frac{3}{4}) + o(1)$; add the same change for the interval from $\varphi = -\pi/3$ to $\varphi = 0$. For large n, it follows that the circle of radius $r = (3\pi(n - \frac{1}{8}) + \frac{3\pi}{2})^{2/3}$ encloses exactly $2n$ zeros of $\mathbf{A}$. ∎

6.3.2 Asymptotics in the Critical Domain

Before we venture into a more detailed global classification of solutions to $\Delta' = 0$ (Section 6.3.4 below), it is worthwhile to get a good understanding of the behaviour of that set near the critical line $t = \sqrt{3}\,r$. Unlike in the case $\varepsilon > 0, t \to \infty$, where there was a natural scaling $|t - \sqrt{3}\,r| \approx r^{-1/2}$, we will get logarithmic terms for $\varepsilon = 0$ and rather need to look at a domain $|t - \sqrt{3}\,r| \leq O(\ln r/\sqrt{r})$.

We will make the substitutions

$$A = \frac{1}{2i}(\zeta_+ - \zeta_-) \qquad \zeta_+ = -B + iA$$
$$B = \frac{-1}{2}(\zeta_+ + \zeta_-) \qquad \zeta_- = -B - iA\,, \tag{6.25}$$

which, in the case of real δ reduces to the decomposition into real and imaginary part, but is useful also otherwise, and then A, B are complex themselves.

When referring to the "critical domain" in the future, we will mean a domain in which $|t - \sqrt{3}\delta|$ is bounded. In particular, $\sigma := \operatorname{Im}\delta$ and $\tau := t - \sqrt{3}\,r$ are bounded ($r := \operatorname{Re}\delta$). In this domain, A will be of order $r^{3/2}$, whereas B will be of order $\leq O(r^{1/2})$ as a consequence of the assumption. Actually interesting phenomena occur on the scale $B = O(\ln r)$. In terms of r, τ and σ, we have

$$\begin{aligned} A &= \frac{4}{3}\sqrt{2}\,r^{3/2} + \frac{\sqrt{6}}{2}\tau r^{1/2} + i\frac{\sqrt{2}}{2}\sigma r^{1/2} + O(r^{-1/2}) \\ B &= \frac{\sqrt{2}}{2}\,r^{1/2}(\tau - i\sqrt{3}\,\sigma) + O(r^{-1/2}) \\ A + B/\sqrt{3} &= \frac{4}{3}\sqrt{2}\,r^{3/2} + \frac{2\sqrt{6}}{3}\tau r^{1/2} + O(r^{-1/2}) \\ \frac{4}{3}\sqrt{2}\,r^{3/2} &= A + O(A^{1/3}) \end{aligned} \tag{6.26}$$

$$\begin{aligned} \left(\frac{2}{3}\right)^{1/3}\partial_\delta &= \frac{1}{2}e^{i\pi/6}A^{1/3}\left(1 + \frac{iB}{A}\right)^{1/3}(-i\partial_A - \partial_B) \\ &\quad + \frac{1}{2}e^{-i\pi/6}A^{1/3}\left(1 - \frac{iB}{A}\right)^{1/3}(i\partial_A - \partial_B) \\ &= \frac{1}{2}A^{1/3}(\partial_A - \sqrt{3}\,\partial_B) + O(BA^{-2/3}) \\ \left(\frac{2}{3}\right)^{1/3}\partial_t &= \frac{1}{2}e^{i\pi/6}A^{1/3}\left(1 + \frac{iB}{A}\right)^{1/3}(\partial_A - i\partial_B) \\ &\quad + \frac{1}{2}e^{-i\pi/6}A^{1/3}\left(1 - \frac{iB}{A}\right)^{1/3}(\partial_A + i\partial_B) \\ &= \frac{1}{2}A^{1/3}(\sqrt{3}\,\partial_A + \partial_B) + O(BA^{-2/3}) \end{aligned} \tag{6.27}$$

It is a straightforward matter to calculate both Δ' and its derivatives by means of these formulas to the necessary orders of the asymptotics. We get:

$$\begin{aligned} \frac{-i\pi}{\sqrt{24}}\left(\frac{3A}{4\sqrt{2}}\right)^{1/3}\Delta'(\delta,t) &= -\sqrt{2\pi A}\exp[B]\sin\left[A - \frac{\pi}{12}\right] + \sin[2A] - \frac{1}{2}\exp[2B] \\ &\quad + \sqrt{\frac{2\pi}{A}}\,\frac{12B-7}{72}\exp[B]\cos\left[A - \frac{\pi}{12}\right] - \frac{17}{36A}\cos[2A] \\ &\quad + \frac{2B}{3\sqrt{3}A}\sin[2A] - \frac{\sqrt{3}(B-1)}{9A}\exp[2B] + \frac{2\sqrt{3}}{9A} \\ + O\Big(\frac{1+B^2}{A^2}\Big)&\left(\sqrt{A}\cosh[\operatorname{Im}A]\exp[\operatorname{Re}B] + \cosh[2\operatorname{Im}A] + \exp[2\operatorname{Re}B]\right) \end{aligned} \tag{6.28}$$

$$
\begin{aligned}
4\Big(\frac{3A}{2}\Big)^{-1/3}\partial_\delta \Big\{\frac{-i\pi}{\sqrt{24}}\Big(\frac{3A}{4\sqrt{2}}\Big)^{1/3}\Delta'(\delta,t)\Big\} = &\\
4\sqrt{2\pi A}\exp[B]\sin\Big[A-\frac{\pi}{4}\Big] + 4\cos[2A] + \frac{4\sqrt{3}}{2}\exp[2B] &\\
-\sqrt{\frac{2\pi}{A}}\exp[B]\Big(\frac{36B+5}{18}\cos\Big[A-\frac{\pi}{4}\Big] + \frac{2}{3}\sin\Big[A-\frac{\pi}{12}\Big]\Big) &\\
+\frac{20\sqrt{3}B}{9A}\cos[2A] + \frac{5}{9A}\sin[2A] + \frac{6(B-1)}{9A}\exp[2B] &\\
+O\Big(\frac{1+B^2}{A^2}\Big)\Big(\sqrt{A}\cosh[\operatorname{Im} A]\exp[\operatorname{Re} B] + \cosh[2\operatorname{Im} A] + \exp[2\operatorname{Re} B]\Big) &
\end{aligned}
\tag{6.29}
$$

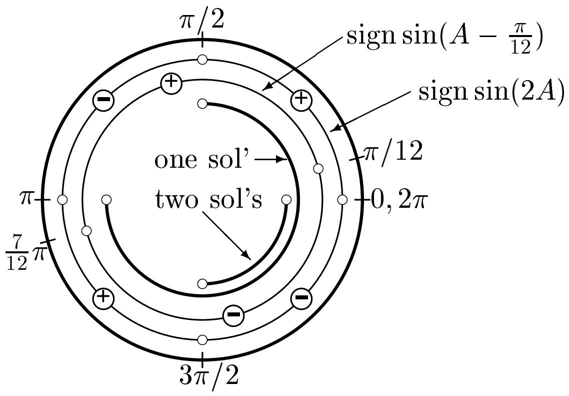
Let us denote by

$$
\Delta^*(A,B) := -\sqrt{2\pi A}\exp[B]\sin\Big[A-\frac{\pi}{12}\Big] + \sin[2A] - \frac{1}{2}\exp[2B] \tag{6.30}
$$

the leading term of (6.29) and discuss its zero set: it is parametrized by

$$
B = f_\pm(A) := \ln\Big[-\sqrt{2\pi A}\,\sin\Big(A-\frac{\pi}{12}\Big) \pm \sqrt{2\pi A\sin^2\Big(A-\frac{\pi}{12}\Big) + 2\sin(2A)}\,\Big]\,. \tag{6.31}
$$

One can easily see that $f_+(k\pi + \frac{\pi}{12}) = 0$ for $k \in \mathbb{N}$. The domain of definition of the functions $f_\pm$ can be read off from the little figure on the right: In (6.31), the term under the square root is positive for $A > 2/\pi$, because $\sin(2A) \geq -2\sin^2(A - \frac{\pi}{12})$, and it is trivially positive for $A \leq 2/\pi$. The sign of the term in brackets is therefore determined by the signs of $\sin(2A)$ and $\sin(A-\frac{\pi}{12})$. Distinction of the appropriate cases shows that f_+ is defined on $]\pi, \frac{5}{2}\pi[$ modulo 2π, and f_- on $]\frac{3}{2}\pi, 2\pi[$.

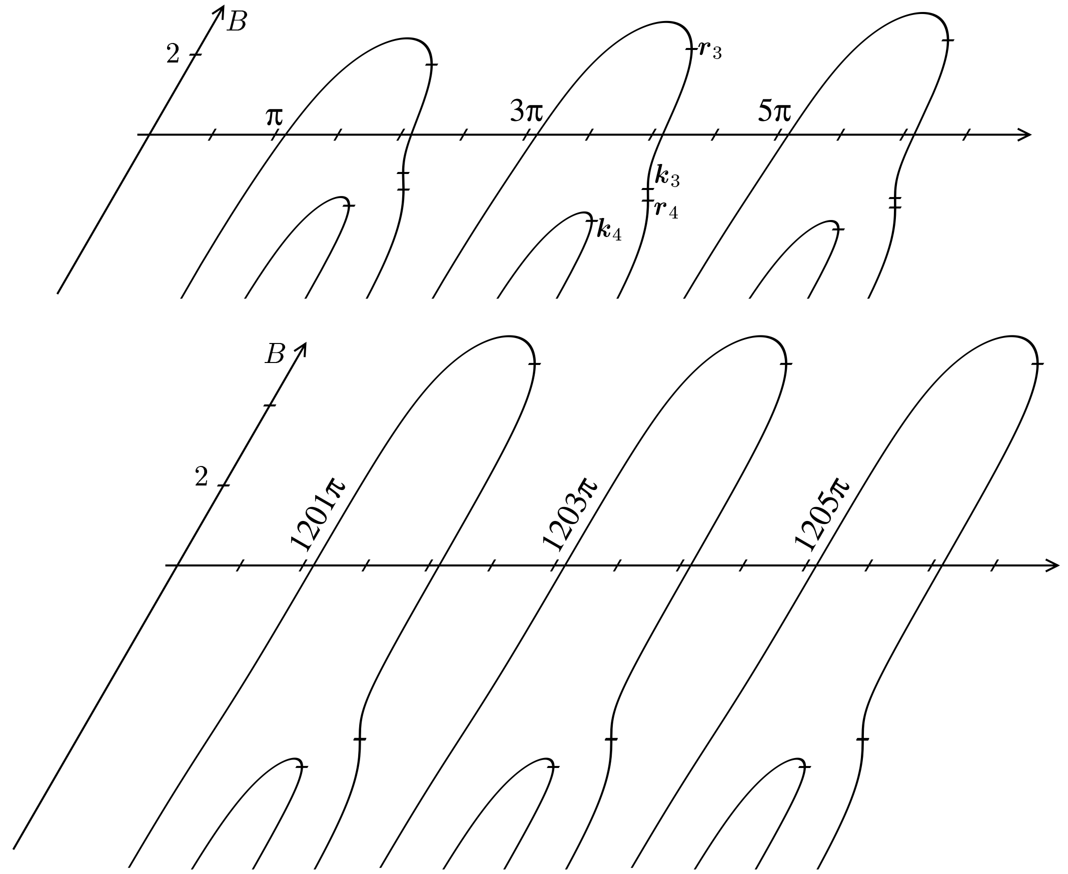


On the zero set of the full Δ', it holds still

$$
\begin{aligned}
B = \ln\Big[&-\sqrt{2\pi A}\Big(\sin\Big(A-\frac{\pi}{12}\Big) + O\Big(\frac{B}{A}\Big)\Big) \pm \\
&\pm\sqrt{2\pi A\Big(\sin\Big(A-\frac{\pi}{12}\Big)+O\Big(\frac{B}{A}\Big)\Big)^2 + 2\sin(2A) + O\Big(\frac{B}{A}\Big)}\,\Big] + O\Big(\frac{B}{A}\Big)\,.
\end{aligned}
$$

The real solutions (6.31) are displayed in Figure 6.5. The curves f_+ and f_- can be separated, e.g., by the curve $\exp[B] = -\sqrt{A}\cos 2A$, $A \in](2n-\frac{3}{4})\pi, (2n-\frac{1}{4})\pi[$, also in the case of Δ' with the error terms included. Separating them within the complex solution set should not be possible, as we do expect a complex connection.

For $A = k\pi + \frac{\pi}{12}$ and the case of the upper sign in (6.31), we get $\exp[B] = 1 + O(A^{-1/2})$ and therefore $\operatorname{sign}\partial_\delta\Delta'(\delta,t) = (-1)^{k+1}$. For $A = 2k\pi + \frac{\pi}{4}$, we find $\exp[B] = (2/\pi A)^{1/2}(1+O(A^{-1}))$ and thus $\partial_\delta\Delta'(\delta,t) > 0$. As $A \to$ something near $2k\pi + \frac{\pi}{2}$ and $B \to -\infty$, we encounter a point where $\exp[B] = 1/A$ and thus $\partial_\delta\Delta'(\delta,t) < 0$. By the intermediate value theorem, we get (at least) three points on the f_+ branch where $\partial_\delta\Delta'(\delta,t) = 0$, and similarly at least one such point on the f_- branch. It should be possible to show that there are no more solutions to $\partial_\delta\Delta'(\delta,t) = 0$



The real solution branches for $n \in \{1,2,3\}$ and $n \in \{601,602,603\}$ in the same scale. The abscissa displays $\operatorname{Re} A$, provided the projection is parallel to the B-axis (which has an angle $\pi/3$ with the abscissa). At the same time, an orthogonal projection on the abscissa yields $A + B/\sqrt{3}$, which is always real. The graphics could be enhanced by an axis orthogonal to the paper plane which would then show both $\operatorname{Im} B$ and $\operatorname{Im} A$ (with different scales and opposite directions).

The branch points are marked by little horizontal lines (two of which have become too close to be distinguished in the lower part of the figure).

Figure 6.5: The real zero set of Δ^*

on either branch, by considering the second derivative, but we have no need to do so, as this will follow from the global analysis.

As it happens, already the zero set of Δ^*, which is displayed in Figure 6.5, has three points where $f'_+(A) = -\sqrt{3}$, and they approximate the corresponding points where $\partial_\delta \Delta' = 0$. So we can label corresponding points alike in Figure 6.5 and Figure 6.6 on page 112. However, the fine splitting between the points $\boldsymbol{r}_{2j}$ and $\boldsymbol{k}_{2j-1}$ is not described in a quantitatively correct way in terms of Δ^* alone, but is influenced by the second order of Δ'.

We have calculated the location of the branch points with the help of Mathematica:

$$\begin{aligned}
\boldsymbol{r}_{2n-1}: \quad & A = (2n - \tfrac{1}{12})\pi - (12\ln(2\pi\sqrt{n}) - 5)/(144\pi n) + O((\ln n)^2 n^{-2}) \\
& B = \ln(2\pi\sqrt{n}) - p_+/n + O((\ln n)^2 n^{-2}) \\
& r = \tfrac{1}{2}\left[(3n - \tfrac{1}{8})\pi\right]^{2/3} - \tfrac{\sqrt{3}}{2}\ln(2\pi\sqrt{n}) \Big/ \left[(3n - \tfrac{1}{8})\pi\right]^{1/3} + O((\ln n)/n^{4/3}) \\
& t = \tfrac{\sqrt{3}}{2}\left[(3n - \tfrac{1}{8})\pi\right]^{2/3} + \tfrac{1}{2}\ln(2\pi\sqrt{n}) \Big/ \left[(3n - \tfrac{1}{8})\pi\right]^{1/3} + O((\ln n)/n^{4/3}) \\
& p_+ = \frac{8\pi\sqrt{3}\ln(2\pi\sqrt{n}) + 4\pi\sqrt{3} - 3\pi^2 - 36}{144\pi^2}
\end{aligned} \tag{6.32}$$

$$\begin{aligned}
\boldsymbol{r}_{2n}: \quad & A = (2n + \tfrac{1}{4})\pi + a/\sqrt{n} + O(n^{-1}) \\
& B = -\ln(\pi\sqrt{n}) - a\sqrt{3}/\sqrt{n} + O((\ln n)n^{-1}) \\
& r = \tfrac{1}{2}\left[3(n + \tfrac{1}{8})\pi\right]^{2/3} + \left[\tfrac{\sqrt{3}}{2}\ln(\pi\sqrt{n}) + 2an^{-1/2}\right] \Big/ \left[3(n + \tfrac{1}{8})\pi\right]^{1/3} + O((\ln n)^2/n^{4/3}) \\
& t = \tfrac{\sqrt{3}}{2}\left[3(n + \tfrac{1}{8})\pi\right]^{2/3} - \tfrac{1}{2}\ln(\pi\sqrt{n}) \Big/ \left[3(n + \tfrac{1}{8})\pi\right]^{1/3} + O((\ln n)^2/n^{4/3}) \\
& a = \sqrt{\frac{18 - \sqrt{3}\pi + 6\sqrt{3}\pi\ln(\pi\sqrt{n})}{72\pi^2}}
\end{aligned} \tag{6.33}$$

$$\begin{aligned}
\boldsymbol{k}_{2n-1}: \quad & A = (2n + \tfrac{1}{4})\pi - a/\sqrt{n} + O(n^{-1}) \\
& B = -\ln(\pi\sqrt{n}) + a\sqrt{3}/\sqrt{n} + O((\ln n)n^{-1}) \\
& r = \tfrac{1}{2}\left[3(n + \tfrac{1}{8})\pi\right]^{2/3} + \left[\tfrac{\sqrt{3}}{2}\ln(\pi\sqrt{n}) - 2an^{-1/2}\right] \Big/ \left[3(n + \tfrac{1}{8})\pi\right]^{1/3} + O((\ln n)^2/n^{4/3}) \\
& t = \tfrac{\sqrt{3}}{2}\left[3(n + \tfrac{1}{8})\pi\right]^{2/3} - \tfrac{1}{2}\ln(\pi\sqrt{n}) \Big/ \left[3(n + \tfrac{1}{8})\pi\right]^{1/3} + O((\ln n)^2/n^{4/3}) \\
& a = \sqrt{\frac{18 - \sqrt{3}\pi + 6\sqrt{3}\pi\ln(\pi\sqrt{n})}{72\pi^2}}
\end{aligned} \tag{6.34}$$

$$\begin{aligned}
\boldsymbol{k}_{2n}: \quad & A = (2n - \tfrac{1}{12})\pi + (12\ln(2\pi\sqrt{n}) + 5)/(144\pi n) + O((\ln n)^2 n^{-2}) \\
& B = -\ln(2\pi\sqrt{n}) - p_-/n + O((\ln n)^2 n^{-2}) \\
& r = \tfrac{1}{2}\left[(3n - \tfrac{1}{8})\pi\right]^{2/3} + \tfrac{\sqrt{3}}{2}\ln(2\pi\sqrt{n}) \Big/ \left[(3n - \tfrac{1}{8})\pi\right]^{1/3} + O((\ln n)/n^{4/3}) \\
& t = \tfrac{\sqrt{3}}{2}\left[(3n - \tfrac{1}{8})\pi\right]^{2/3} - \tfrac{1}{2}\ln(2\pi\sqrt{n}) \Big/ \left[(3n - \tfrac{1}{8})\pi\right]^{1/3} + O((\ln n)/n^{4/3}) \\
& p_- = \frac{16\pi\sqrt{3}\ln(2\pi\sqrt{n}) - 4\pi\sqrt{3} - 3\pi^2 - 36}{144\pi^2}
\end{aligned} \tag{6.35}$$

We conclude this section with a little lemma that will be useful later:

Lemma 6.7 *If in some real point in the critical domain it holds* $\Delta' = 0 = \partial_B \Delta'$, *then* $\exp[B] \leq O(A^{-7/6})$.

PROOF:
From

$$-\sqrt{2\pi A}\exp[B]\left(\sin\left[A - \tfrac{\pi}{12}\right] + O\left(\tfrac{B}{A}\right)\right) + \sin[2A] + O\left(\tfrac{B}{A}\right) - \tfrac{1}{2}\exp[2B]\left(1 + O\left(\tfrac{B}{A}\right)\right) = 0$$

$$-\sqrt{2\pi A}\exp[B]\left(\sin\left[A - \tfrac{\pi}{12}\right] + O\left(\tfrac{B}{A}\right)\right) + O\left(\tfrac{B}{A}\right) - \exp[2B]\left(1 + O\left(\tfrac{B}{A}\right)\right) = 0$$

we conclude, using $O(B/A) \leq O(A^{-2/3})$,

$$O(A^{-1/6})\exp[B] + O(A^{-2/3}) - \exp[2B]\Big(\frac{1}{2} + O(A^{-2/3})\Big) - \sin[2A] = 0\,.$$

This implies $\exp[2B] < 3$ and $\sin[2A] < O(A^{-2/3})$. But then, reinserting into $\partial_B \Delta' = 0$ implies

$$0 = -\sqrt{2\pi A}\exp[B]\left(\sin\left[A - \tfrac{\pi}{12}\right] + O\left(\tfrac{B}{A}\right)\right) + O\left(\tfrac{B}{A}\right) > \left(\tfrac{1}{4} - O(A^{-2/3})\right)\exp[B]\sqrt{2\pi A} - O(A^{-2/3}) .$$

It follows $\exp[B] < O(A^{-7/6})$ as claimed. ■

6.3.3 Appendix: Technical calculations

Here we give the Mathematica definitions used to compute the asymptotics in the critical domain. (Mma 3.0 for Solaris): The symbol `E2` lacks the term $2/t$ given for E_2 in (6.14). Instead, the "secular" term `2/t * E3` will be handled separately.

```
(* **** the series coefficients **** *)
g1[s_]:=(-1)^s   * (2s-1)!!  *
    Sum[(6k-1)!!/(2k-1)!!^2/6^(3k)/k!/2^(s-k) ,{k,0,s}]
g2[s_]:= (2s-1)!!  *
    Sum[(6k-1)!!/(2k-1)!!^2/6^(3k)/k!/2^(s-k) ,{k,0,s}]
g3[s_]:= -(-1)^s (6s+1)/(6s-1) * (6s-1)!!/(2s-1)!!/6^(3s)/s!
g4[s_]:= - (6s+1)/(6s-1) * (6s-1)!!/(2s-1)!!/6^(3s)/s!

E1:= 1/(Sqrt[6Pi] I t) Sum[
(Exp[-zetaminus] / zetaminus^(s + 1/2) -
         Exp[-zetaplus ] /  zetaplus^(s + 1/2)  ) g1[s],   {s,0,smax}]

E2:= -2/(Sqrt[6Pi] I t) Sum[
(Exp[zetaminus] / zetaminus^(s + 1/2) -
 Exp[zetaplus ] /  zetaplus^(s + 1/2)  ) g2[s],   {s,0,smax}] -
     1/(Sqrt[6Pi] t) Sum[
(Exp[-zetaminus] / zetaminus^(s + 1/2) +
         Exp[-zetaplus ] /  zetaplus^(s + 1/2)  ) g1[s],  {s,0,smax}]
(* the extra term 2/t E3 will be considered separately later *)

E3:= (3/2)^(1/6)/(2Sqrt[Pi] ) Sum[
(Exp[-zetaminus] / zetaminus^(s - 1/6) -
 Exp[-zetaplus ] /  zetaplus^(s - 1/6)  ) g3[s],  {s,0,smax}]

E4:= -(3/2)^(1/6)/(Sqrt[Pi] ) Sum[
(Exp[zetaminus] / zetaminus^(s - 1/6) -
 Exp[zetaplus ] /  zetaplus^(s - 1/6)  ) g4[s],   {s,0,smax}] -
     I (3/2)^(1/6)/(2Sqrt[Pi] ) Sum[
(Exp[-zetaminus] / zetaminus^(s - 1/6) +
 Exp[-zetaplus ] /  zetaplus^(s - 1/6)  ) g3[s],  {s,0,smax}]

zetarules= {
    Exp[sign_.  zetaminus] :> Exp[sign (-B - I Ahide)],
    Exp[sign_.  zetaplus ] :> Exp[sign (-B + I Ahide)],
    zetaminus -> -B - I A,
    zetaplus  -> -B + I A
}
(* Ahide == A, but Mma shouldn't know this when it uses power series in A,
  lest it complain about essential singularities at Infinity *)
```

```
tidy = {
        (n_Integer?Negative)^any_ :> (-n)^any (Cos[Pi any] + I Sin[Pi any]),
        (n_Rational?Negative)^any_ :> (-n)^any (Cos[Pi any] + I Sin[Pi any])
       }

exp2trig[0] = {
    Exp[ 2I Ahide] -> Cos[2Ahide] + I Sin[2Ahide],
    Exp[-2I Ahide] -> Cos[2Ahide] - I Sin[2Ahide]
   }

exp2trig[s_Integer] := {   (* I'll only use this for s == -1 *)
    Exp[ I Ahide] -> Expand[
                 (Cos[Ahide+s Pi/12] + I Sin[Ahide+s Pi/12]) *
                 (Cos[s Pi/12] - I Sin[s Pi/12]) ],
    Exp[-I Ahide] -> Expand[
                 (Cos[Ahide+s Pi/12] - I Sin[Ahide+s Pi/12]) *
                 (Cos[s Pi/12] + I Sin[s Pi/12]) ]
}

(* definition for derivatives: *)
Ahide /:  D[Ahide, A, NonConstants-> {Ahide}] := 1
```

With these definitions, the derivatives and series expansions are calculated:

```
smax = 3

(*
 I calculate the expression
```
$A^{1/3} i(2/3)^{1/2} \times (-\pi/4)(3/2)^{1/3}\Delta'/\sqrt{2} \;=\; A^{1/3}(-i)\pi/\sqrt{24}(2/3)^{-1/3}\Delta'/\sqrt{2}$
```
 It is: nonsecpart + 2 Sqrt[Pi] secpart ==
         nonsecCSE + 2 Sqrt[Pi] Exp[B] secCSE
*)

nonsecpart= Series[A^(1/3) *
     I Pi t (2/3)^(1/6) (E2 E3 - E1 E4)/Sqrt[2] /.
        zetarules, {A,Infinity,smax}]

nonsecEXP=  Map[Collect[#, E^_]&, nonsecpart, {2} ] /.  exp2trig[0] /.  tidy
nonsecCSE= Map[Collect[#,
    {Cos[2Ahide], Sin[2Ahide], Exp[2B]}]&, nonsecEXP, {2}]
nonsecCSE= Map[Together, nonsecCSE, {4}]

secpart= Series[ A^(1/3)*
     I Sqrt[Pi]  (2/3)^(1/6)  E3/Sqrt[2] /.
           zetarules , {A,Infinity,smax}]

secEXP=  Map[Collect[# Exp[-B], E^_]&, secpart, {2} ] /.  exp2trig[-1] /.  tidy
secCSE= Map[Collect[#, {Cos[_], Sin[_]}]&, secEXP, {2}]
secCSE= Map[Together, secCSE/.  DirectedInfinity[_] -> Infinity, {4}] +
        SeriesData[A,Infinity,{},6smax-3,6smax-3,6]

(* Derivatives: *)
```

```
DnonsecA= D[nonsecCSE,A,NonConstants -> Ahide] +
            SeriesData[A,Infinity,{},6smax,6smax,6]
DeBsecA=    D[Exp[B] secCSE,A,NonConstants -> Ahide] +
            SeriesData[A,Infinity,{},6smax-3,6smax-3,6]
DnonsecB= D[nonsecCSE,B]
DeBsecB=    D[Exp[B] secCSE,B]

Dnonsecdelta = (*  1/4 * (3/2)^(1/3) * A^(1/3) * ...   omitted *) \
     -(Sqrt[3]+I) (1+I B/A)^(1/3) (DnonsecB + I DnonsecA) -
     (Sqrt[3]-I) (1-I B/A)^(1/3) (DnonsecB - I DnonsecA)
DeBsecdelta = (*  1/4 * (3/2)^(1/3) * A^(1/3) * ...   omitted *) \
     -(Sqrt[3]+I) (1+I B/A)^(1/3) (DeBsecB + I DeBsecA) -
     (Sqrt[3]-I) (1-I B/A)^(1/3) (DeBsecB - I DeBsecA)

Dnonsect = (*  1/4 * (3/2)^(1/3) * A^(1/3) * ...   omitted *) \
     (Sqrt[3]+I) (1+I B/A)^(1/3) (DnonsecA - I DnonsecB) +
     (Sqrt[3]-I) (1-I B/A)^(1/3) (DnonsecA + I DnonsecB)
DeBsect = (* ‘ 1/4 * (3/2)^(1/3) * A^(1/3) * ...   ’ omitted *) \
     (Sqrt[3]+I) (1+I B/A)^(1/3) (DeBsecA - I DeBsecB) +
     (Sqrt[3]-I) (1-I B/A)^(1/3) (DeBsecA + I DeBsecB)

(* Splitting of near-double-branch-point: *)

eqns=
(Normal/@
{ nonsecCSE / Sqrt[2] + Sqrt[2Pi] Exp[B] secCSE
 Dnonsecdelta/Sqrt[2] + Sqrt[2Pi]DeBsecdelta    } ) /.
    Ahide -> Pi/4 + aa[n] /.
    A -> (2n+1/4)Pi + aa[n] /.
    Exp[B] -> 1/Pi/Sqrt[n] Exp[bb[n]] /.
    Exp[2B] -> 1/Pi^2/n Exp[2bb[n]] /.
    B -> -logpisqn + bb[n]   (* logpisqn == Log[Pi Sqrt[n]] *)

eqns /.  { aa[n] -> SeriesData[n,Infinity, a/@Range[2smax], 1, 2smax, 2],
         bb[n] -> SeriesData[n,Infinity, b/@Range[0,2smax], 0, 2smax, 2] }
```

6.3.4 Conclusions for the Zero Set

In order to analyze the asymptotic expansion from Theorem 6.4, we profit from the results of Section 5.3.4, namely pages 74ff. Like in Theorem 5.14, we let $x := \operatorname{Re}\zeta_+$ and $y := \operatorname{Re}\zeta_-$, and with no loss of generality, we assume $y \geq x$ (equivalently $\operatorname{Im}\delta \geq 0$); hence $|\zeta_+| \geq |\zeta_-|$. In (6.18), abbreviated as $0 = \sum_1^N A_i$, we select one of the terms, say A_1, and exploit $|A_1| \leq \sum_2^N |A_i|$. Given $a \in]0,1[$, we assume $\zeta_\pm$ to be large enough for all O-terms occurring to be less than a in absolute value. We also let $a_1 = a_2 = \ldots = a$ and use these synonyms in order to indicate by the subscript from which of the lines in (6.18) the corresponding term arises. (There is no intent to get optimal quantitative estimates by fine tuning of different values a_i for different O-symbols.)

It is clear from Lemma 5.13 that for every $K > \sqrt{2}$ there exists a constant $\bar{K}$ such that $|\zeta_+| \leq K(-\operatorname{Re}\zeta_+) + \bar{K}\frac{2}{3}(\operatorname{Re}\delta)^{3/2}$. (The locus for the equal sign would be a hyperbola with an opening angle $> \pi/2$.) Based on this observation, we use the following estimates:

$$
\begin{aligned}
(\zeta_+\zeta_-)^{1/6}(\zeta_+^{-2/3} - \zeta_-^{-2/3}) &= \sqrt[3]{12}\,\frac{-it}{(\delta^2+t^2)^{3/4}} \\
|\zeta_+|^{-4/3} \le |\zeta_-|^{-4/3} &\le const\,\frac{|x|^{4/3} + (\operatorname{Re}\delta)^2}{|\delta^2+t^2|^2} \\
O(\zeta_*^{-4/3}) &\le const\,\frac{|x|^{5/3} + (\operatorname{Re}\delta)^{5/2}}{|\delta^2+t^2|^{9/4}} \\
(\operatorname{Re}\delta)^2 &\le |\delta^2+t^2|
\end{aligned}
$$

First we select the $e^{\zeta_- - \zeta_+}$ term from the second line of (6.18). It is exponentially dominant for $y > \max\{x, 0\}$, but if $x, y \to 0$, the first line becomes dominant like a power. It is for this reason that we cannot a priori specify a domain in the x-y-plane (i.e., bounds connecting $\operatorname{Re}\zeta_-$ and $\operatorname{Re}\zeta_+$) independent of bounds for $|\zeta_\pm|$. We assume $t \ge 1$ (because otherwise we are in the domain covered by Theorem 4.8, or in some compact set), and choose an upper bound $M \ge |\delta^2 + t^2|/t$, $M \ge 1$. Then we conclude

$$
\tfrac{1}{2}(1-a_2)e^{y-x} \le \tfrac{1}{2}(1+a_2)e^{x-y} + \tfrac{1}{2}(1+a_4)e^{-x-y} + a_3(1+|x|^{4/3}) + (1+a_1)6M(e^{-x}+e^{-y})
$$

hence

$$
\begin{aligned}
e^{2y} &\le \frac{1+a}{1-a}(e^{2x}+1) + \frac{2}{1-a}\Big(a(1+|x|^{4/3})e^x + 12(1+a)M\Big)e^y \\
&\le \frac{1+3a}{1-a}(e^{2x}+1) + \frac{24(1+a)M}{1-a}e^y
\end{aligned}
\tag{6.36}
$$

Similarly, when we select $e^{-\zeta_- - \zeta_+}$ (which is dominant for $x, y < 0$), we conclude

$$
\tfrac{1}{2}(1-a_4)e^{-x-y} \le \tfrac{1}{2}(1+a_2)(e^{x-y}+e^{y-x}) + a_3(1+|x|^{4/3}) + (1+a_1)6M(e^{-x}+e^{-y})
$$

hence

$$
\begin{aligned}
e^{2x} + e^{2y} &\ge \frac{1-a}{1+a} - \frac{2}{1+a}\Big(a(1+|x|^{4/3})e^{x+y} + (1+a)6M(e^x+e^y)\Big) \\
&\ge \frac{1-3a}{1+a} - 12M(e^x+e^y)
\end{aligned}
\tag{6.37}
$$

where the first inequality holds in the domain $x \le y \le 0$, the second by symmetry for $x, y \le 0$ again. The qualitative figure is like in (5.30), however we get some set $Y_0(a, M)$ dependent also on M instead of $Y(a)$ there.

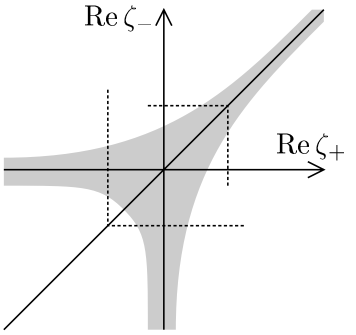
We have already proved the first part of the following theorem (which is the analog to Theorem 5.14):

Theorem 6.8

1. *For every $a < 1$, there exists a compact set $K(a)$ such that for all $(\delta, t) \in \mathcal{Z}_0^+ \setminus K(a)$, it holds:*

   $$r := \operatorname{Re}\delta > 0\,,\quad t > |\operatorname{Im}\delta|\,,\quad (\operatorname{Re}\zeta_+, \operatorname{Re}\zeta_-) \in Y_0(a, M) \quad \textit{with} \quad M := \max\{|\delta^2+t^2|/t, 1\}$$

   where



$$Y_0(a,M) = \Big\{(x,y)\ \Big|\ e^{2y} \le \frac{1+3a}{1-a}(e^{2x}+1) + \frac{24(1+a)M}{1-a}e^{y}\,,\quad e^{2x} \le \frac{1+3a}{1-a}(e^{2y}+1) + \frac{24(1+a)M}{1-a}e^{x}\,,\quad e^{2y}+e^{2x} \ge \frac{1-3a}{1+a} - 12M\,(e^{y}+e^{x})\Big\} \tag{6.38}$$

If b, $\tilde b$ satisfy the conditions $b^2\tilde b^2 > 3(1+\tilde b^2)$ and $b \ge 48M$, then there exists some number a such that every $(x,y) \in Y_0(a,M)$ satisfies at least one of the conditions $\max\{x,y\} \le \ln b$ ("wing condition") and $\min\{x,y\} \ge -\ln \tilde b$ ("body condition").

There exists a number k dependent only on an upper bound for a and lower bounds for $r := \operatorname{Re}\delta$ and t such that for any $(\delta,t) \in \mathcal{Z}_0$ satisfying these bounds, $M \le k(1+r^2)$ and $\frac{2(1+a)}{1-3a} + \frac{24(1+a)}{1-3a}M < k(1+r^2)$.

2. *For every b, solutions $(\delta,t) \in \mathcal{Z}_\varepsilon^+ \setminus K(a)$ satisfying the wing condition $\max\{x,y\} \le \ln b$ lie in the "wing domain"*

$$V_0(b) := \Big\{(\delta,t)\ \Big|\ -\sqrt{2}(\ln b)r^{-1/2} - O((\ln b)^2 r^{-2}) < t - \sqrt{3}\,r - |\operatorname{Im}\delta| < \; < \sqrt{2}r^{-1/2}\ln k(1+r^2) + O((\ln r)^2 r^{-2})\Big\} \tag{6.39}$$

for the constant k just given above.

3. *For every $\tilde b$, solutions $(\delta,t) \in \mathcal{Z}_0^+ \setminus K(a)$ satisfying the body condition $\min\{x,y\} \ge -\ln\tilde b$ lie in the "body domain"*

$$I_0(\tilde b) := \Big\{(\delta,t)\ \Big|\ |\operatorname{Im}\delta| + t < \sqrt{3}\,r + \sqrt{2}\,(\ln\tilde b)r^{-1/2} + O((\ln\tilde b)^2 r^{-2})\,,\quad |\operatorname{Im}\delta| < O\Big((\ln\operatorname{Re}\delta)(\operatorname{Re}\delta/C)^{1/4}\Big)\Big\}\,. \tag{6.40}$$

and, provided $t > (C|\delta|)^{1/4}$, they satisfy

$$|\sinh(\zeta_- - \zeta_+)| \le \frac{1+a}{2}\tilde b^2 + aO(1+\tilde b M) + \frac{5}{3}\tilde b M\,. \tag{6.41}$$

4. *There exists $a > 0$ such that solutions $(\delta,t) \in \mathcal{Z}_0^+ \setminus K(a)$ satisfying $\operatorname{Im}\delta \ge 0$ do not meet any of the "walls" given by $\operatorname{Im}\zeta_- = -(2n+\frac{3}{4})\pi$, where n is a sufficiently large integer.*

   Continue to assume $\operatorname{Im}\delta \ge 0$: For any integer $n \ge 0$ and any t sufficiently large (depending on n), or else, for n sufficiently large with no condition on t, there are exactly two zeros of $\delta \mapsto \Delta'(\delta,t)$ for which $\operatorname{Im}\zeta_- \in\,]-(2n+\frac{11}{4})\pi, -(2n+\frac{3}{4})\pi[$; if $\operatorname{Re}(\zeta_+ - \zeta_-) \le -\ln 2$, then one of them has $\operatorname{Re}\zeta_- > 0$, the other $\operatorname{Re}\zeta_- < 0$.

   (The same claims hold for $\operatorname{Im}\delta \le 0$ with the role of ζ_+ and ζ_- exchanged.)

5. *For every $\sigma > 1/2$ and every $C > 0$, there exist some number a such that solutions outside $(\delta,t) \in \mathcal{Z}_0^+ \setminus K(a)$ satisfying $t > (C|\delta|)^{1/4}$, $\operatorname{Im}\delta \ge 0$ do not meet any of the "walls" given by $\operatorname{Im}(\zeta_+ - \zeta_-) = (n+\frac{1}{2})\pi$, $\operatorname{Re}\zeta_- \ge \sigma\ln n$, n a positive integer.*

6. *Real solutions satisfying $t > (C\delta)^{1/4}$ do not meet the lines*

$$T_{0\mathrm{r}}(n) := \{(\delta,t) \mid \delta \text{ real},\ \operatorname{Im}\zeta_+ = (n+\tfrac{3}{4})\pi\,,\ t < \sqrt{3}\,r - (3.7 + \tfrac{7}{6}\sqrt{2}\ln r)r^{-1/2}\}$$
$$T_{0\mathrm{r}}(n-\tfrac{1}{2}) := \{(\delta,t) \mid \delta \text{ real},\ \operatorname{Im}\zeta_+ = (n+\tfrac{1}{4})\pi\,,\ t < \sqrt{3}\,r - (3.7 + \tfrac{7}{6}\sqrt{2}\ln r)r^{-1/2}\}\,,$$

provided n is a sufficiently large integer. For even n, $T_{0\mathrm{r}}(n)$ does not meet the lines $\operatorname{Im}\zeta_+ = (n+\frac{3}{4})\pi$, even without the restriction on t. Real solutions are restricted to the sector $t < \sqrt{3}\,r + \gamma(\ln r)r^{-1/2}$ for some appropriate $\gamma > 0$, provided r is sufficiently large.

PROOF:
We have seen (6.38), and now we want to classify its points into body and wing domain. With no loss of generality, let $y \geq x$. Let us assume that for arbitrarily small a, there is a point which satisfies neither the wing nor the body condition.

We start by insisting on $a \leq 1/7$ and $b \geq 48M$. In particular, this implies $b \geq 36(1+a)M/(1-a)$, and $e^{2y} - 24(1+a)Me^y/(1-a) > e^y(b - 24(1+a)M/(1-a)) > b^2 - 4aMb/(1-a)$ for $e^y > b$. But then, by (6.38),

$$\frac{b^2}{3} \leq b^2 - \frac{24(1+a)Mb}{1-a} < \frac{1+3a}{1-a}\,(e^{2x}+1) < \frac{1+3a}{1-a}\,(\tilde{b}^{-2}+1)\;.$$

If this holds for arbitrarily small a, the condition $b^2\tilde{b}^2 > 3(1+\tilde{b}^2)$ is violated.

Now let us turn to the estimate for M. It can easily be checked that for every $(x,y) \in Y_0(a,M)$ with $y \geq x$, it holds

$$e^{-y} \leq \frac{2(1+a) + 24(1+a)M}{1-3a}\;. \tag{6.42}$$

Use $M = |\delta - it||\delta + it|/t \leq |\delta - it|\,(2 + |\delta - it|/t) \leq (1 + |\delta - it|)^2 \leq \frac{5}{4}(1+|\zeta_-|)^2$ for $t \geq 1$. (If, however, $M = 1$, this estimate is trivially true.) We conclude from (6.42)

$$\operatorname{Re}\zeta_- \geq -\ln\frac{2(1+a)}{1-3a} - \ln\Big(1 + 15(1+|\zeta_-|)^2\Big)\;,$$

hence for a small and r (and thus $|\zeta_-|$) bounded away from zero, we get a bound $\alpha \leq \alpha_* < \pi/4$ for the quantity α in Lemma 5.13 and therefore $|\delta - it| \leq O(|\operatorname{Re}\delta|)$. In particular, $M \leq 1 + |\delta - it|^2 = O(r^2)$. This proves part 1.

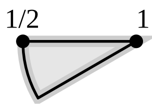
ESTIMATES FOR THE WING DOMAIN:

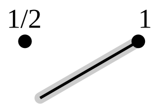
Now use (6.42) with our estimate for M, i.e., $e^{-y} \leq k(1+r^2)$. In the wing domain, we also have $e^x \leq e^y \leq b$ and hence $e^{2x-y} \leq b$. This, together with Lemma 5.13, already implies (6.39).

Having (6.18) in view, we notice that $\zeta_+^{-2/3} - \zeta_-^{-2/3} = \zeta_-^{-2/3}(-2it)/(\delta + it)$; moreover, for

$$(\delta + it)/(-2it) = i(\delta - it)/(2t) - 1 =: -\theta\;,$$

the wing estimate (6.39) tells us that $t > \sqrt{3}\,r - o(1)$ and $-\sqrt{3}\,r - o(1) < \operatorname{Im}(\delta - it) < -\sqrt{3}\,r + o(1) < 0$. Therefore, for solutions in the wing domain, θ lies in a neighbourhood of size $o(1)$ of the segment $1 - e^{i\pi/6}[0,\frac{1}{2}]$. Independent of the solution property, if only $\operatorname{Re}\zeta_- \leq O(\ln|\operatorname{Im}\zeta_-|)$ and $\operatorname{Re}\delta \geq 0$, we still get that θ lies in an $o(1)$-neighbourhood of the sector $1 - e^{i[0,\pi/6]}[0,\frac{1}{2}]$:

$\theta \in$ 

for $\operatorname{Re}\zeta_- \leq O(\ln|\operatorname{Im}\zeta_-|)$, $\operatorname{Re}\delta \geq 0$ — $\theta \in$ 

for solutions in the wing domain

We can cancel $e^{-\zeta_+}(\zeta_+\zeta_-)^{1/6}(\zeta_+^{-2/3} - \zeta_-^{-2/3})$ from (6.18) and obtain (grouping the main terms on the left and small error terms on the right) that $\Delta' = 0$ amounts to

$$\begin{aligned} -\frac{\sqrt{6\pi}}{2}\zeta_-^{1/2}\Big(\frac{i(\delta - it)}{2t} - 1\Big) + \frac{\sqrt{6\pi}}{2}\,e^{\zeta_+ - \zeta_-}\,(\zeta_-/\zeta_+)^{1/6}\zeta_-^{1/2}\Big(\frac{i(\delta - it)}{2t} - 1\Big) + \frac{i}{2}e^{\zeta_-} - \frac{1}{2}e^{-\zeta_-} &= \\ = \Big(\frac{i}{2} + a_2\Big)e^{2\zeta_+ - \zeta_-} + a_1 + a_3 + (a_2 + a_4)O(\cosh\operatorname{Re}\zeta_-) \end{aligned}$$

Rewriting the left hand side of this equality, we obtain

$$\begin{aligned}\frac{\sqrt{6\pi}}{2}\zeta_-^{1/2}\theta\Big(1-e^{\zeta_+-\zeta_-}(\zeta_-/\zeta_+)^{1/6}\Big)-e^{-i\pi/4}\cosh\Big(\zeta_--\frac{i\pi}{4}\Big)&=\\ =\Big(\frac{i}{2}+a_2\Big)e^{2\zeta_+-\zeta_-}+a_1+a_3+(a_2+a_4)O(\cosh\operatorname{Re}\zeta_-)&\end{aligned} \tag{6.43}$$

with θ as above and consequently $\arg\theta\in[-\pi/6-o(1),0[$.

We now assume a solution to satisfy $\operatorname{Im}\zeta_-=-(2n+\frac{3}{4})\pi$ with $n\to\infty$ a positive integer and try to derive a contradiction. In a first step, (6.43) implies $|\cosh(\zeta_--i\pi/4)|\le O(|\zeta_-|^{1/2})$ and therefore $\arg\zeta_-=-\pi/2+o(1)$.

The argument of $\Big(1-e^{\zeta_+-\zeta_-}(\zeta_-/\zeta_+)^{1/6}\Big)$ clearly lies in $]-\pi/2,\pi/2[$, whereas $-e^{-i\pi/4}\cosh(\zeta_--i\pi/4)=e^{-i\pi/4}\cosh\operatorname{Re}\zeta_-$. Therefore the two terms on the left hand side of (6.43) cannot cancel, because of the angle they include. Hence, this equation bounds each term on the left hand side, in particular $\zeta_-^{1/2}$. (θ is clearly bounded away from 0, and we will see in a minute – Lemma 6.9 – that the same holds for the term in the parenthesis.) To get a bound for ζ_+ as well, assume to the contrary that $|\zeta_+|\to\infty$, hence $t\to\infty$, $\theta\to1$, $e^{\zeta_+-\zeta_-}\to0$ and (6.43) reduces to

$$\frac{\sqrt{6\pi}}{2}\zeta_-^{1/2}+e^{-i\pi/4}\cosh(\operatorname{Re}\zeta_-)=O(a)\ .$$

But $|\frac{\sqrt{6\pi}}{2}\zeta_-^{1/2}+e^{-i\pi/4}\cosh(\operatorname{Re}\zeta_-)|\ge\frac{\sqrt{6\pi}}{2}|\zeta_-|^{1/2}\ge\frac{\sqrt{6\pi}}{2}\sqrt{(2n+\frac{3}{4})\pi}\ge3\sqrt{2}\,\pi/4$, so we get a contradiction for small a. Now, ζ_+ and ζ_- being bounded, so are δ and t.

This finishes the first half of part 4. For the second half, we first assume n large enough such that $|\zeta_\pm|$ are also large enough. We want to apply Rouché's theorem again and compare the difference between the lhs and the rhs of (6.43) with $g(\zeta_-):=\frac{1}{2}\sqrt{6\pi}\,\zeta_-^{1/2}-e^{-i\pi/4}\cosh(\zeta_--i\pi/4)$. That first expression is indeed an analytic function of ζ_-, when, for given t, we express ζ_+ in terms of ζ_-. We are required to show that

$$\frac{\Big|\frac{1}{2}\sqrt{6\pi}\,\zeta_-^{1/2}\,\Big[1-\theta(1-e^{\zeta_+-\zeta_-}(\zeta_-/\zeta_+)^{1/6})\Big]\Big|+O(a)\cosh\operatorname{Re}\zeta_-+\frac{1}{2}}{\Big|\frac{1}{2}\sqrt{6\pi}\,\zeta_-^{1/2}-e^{-i\pi/4}\cosh(\zeta_--i\pi/4)\Big|}<1 \tag{6.44}$$

on the lines $\operatorname{Im}\zeta_-=-(2n+\frac{3}{4})\pi$, on the half-line $\operatorname{Re}\zeta_-=0$, $\operatorname{Im}\zeta_-\le-const$, and in the domain where $|\operatorname{Re}\zeta_-|$ is large enough (depending on n). Then the counting of zeros can be done with the function g. It will be crucial to know that the term in brackets has an absolute value <1, provided $\operatorname{Im}\zeta_-=-(2n+\frac{3}{4})\pi$. This is what we show next.

Lemma 6.9 *Let* $\theta:=(\delta+it)/(2it)$, $\operatorname{Im}\delta\ge0$, $\operatorname{Im}\zeta_-=-(2n+\frac{3}{4})\pi$, *and* $\operatorname{Re}\zeta_-\le O(\ln|\operatorname{Im}\zeta_-|)$. *Then,*

$$\Big|1-\theta(1-e^{\zeta_+-\zeta_-}(\zeta_-/\zeta_+)^{1/6})\Big|<0.75\ .$$

PROOF OF THE LEMMA:
Note that $(\zeta_-/\zeta_+)^{1/6}=(1/\theta-1)^{1/4}e^{-i\pi/4}$. This transforms our claim into

$$\Big|1+\theta(1/\theta-1)^{1/4}e^{-i\pi/4}e^{\zeta_+-\zeta_-}-\theta\Big|<0.75\ . \tag{6.45}$$

As θ is contained in the sector given above, $\theta(1/\theta-1)^{1/4}=\theta^{3/4}(1-\theta)^{1/4}$ is contained in the sector

$$\Big|\theta^{3/4}(1-\theta)^{1/4}\Big|<0.61\ ,\quad\arg\Big[\theta^{3/4}(1-\theta)^{1/4}\Big]\in\,]-\frac{\pi}{10},\frac{\pi}{24}]\ .$$

Indeed, if $|1-\theta| =: \rho$, then $|\theta| \le (1+\rho^2-\sqrt{3}\rho)^{1/2}$ by the law of cosines, and

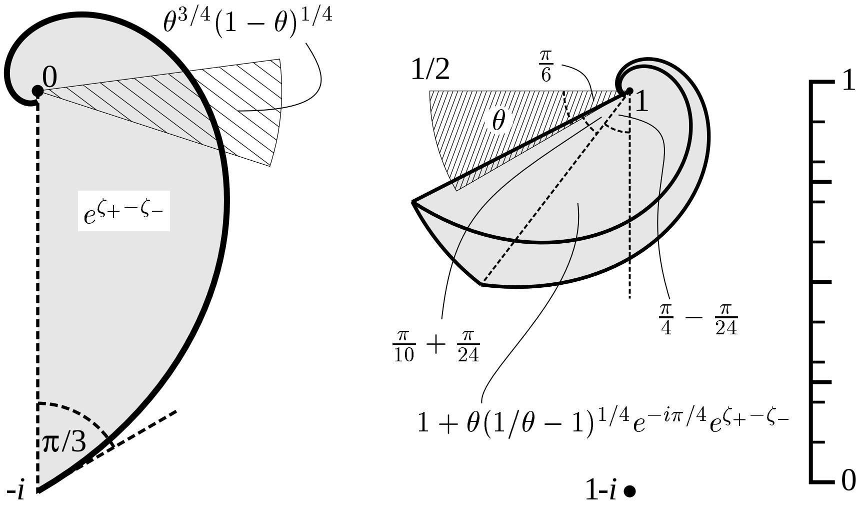
$$\max\{\rho^{1/4}(1+\rho^2-\sqrt{3}\rho)^{3/8} \mid \rho \in [0,\frac{1}{2}]\} \doteq 0.600477 \ .$$

Moreover, $\arg(1-\theta) \in [0,\frac{\pi}{6}]$, and $\arg\theta \in [-\arctan\frac{1/4}{1-\sqrt{3}/4}, 0]$, hence the claim about the argument.

In Lemma 5.13, $\mathrm{Re}\,\zeta_- \le O(\ln|\mathrm{Im}\,\zeta_-|)$ implies $\alpha \ge \frac{\pi}{6} - o(1)$, and we have $\mathrm{Re}(\zeta_+ - \zeta_-) \le (-\tan\alpha)\,\mathrm{Im}(\zeta_+ - \bar\zeta_-)$. Together with $\mathrm{Im}(\bar\zeta_- - \zeta_-) \equiv \frac{3}{2}\pi \pmod{2\pi}$, one concludes that $e^{\zeta_+ - \zeta_-}$ lies "in the interior" of a logarithmic spiral, i.e., in the set

$$\Big\{w \;\Big|\; |w| \le \exp\Big[-(\sqrt{3}/3 - o(1))(\pi/2 + \arg w)\Big] \ , \ \arg w \ge -\pi/2\Big\}$$

with α as specified. See the graphics.



The geometric situation is laid out in the figure in correct scale, and from the figure, as well as by careful reasoning based on its analytic description, one can confirm (6.45). The figure displays closed sets, and the corresponding expressions lie in $o(1)$-neighbourhoods of these sets. The largeness of ζ_- enters here, too.

This completes the proof of Lemma 6.9. ∎

Proof of Theorem 6.8 Continued:

Now consider the lines $\mathrm{Im}\,\zeta_- = -(2n+\frac{3}{4})\pi$. We may assume $\mathrm{Re}\,\zeta_- \le O(\ln|\mathrm{Im}\,\zeta_-|)$ here, because the opposite case is dealt with below.

There, the denominator of (6.44) is

$$|\tfrac{1}{2}\sqrt{6\pi}\,\zeta_-^{1/2} + e^{-i\pi/4}\cosh(\mathrm{Re}\,\zeta_-)| > \Big[6\pi|\zeta_-|/4 + \cosh^2(\mathrm{Re}\,\zeta_-)\Big]^{1/2} \ ,$$

because both terms in the sum subtend an angle $< \pi/2$.

On the other hand, we have just seen that the bracket in the numerator is less than $\frac{3}{4}$ in absolute value. We may also make the $O(a)$ term arbitrarily small. Then, condition (6.44) obtains the form $\frac{3}{4}A + O(a)B + \frac{1}{2} < (A^2+B^2)^{1/2}$, which will be satisfied if A^2+B^2 is large enough, provided $O(a)$ is sufficiently small (at least $O(a) < (7/16)^{1/2}$, as a matter of fact).

For the half-line $\mathrm{Re}\,\zeta_- = 0$, the denominator of (6.44) is $\ge \frac{1}{2}\sqrt{6\pi}|\zeta_-|^{1/2} - 1$. However, for the numerator, we cannot exploit an angle information for the logarithmic spiral as before. Instead, we have the estimate $\left|e^{\zeta_+-\zeta_-}\right| \le \frac{1}{2}$ to control the bracket term in (6.44). In the right half of the figure, we have to replace the logarithmic spiral by a circle of radius $\frac{1}{2}\,0.61$, and them the numerator of (6.44) is $\le \frac{5}{6}\,\frac{1}{2}\sqrt{6\pi}|\zeta_-|^{1/2} + O(1)$, so (6.44) holds again for sufficiently large $|\zeta_-|$.

If $|\mathrm{Re}\,\zeta_-| > \ln(\sqrt{6\pi}|\zeta_-|^{1/2})$, the cosh terms dominate in (6.44), and the argument succeeds again, by the smallness of the $O(a)$ term. That hypothesis is satisfied, if $|\mathrm{Re}\,\zeta_-| > \mathit{const}\ \ln n$.

This concludes the proof that Rouché's theorem does in fact apply. Now the number of zeros of $g(\zeta_-) = \frac{1}{2}\sqrt{6\pi}\,\zeta_-^{1/2} - e^{-i\pi/4}\cosh(\zeta_- - i\pi/4)$ in the strip $\mathrm{Im}\,\zeta_- \in \,]-(2n+\frac{11}{4})\pi, -(2n+\frac{3}{4})\pi[$ can be determined by a homotopy argument, i.e., by considering $g_s(\zeta_-) = s\,\zeta_-^{1/2} - e^{-i\pi/4}\cosh(\zeta_- - i\pi/4)$ with s running from 0 to $\frac{1}{2}\sqrt{6\pi}$. Zeros cannot cross the boundary during the homotopy.

We are left with the counting of zeros for those strips with small n (which have been excluded above). Here, δ (and thus $\mathrm{Re}\,\delta$) is restricted to a bounded set arising from the exceptional set $K(a)$. We are in the non-asymptotic case where $\delta - it$ is bounded, but $t \to \infty$. For t large enough, Theorem 6.6 tells us that ζ_- will lie in the regions specified. This concludes the second half of part 4.

<u>Estimates for the Body Domain:</u>

Let us now consider the body part of $Y_0(a, M)$, where $\mathrm{Re}\,\zeta_\pm > -\ln\tilde{b}$. The procedure is similar to the one on page 76. From Lemma 5.13, we get the first part of (6.40) immediately.

Next we have $e^{-2x}, e^{-(x+y)} < \tilde{b}^2$, and from (6.38)

$$
\begin{aligned}
e^{2y} &\le \frac{1+3a}{1-a}e^{2x}(1+\tilde{b}^2) + \frac{1}{2}\left(\left(\frac{24(1+a)M}{1-a}\right)^2(\tilde{b}^2e^{2x}) + e^{2y}\right)\\
e^{y-x} &\le \left[2\frac{1+3a}{1-a}(1+\tilde{b}^2) + \left(\frac{24(1+a)M}{1-a}\right)^2\tilde{b}^2\right]^{1/2}
\end{aligned}
$$

for $y > x$. Consequently, for $a \le 1/3$, we get $e^{|x-y|} \le [6(1+\tilde{b}^2) + 48^2M^2\tilde{b}^2]^{1/2} =: f(\tilde{b}, M)$.

Selecting the second line in (6.18), we obtain

$$
\begin{aligned}
|\sinh(\zeta_- - \zeta_+)| &\le \tilde{b}^2(1+a_4)/2 + a_2 f(\tilde{b}, M) + a_3 + \tilde{b}(\sqrt{6\pi} + a_1)|\delta^2 + t^2|/(\sqrt[3]{18}\,t)\\
&\le \tilde{b}^2(1+a)/2 + a\left(\sqrt{6} + \sqrt{6}\,\tilde{b} + 2\tilde{b}M + 1 + \tilde{b}M/\sqrt[3]{18}\right) + \sqrt{6\pi}/\sqrt[3]{18}\,\tilde{b}M\\
&\le \tfrac{1+a}{2}\tilde{b}^2 + a\,O(1+\tilde{b}M) + \tfrac{5}{3}\tilde{b}M\ ,
\end{aligned}
$$

i.e., (6.41). Just as for (5.43), we conclude from $\exp|x-y| \le f(\tilde{b}, M)$ that $\ln f(\tilde{b}, M) \ge t|\mathrm{Im}\,\delta|/(2(|\delta|^2+t^2))^{1/4}$, and furthermore

$$
|\mathrm{Im}\,\delta| < O\Big((\ln \mathrm{Re}\,\delta)(\mathrm{Re}\,\delta/C)^{1/4}\Big)\ , \tag{6.46}
$$

and therefore solutions in the diagonal region must be nearly real. This proves part 3 of the theorem.

For δ real, we have $\zeta_+ = \bar{\zeta}_-$, and (6.18) together with (6.38) implies

$$
\begin{aligned}
&\Bigg|2\sin(2\,\mathrm{Im}\,\zeta_+) - \exp(-2\,\mathrm{Re}\,\zeta_+) + \\
&\qquad + \frac{2\sqrt{\boldsymbol{6\pi}}(\delta^2+t^2)^{3/4}}{(12)^{1/3}\,t}\exp(-\mathrm{Re}\,\zeta_+)|\zeta_+|^{1/6}\sin\Big(\frac{1}{6}\arg\zeta_+ - \mathrm{Im}\,\zeta_+\Big)\Bigg| \le \\
&\le (a_2 + a_4)(1 + \exp(-2\,\mathrm{Re}\,\zeta_+)) + a_3 + a_1\exp(-\mathrm{Re}\,\zeta_+)|\zeta_+|^{1/6}\ .
\end{aligned} \tag{6.47}
$$

Clearly, if $\mathrm{Im}\,\zeta_+ = (n+\frac{3}{4})\pi$ with <u>even</u> n, then all terms inside the absolute value signs on the left hand side have the same signs (namely negative), and (6.47) cannot be satisfied at all, provided a is small enough (this repeats one detail of part 4). If, however, $\mathrm{Im}\,\zeta_+ = (n +$

$\frac{3}{4})\pi$ with odd n or else $\operatorname{Im}\zeta_+ = (n+\frac{1}{4})\pi$, then (6.47) still cannot be satisfied if $\operatorname{Re}\zeta_+$ is large enough for the sine term to dominate the two other terms; this latter is the case for $\operatorname{Re}\zeta_+ > \gamma_0$ with $\gamma_0 := \max\left\{\frac{1}{2}\ln 2, 3\sqrt{6\pi}(2r)^{7/6}/(12)^{1/3}\right\}$. The domain thus specified contains $\{t < \sqrt{3}\,r + \sqrt{2}\gamma_0 r^{-1/2}\}$. On the other hand, if $\exp(-\operatorname{Re}\zeta_+) > 2 + 2\sqrt{6\pi}|\zeta_+|^{1/6}/(12)^{1/3}$, i.e., $t > \sqrt{3}r + \gamma(\ln r)r^{-1/2}$ for some γ, then (6.47) cannot be satisfied whatever value $\operatorname{Im}\zeta_+$ may have, because the $\exp(-2\operatorname{Re}\zeta_+)$ term dominates.

This establishes part 6.

The constructed lines of constant $\operatorname{Im}\zeta$ separating real solutions correspond to curves $t = t_n(\delta)$ disconnecting the sector $t \le \sqrt{3}\,\delta$. Again, by the intermediate value theorem (applied to the real valued function $\delta \mapsto i\tilde{\Delta}(\delta, t, \varepsilon)$), there is indeed at least one real solution between each of these lines, at least up to the maximal value of t where $T_{0\mathrm{r}}(n+\frac{1}{2})$ ends.

For n an integer or half an odd integer, we extend the curves $T_{0\mathrm{r}}(n)$ to surfaces. We want to show that these surfaces are still disjoint from $\mathcal{Z}_0^+$ for n large enough. For that purpose, we rearrange (6.43) above slightly:

$$\frac{\sqrt{6\pi}}{2}\zeta_-^{1/2}\theta\Big(1 - e^{\zeta_+ - \zeta_-}\,(\zeta_-/\zeta_+)^{1/6}\Big) + \frac{i}{2}\Big(1 - e^{2\zeta_+ - 2\zeta_-}\Big)e^{\zeta_-} =$$
$$= \frac{1}{2}e^{-\zeta_-} + a_1 + a_3 + (a_2 + a_4)O(\cosh\operatorname{Re}\zeta_-)$$

If $\operatorname{Im}(\zeta_+ - \zeta_-) \equiv \frac{\pi}{2} \pmod{\pi}$, then $1 - e^{2\zeta_+ - 2\zeta_-}$ is real ≥ 1, so this equation can be used for an upper estimate of $|e^{\zeta_-}|$. Clearly, $\left|1 - e^{\zeta_+ - \zeta_-}(\zeta_-/\zeta_+)^{1/6}\right| \le 2$. Therefore, we get

$$e^{\operatorname{Re}\zeta_-} \le K|\zeta_-|^{1/2} \le K\Big((n+\tfrac{1}{2})\pi + |\operatorname{Re}\zeta_-|\Big)^{1/2}$$

for some constant $K \approx 2\sqrt{6\pi}$. This contradicts $\operatorname{Re}\zeta_- \ge \sigma\ln n$ for n large enough, provided $\sigma > \frac{1}{2}$; or else, given any smaller n, it gives an upper bound for $\operatorname{Re}\zeta_-$. In other words, solutions can meet walls only for ζ_- inside some compact set. This yields an upper bound for $\operatorname{Re}\delta$, then – cf. (5.43) – for $\operatorname{Im}\delta$, and finally from $\operatorname{Re}\zeta_- \ge 0$ for t. This concludes the proof of Part 5 and therefore of Theorem 6.8. ■

6.4 Structure of the Solution Set $\mathcal{Z}_0$

Similar to Theorem 5.15, we can now put together the information and get the structure of the set $\mathcal{Z}_0$ in all detail outside some compact set. Actually, the claims of the following theorem appear to hold without the assumption j sufficiently large, if one may believe the numerical calculations. An analytic proof is however not available for the time being. Presumably, it would be possible to reinforce the proof via asymptotic expansions with quantitative error estimates valid already for $j = 1$ and possibly some computer supported interval arithmetics to get an analytic proof.

The numbering in our theorem is meant to parallel to the one in Theorem 5.15; this is why a void statement is included.

Theorem 6.10

1. *For all real solutions (i.e., δ real) to the equation $\Delta'(\delta, t) = 0$ with $t > 0$, it holds $t < \sqrt{3}\delta + O((\ln\delta)\delta^{-1/2})$.*

2. *In the whole domain $t < \sqrt{3}\operatorname{Re}\delta - O((\ln\delta)(\operatorname{Re}\delta)^{-1/2})$ (except possibly a compact set) there are no other solutions beyond the real ones arising from the arcs R_{n0} and R_{n1}.*

   For large enough j, the arcs $R_{2j-1,0}$ and $R_{2j,1}$ (defined in Theorem 5.10) belong to one branch, $\tilde{R}_{2j-1}$, and the arcs $R_{2j-1,1}$ and $R_{2j,0}$ belong to another branch, $\tilde{R}_{2j}$.

   There is exactly one point $\boldsymbol{k}_{2j}$ on $\tilde{R}_{2j}$ from which non-real solutions bifurcate. t has a nondegenerate maximum on $\tilde{R}_{2j}$ at $\boldsymbol{k}_{2j}$.

   There are exactly three points $\boldsymbol{r}_{2j-1}$, $\boldsymbol{r}_{2j}$, and $\boldsymbol{k}_{2j-1}$ on $\tilde{R}_{2j-1}$ from which non-real solutions bifurcate. t has a nondegenerate maximum on $\tilde{R}_{2j}$ at $\boldsymbol{r}_{2j}$ and $\boldsymbol{r}_{2j-1}$ and a nondegenerate minimum at $\boldsymbol{k}_{2j-1}$.

3. *void*

4. *For every j, there exist branches $\tilde{C}_{2j-1}$ and $\tilde{C}_{2j}$ (topologically lines) of solutions, on which $t \to \infty$ at both ends. In the case of $\tilde{C}_{2j-1}$, one has $\delta \sim \bar{z}_{j-} + it + o(1)$ or $\delta \sim z_{j-} - it + o(1)$ as $t \to \infty$. In the case of $\tilde{C}_{2j}$, one has $\delta \sim \bar{z}_{j+} + it + o(1)$ and $\delta \sim z_{j+} - it + o(1)$ instead.*

   Here $z_{j\pm} = A_{j\pm}e^{2i\pi/3}$ where $A_{j\pm}$ is a zero of $\int^\infty \operatorname{Ai}(x)dx$ as in Theorem 6.6.

   There is exactly one point $\mathbf{c}_n$ on $\tilde{C}_n$ from which other solutions bifurcate (real ones, in fact). t has a nondegenerate minimum there. In all of the domain $t > \sqrt{3}\operatorname{Re}\delta + O((\ln\operatorname{Re}\delta)/(\operatorname{Re}\delta)^{-1/2})$, except possibly in a compact set, there are only those solutions belonging to the branches $\tilde{C}_n$.

5. *For j sufficiently large, $\tilde{C}_{2j-1}$, $\tilde{C}_{2j}$, and $\tilde{R}_{2j-1}$, $\tilde{R}_{2j}$ belong to the same connected component, called $\hat{K}_{2j}$, of $\mathcal{Z}_\varepsilon^+$, and the $\hat{K}_{2j}$ are different for different j. (Expressions $\hat{K}_k$ for odd k will not be defined.)*

6. *For j sufficiently large, $\hat{K}_{2j}$ consists only of the branches mentioned in 5, plus one compact branch $\tilde{O}_{2j}$ connecting $\boldsymbol{k}_{2j}$ and $\boldsymbol{k}_{2j-1}$; in particular, $\mathbf{c}_k = \boldsymbol{r}_k$. (No $\tilde{O}_k$ for odd k will be defined.)*

PROOF:

Theorem 6.8,1, in particular (6.38), immediately yields the location constraint for real solutions given in Part 1.

As in the proof of Theorem 5.15, we can follow the arcs R_{n0} or R_{n1} from the lower domain into the upper domain according to Figure 5.5. They belong to certain real branches, which are confined to a compact set by part 1 and Theorem 6.8,6, as long as they do not return to the lower domain. With an appropriate choice of $\tilde{b}$, the real branches (in the upper domain) obey the body domain estimate from Theorem 6.8,3, and they cannot intersect the surfaces $\operatorname{Im}(\zeta_+ - \zeta_-) = (n + \frac{1}{2})\pi$, $\operatorname{Re}\zeta_- \geq \sigma \ln n$ according to Theorem 6.8,5.

The condition $(n + \frac{1}{2})\pi = \operatorname{Im}(\zeta_+ - \zeta_-) = \int_{(\operatorname{Im}\delta)-t}^{(\operatorname{Im}\delta)+t} \frac{d}{ds}\operatorname{Im}\frac{2}{3}(r + is)^{3/2}\,ds$ determines a surface that is a graph $t = t(\delta)$ over the half plane $\operatorname{Re}\delta > 0$, because the integrand $\frac{d}{ds}\operatorname{Im}\frac{2}{3}(r + is)^{3/2}$ is strictly increasing in s according to Lemma 5.13. Therefore these surfaces dissect the body domain (6.40) into tubes, with one real arc in each tube.

The planes $t = const$ intersect the tubes in bounded domains, and as in the proof of Theorem 5.15, the number of solutions to $\Delta'(\cdot, t) = 0$ in a tube is a constant function of t, namely it is 1; hence no solutions can branch off inside the tube.

When the real branches leave the tube, they satisfy $0 < \operatorname{Re}\zeta \leq O(\ln(\operatorname{Im}\zeta))$ with $\zeta := \frac{2}{3}(r+it)^{3/2}$; by Lemma 5.13, this is already in the critical domain. In the critical domain, however, we have seen in Section 6.3.2 how the solution curves combine, namely the arcs $R_{2j-1,0}$ and $R_{2j,1}$ connect into one real branch, which we will denote $\tilde{R}_{2j-1}$, whereas $R_{2j-1,1}$ and $R_{2j,0}$ form another branch, $\tilde{R}_{2j}$. This gives the first two statements of part 2.

Next remember the construction of walls in Thm 6.8,4. It shows that $\tilde{R}_k$ and $\tilde{R}_l$ belong to different connected components of the full complex solution set $\mathcal{Z}_0$, unless $k = l$ or $\{k, l\} = \{2j-1, 2j\}$ for some j.

Clearly, $\tilde{R}_{2j}$ must have at least one maximum; we now show that it is the only one. Generally, if we follow branches along decreasing t, they are controlled by the asymptotics (6.31) inside the critical domain, i.e., until they enter the tubes again. Therefore, the lowest critical point on $\tilde{R}_{2j} \cup \tilde{R}_{2j-1}$ (i.e., the one with smallest t; let's call it $\boldsymbol{k}_{2j}$) must be a nondegenerate maximum: otherwise a branch bifurcating to smaller t would have to exist by Weierstraß' preparation theorem, and this branch would have to return to $\tilde{R}_{2j} \cup \tilde{R}_{2j-1}$ (contradicting the minimality property of $\boldsymbol{k}_{2j}$). The branch among $\tilde{R}_{2j}$, $\tilde{R}_{2j-1}$ on which $\boldsymbol{k}_{2j}$ lies cannot have another critical point on it, so $\boldsymbol{k}_{2j}$ is its global maximum. By the position of $\tilde{R}_{2j}$ "inside" $\tilde{R}_{2j-1}$, $\boldsymbol{k}_{2j}$ can only lie on the former.

Thus we have shown that on $\tilde{R}_{2j}$, there is exactly one branch point $\boldsymbol{k}_{2j}$, a nondegenerate global maximum, as specified in the theorem.

We have seen in Section 6.3.2 that there are (at least) three distinct critical points on $\tilde{R}_{2j-1}$, so one of them has to be local minimum. If several minima should exist on $\tilde{R}_{2j-1}$, we select one with smallest t and call it $\boldsymbol{k}_{2j-1}$. One complex branch has to bifurcate from it in the direction of decreasing t, and it has to return (for the time being, possibly after further bifurcations) to one of the real solution branches $\tilde{R}_{2j} \cup \tilde{R}_{2j-1}$. The only possibility for return is at $\boldsymbol{k}_{2j}$. The minimum $\boldsymbol{k}_{2j-1}$ has to be nondegenerate, because otherwise more than one branch would have to bifurcate there in the direction of decreasing t, and there would be no point left where it could finally return to the real solution set. (Equivalently, the total number of solutions between the walls constructed in Theorem 6.8,4 is 4, independent of t, and taking into account the other real branch, we must have a nondegenerate minimum here.)

The same counting argument guarantees that no further minima (and therefore only two maxima) can exist on $\tilde{R}_{2j-1}$, and they have to be nondegenerate. This completes part 2. At the same time, we have shown that $\tilde{R}_{2j}$ and $\tilde{R}_{2j-1}$ are indeed connected through one branch (or possibly a cascade of branches), which proves part 5, as far as the real solutions are concerned. We have not defined yet which of the two maxima on $\tilde{R}_{2j-1}$ is to be called $\boldsymbol{r}_{2j-1}$ and which $\boldsymbol{r}_{2j}$. We decide this now by defining as $\boldsymbol{r}_{2j-1}$ the one with the larger coordinate t.

It is conceivable, a priori, that when we follow the complex branch from $\boldsymbol{k}_{2j-1}$ along decreasing t to $\boldsymbol{k}_{2j}$, we encounter further bifurcations. However, as the total number of solutions is 4 in the domain under consideration, two of which are real and lie on $\tilde{R}_{2j-1}$, the only possibility for this to happen is that the complex branch returns to a real solution <u>before</u> encountering $\boldsymbol{k}_{2j}$. Then there would have to be a compact branch of real solutions in the critical domain (bifurcating from this real solution). We next exclude this possibility:

If a compact branch of real solutions existed in the critical domain (elsewhere it couldn't exist anyway), it could not extend down into the tubes which separate the reals arcs R_{n0} and R_{n1}. In other words, on the whole compact branch, the coordinate t would have to be no smaller than it is in the point $\boldsymbol{k}_{2j}$. We have seen in (6.35) that this means $t - \sqrt{3}\,r = -2\ln(2\pi\sqrt{j})/\sqrt{2r} + O((\ln r)/r^2)$ and thus $B > -\ln(2\pi\sqrt{j}) + O(j^{-1/2})$, hence $\exp[B] > const\, j^{-1/2}$. On the other hand, on a compact real branch there would have to be a point with maximal A. But the condition $\Delta' = 0 = \partial_B \Delta'$ in such a point implies $\exp[B] < O(j^{-7/6})$ by Lemma 6.7. This contradiction proves the nonexistence of real compact branches and therefore almost that $\boldsymbol{k}_{2j}$ and $\boldsymbol{k}_{2j-1}$ are connected by one single branch, $\tilde{O}_{2j}$, as stated in part 6. We have not shown that the branch is actually an imbedded circle: it might contain degenerate real points where it has self-intersections. However, then one would have $\Delta' = 0 = \partial_A \Delta' = \partial_B \Delta$ at that point (in the critical domain. This is however also excluded by Lemma 6.7, because inserting $\sin[2A] = O(A^{-2/3})$ and

$\exp[B] = O(A^{-7/6})$ into the condition $\partial_A\Delta'0$ from formal (legitimate as explained) differentiation of (6.28) yields the contradiction $\cos[2A] = O(A^{-2/3})$.

According to Theorem 6.8,4, solutions do not meet the walls $\operatorname{Im}\zeta_- = -(2j-\frac{5}{4})\pi$ for $j \geq 1$. For given t (large, dependent on n), there are exactly two solutions in the strip $\operatorname{Im}\zeta_- \in\,]-(2j+\frac{3}{4})\pi, -(2j-\frac{5}{4})\pi[$, $\operatorname{Im}\delta > 0$. For t large enough for $\operatorname{Re}(\zeta_+ - \zeta_-)$ to be $\leq -\ln 2$, the one with $\operatorname{Re}\zeta_- > 0$ forms an arc that will be called $C_{2j,+}$, whereas the one with $\operatorname{Re}\zeta_- < 0$ forms an arc that will be called $C_{2j-1,+}$. Their complex conjugate arcs ($\bar{\delta}$ instead of δ) will be called $C_{2j,-}$, $C_{2j,-}$ respectively.

We can follow these arcs along decreasing t until a branch point occurs. As long as the arcs are isolated in their half strip, branching cannot happen at all. For j large enough, the branches can only escape from their isolating half strip $\operatorname{Im}\zeta_- \in\,]-(2j+\frac{3}{4})\pi, -(2j-\frac{5}{4})\pi[$, $\operatorname{Re}\zeta_- \gtrless 0$, if $\operatorname{Re}(\zeta_+ - \zeta_-) > -\ln 2$ (Theorem 6.8,4). But then we are already in the critical domain, because Lemma 5.13 concludes $\operatorname{Im}\delta \leq O(r^{-1/2})$.

This proves the last statement of part 4. Following these arcs for increasing t puts one automatically in the non-asymptotic case, where we obtain the asymptotic behaviour claimed in the first sentence of part 4 from Theorem 6.6.

We now discuss how these arcs assemble to complex branches and how they connect to the real branches.

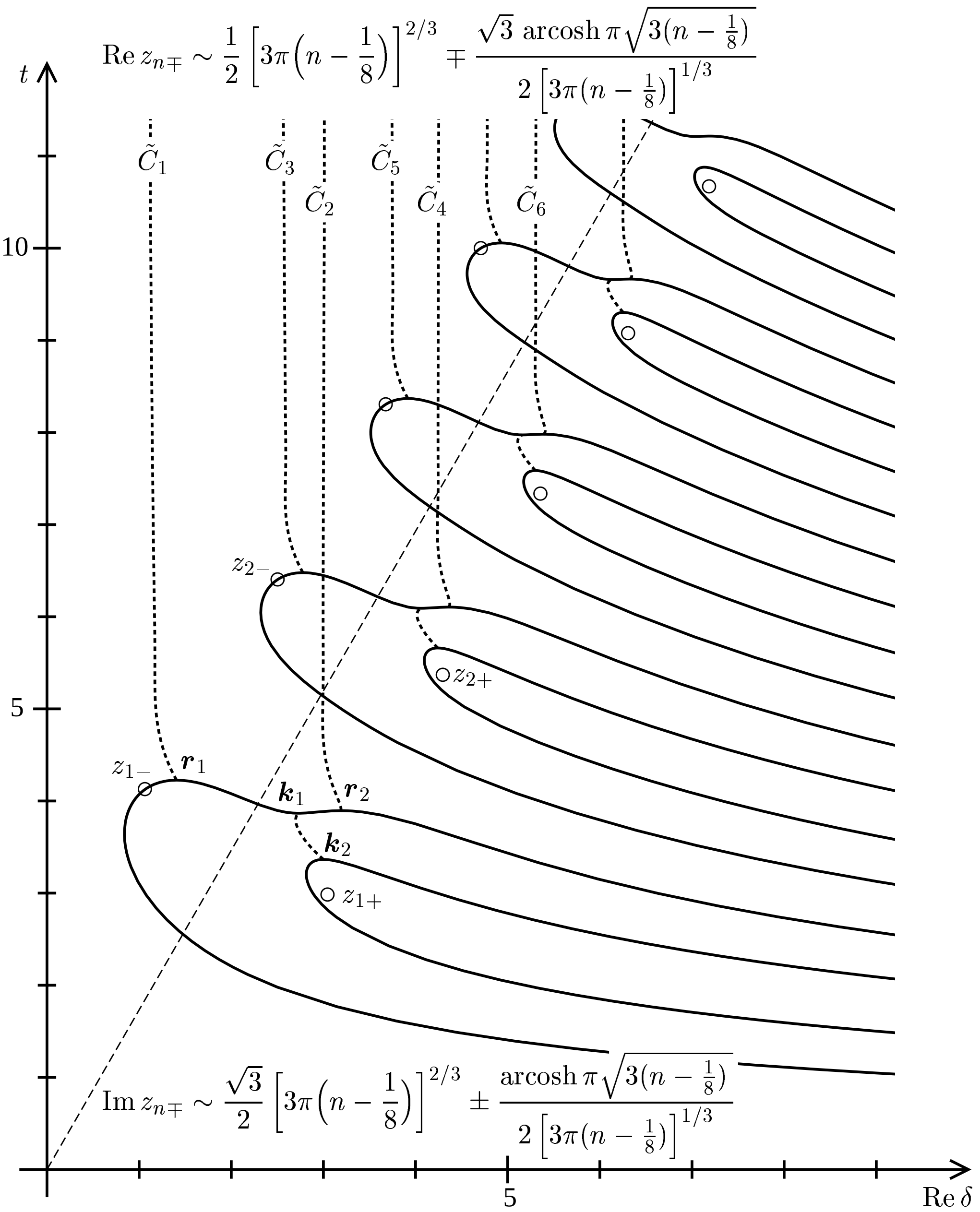
It is immediate from the construction of the walls $\operatorname{Im}\zeta_- = -(2j-\frac{5}{4})\pi$ constructed, that each of $C_{2j-1,-}$, $C_{2j,-}$, $C_{2j-1,+}$, $C_{2j,+}$ (which have to connect to <u>some</u> real branches) can only belong to the connected component $\hat{K}_{2j}$. Combinatorically, there are only three possibilities: (1) $C_{2j-1,\pm}$ form one branch by connecting in a real point $\boldsymbol{c}_{2j-1}$, and $C_{2j,\pm}$ form another by connecting in a real point $\boldsymbol{c}_{2j}$. (2) $C_{2j-1,+}$ and $C_{2j,+}$ form one branch, and $C_{2j-1,-}$ and $C_{2j,-}$ form its complex conjugate. (3) $C_{2j-1,+}$ connects to $C_{2j,-}$, and $C_{2j-1,-}$ connects to $C_{2j,+}$; this would result in two branches that cross at a real point. We want to exclude the latter two possibilities. In either of the possibilities (2) and (3), two branches are formed whose lowest points are at the same level t. This level must be higher than the level t at the point $\boldsymbol{r}_{2j-1}$, because further bifurcations into branches with lower t would finally have to lead to a branch point on the real branches. But then the crossing of $\operatorname{Re}\zeta_- = 0$ must occur at an even higher level. By (6.32), this implies $B > O(j^{1/2})$ in that crossing point.

On the other hand, we can use the fine asymptotics given in (6.28) for the crossing point: For a solution to satisfy $\operatorname{Re}\zeta_- = 0$, it is necessary that $\operatorname{Re}(\zeta_+ - \zeta_-) \geq -\ln 2$ and thus $\operatorname{Im}\delta \leq O(r^{-1/2})$, as we have seen In Thm. 6.8,4. This bounds $\operatorname{Im}A$ and $\operatorname{Im}B$ by (6.26). At the same time, $\operatorname{Re}\zeta_- = 0$ implies $\tau = \sigma$, and therefore B itself is bounded.

This contradiction proves that the complex arcs $C_{n\pm}$ and their conjugates combine into branches according to possibility (1). Critical points on these branches have to be real, because otherwise, together with their complex conjugates and the other branch, the total number of four solutions at each level would be exceeded. Therefore the first branch point encountered when descending from $t = \infty$ along the arcs is already a real local minimum $\boldsymbol{c}_{2j}$ or $\boldsymbol{c}_{2j-1}$, and it is a nondegenerate minimum by the usual argument. Crossings (where $\Delta' = 0 = \partial_B\Delta' = \partial_A\Delta'$) have already been excluded above. This proves part 4.

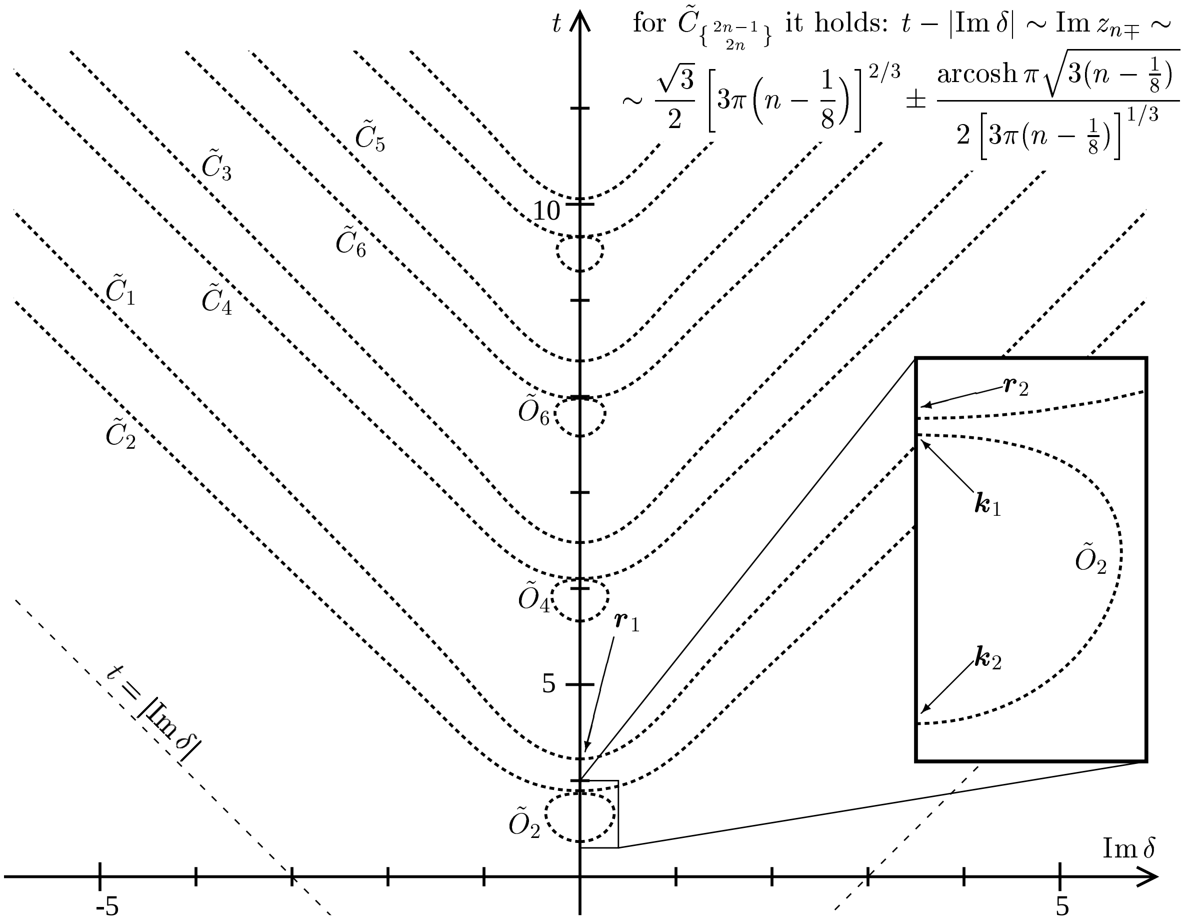
Moreover, $\boldsymbol{c}_n = \boldsymbol{r}_n$, because otherwise compact real branches would intervene between these two points. The claim about the connected components (part 5) is also proved now. This ends the proof of Theorem 6.10. ■

Figure 6.6 shows a numerically calculated solution set, projected into the (r, t) plane. The projection of the complex branches into the $(\operatorname{Im}\delta, t)$ plane is shown in Figure 6.7.



The dotted lines represent complex branches, and it is their projection onto the real plane that is drawn. The solid lines are real branches. The little circles represent numbers $r+it = e^{-2\pi i/3} z_k$, where the z_k are solutions to $\int_z^\infty \operatorname{Ai}(y)\,dy = 0$. For the open complex branches, $\delta - it$ converges to the complex conjugate of these points as $t \to \infty$. Although every other among these points appears to lie on the set $\mathcal{Z}_0$, this is not exactly true. For example, the point on $\mathcal{Z}_0$ that is closest to $z_{1-} = 1.0626258 + 4.1288430i$ (i.e., the one calculated by the gradient method) is the point $1.0645180 + 4.1261998i$;

Figure 6.6: The set $\mathcal{Z}_0^+$ in the $\operatorname{Re}\delta$-t plane



Beware: The "obvious" ordering of the projection of complex branches into the $(\mathrm{Im}\,\delta, t)$ plane as suggested by this figure will not persist for arbitrarily large orders (curves far beyond the scope of this picture), as the asymptotic (6.23) above shows.

Figure 6.7: The complex branches of $\mathcal{Z}_0^+$ in the $(\mathrm{Im}\,\delta, t)$ plane

6.5 Appendix: Some Eigenfunctions for $\varepsilon = 0.004$

Having obtained the branch structure of the spectrum, it is a routine matter to calculate some eigenfunctions corresponding to these eigenvalues. They can be expressed by quadratures by means of the method of variation of constants. We offer a collection of plots of numerically calculated eigenfunctions in an arbitrary normalization, as they follow each other along the branches of the spectrum. The real part of the eigenfunctions is plotted with solid lines, the complex part is plotted in dotted lines. The distinction according to real and imaginary part is somewhat arbitrary, due to the arbitrariness in the choice of a (complex) normalization constant.

The phenomenon that the number of humps characterizes the arcs R_{j0}, R_{j1} at the "nearly self-adjoint end" (in the sense briefly explained in the very beginning of Section 4.3) is clearly visible, but needs still rigorous formulation. The calculations are for $\varepsilon = 0.004$. The eigenfunctions for the sample eigenvalues given in Figure 6.8 are shown in Figure 6.9.

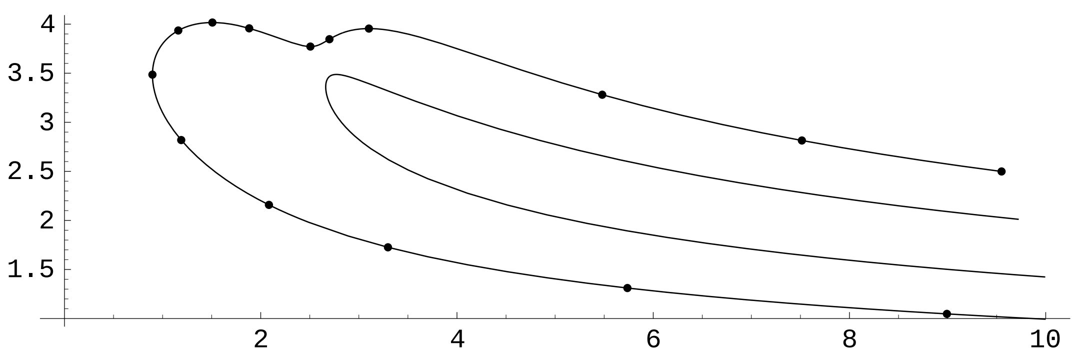


Figure 6.8: Sample points used in Figure 6.9

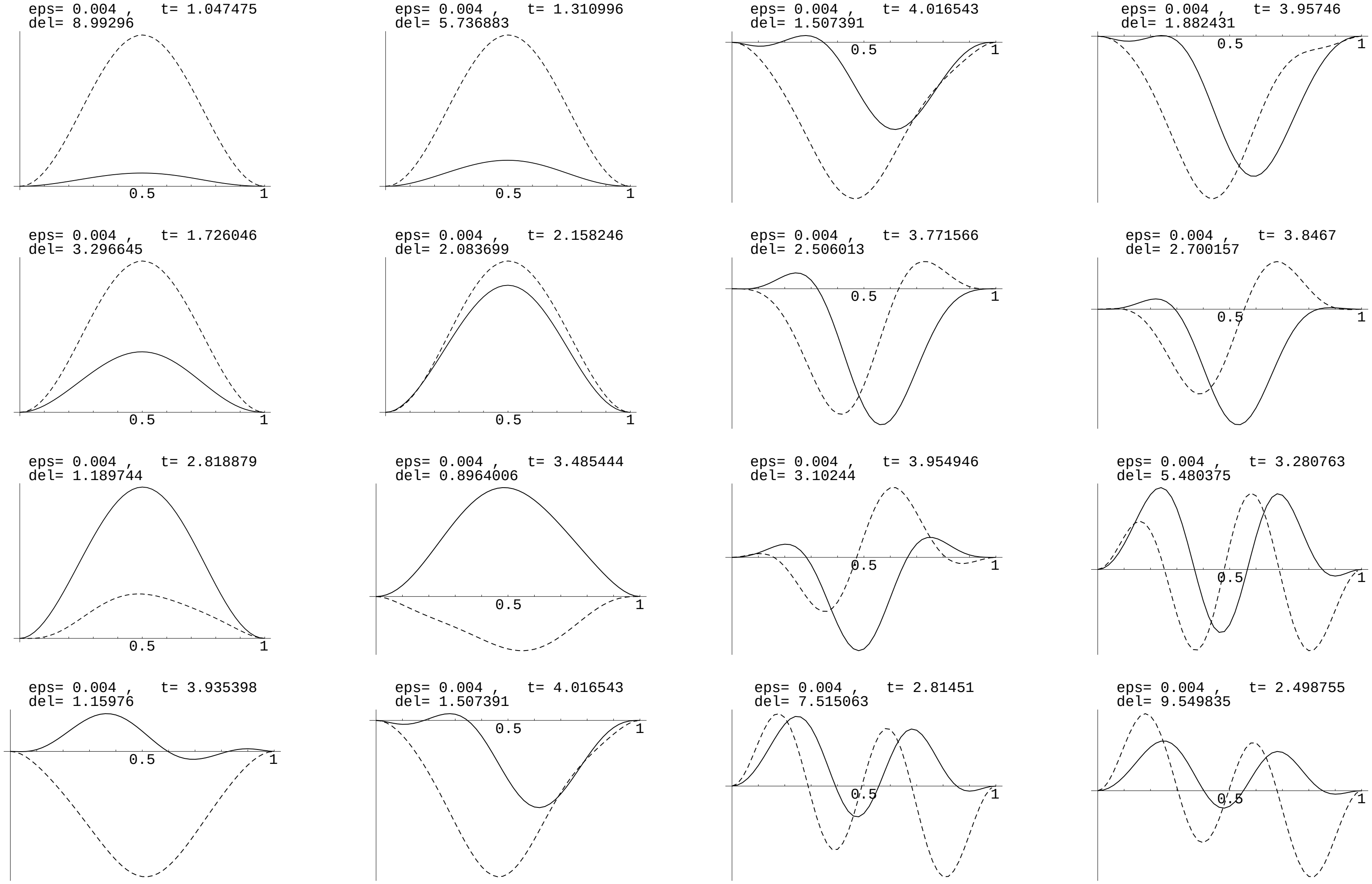


Figure 6.9: Eigenfunctions along the branch $\tilde{R}_1$ for $\varepsilon = 0.004$

Chapter 7

Algebraic PDEs for the Spectral Condition

7.1 Simple Consequences of the Airy ODE

Let Ci be any solution of Airy's differential equation, and let $F(\delta,t,\varepsilon) := \mathrm{Ci}(\delta+itz)\exp(4\varepsilon t^3 z)$, or any linear combination thereof, like e.g. $\int_{-1}^{1}\mathrm{Ci}(\delta+itz)\exp(4\varepsilon t^3 z)\,dz$. We shall also use the variables $\tau = t, q = 4\varepsilon t^3$, and when we refer to a partial derivative with respect to τ, we mean that q and δ are fixed, whereas a partial derivative with respect to t implies that δ and ε are fixed. We denote $\tilde{F}(\delta,\tau,q) := F(\delta,t,\varepsilon)$, i.e., $\tilde{F}(\delta,\tau,q) = \mathrm{Ci}(\delta+i\tau z)\exp(qz)$ or a linear combination thereof.

The following results are valid, as can be easily checked:

$$\begin{aligned} \tilde{F}_\tau &= i\tilde{F}_{\delta q} \\ \tilde{F}_{\tau\tau} &= -\delta\tilde{F}_{qq} - i\tau\tilde{F}_{qqq} \\ \tilde{F}_{\tau\delta} &= i\delta\tilde{F}_q - \tau\tilde{F}_{qq} \\ \tilde{F}_{\delta\delta} &= \delta\tilde{F} + i\tau\tilde{F}_q \end{aligned} \tag{7.1}$$

Using the transformation laws

$$\begin{array}{ll} \partial_t = \partial_\tau + 12\varepsilon t^2\partial_q & = \partial_\tau + 12\varepsilon\tau^2\partial_q \\ \partial_\varepsilon = 4t^3\partial_q & = 4\tau^3\partial_q \\ \hline \partial_\tau = \partial_t - 3q/(4\tau^4)\partial_\varepsilon & = \partial_t - (3\varepsilon/t)\partial_\varepsilon \\ \partial_q = 1/(4\tau^3)\partial_\varepsilon & = 1/(4t^3)\partial_\varepsilon \end{array}$$

we conclude

$$\begin{aligned} tF_t &= (12it^2\delta\varepsilon - 1)F - \delta F_\delta - 12it^2\varepsilon F_{\delta\delta} + F_{\delta\delta\delta} \\ F_\varepsilon &= 4i\delta t^2 F - 4it^2 F_{\delta\delta} \\ tF_t - 3\varepsilon F_\varepsilon &= -F - \delta F_\delta + F_{\delta\delta\delta} \end{aligned} \tag{7.2}$$

These relations do not exploit the full information of $F(\delta,t,\varepsilon) = \int_{-1}^{1}\mathrm{Ci}(\delta+itz)\exp(4\varepsilon t^3 z)\,dz$. Any other function with another measure of integration instead of dz would satisfy the same relations.

7.2 Linear Inhomogeneous PDEs satisfied by Δ

In this section, we employ the Airy ODE in a more systematic way to find algebraic differential equations satisfied by $\Delta(\delta,t,\varepsilon)$. The author's original hope that these might open an elegant access to some facts about the spectrum of the OS equation – somehow reminiscent of the ideas employed in Lemma 5.3 – has not been substantiated so far. But the results seem worth mentioning all the same.

It is in principle straightforward to produce this kind of differential equations, but still the practical calculation is not quite trivial as it needs some algorithmic care to avoid blowing up the computer algebra capacity, and to achieve an output in a human-intelligible form.

We use Ci for a 2-vector valued solution of Airy's ODE, e.g., $\mathrm{Ci} = \left[\begin{smallmatrix}\mathrm{Ai}\\ \mathrm{Bi}\end{smallmatrix}\right]$, and denote the determinant of two 2-vectors by $\wedge$. hyp is used for either sinh or cosh.

Any (repeated) derivative of $\int_{-1}^{1} \mathrm{Ci}(\delta + itz)\,\mathrm{hyp}(4\varepsilon t^3 z)\,dz$ with respect to any of δ, t, ε leads to integrals of the type $\int_{-1}^{1} z^n\,\mathrm{Ci}^{(m)}(\delta + itz)\,\mathrm{hyp}(4\varepsilon t^3 z)\,dz$. One may conceive an algorithmic scheme to treat these integrals as follows: Derivatives of Ci are eliminated from under the integral by repeated integration by parts. Then powers of z are reduced according to $z\,\mathrm{Ci}(\delta + itz) = [-\delta\,\mathrm{Ci}(\delta + itz) + \mathrm{Ci}''(\delta + itz)]/(it)$, with subsequent treatment of Ci'' by integration by parts again. On the other hand, higher derivatives of Ci in the integrated terms are eliminated by the Airy ODE.

All derivative operators operate therefore in a free rank-6-module $\mathcal{V}$ over an integral domain $\mathbb{S}$, or else you may prefer to enlarge $\mathcal{V}$ to a 6-dimensional vector space over the quotient field of $\mathbb{S}$. Namely,

$$\mathbb{S} := \mathbb{Q}[i](t)[\delta, \varepsilon, S, C]/(C^2 - S^2 = 1) \quad \text{with} \quad C = \cosh(4\varepsilon t^3)\,,\ S = \sinh(4\varepsilon t^3)\,,$$

in words: the ring of polynomials in $\delta, \varepsilon, S, C$ over the field of rational functions (in t) over the number field $\mathbb{Q}[i]$ of Gaussian rationals, modulo the ideal generated by $C^2 - S^2 - 1$. $\mathcal{V}$ is spanned by

$$\mathcal{B} = \left\{ \begin{array}{lcl} \langle + \rangle := \int_{-1}^{1} \mathrm{Ci}(\delta + itz) \cosh 4\varepsilon t^3 z\,dz & , & \langle - \rangle := \int_{-1}^{1} \mathrm{Ci}(\delta + itz) \sinh 4\varepsilon t^3 z\,dz \\ {[+]} := \mathrm{Ci}(\delta + it) + \mathrm{Ci}(\delta - it) & , & {[-]} := \mathrm{Ci}(\delta + it) - \mathrm{Ci}(\delta - it) \\ {[\acute{+}]} := \mathrm{Ci}'(\delta + it) + \mathrm{Ci}'(\delta - it) & , & {[\acute{-}]} := \mathrm{Ci}'(\delta + it) - \mathrm{Ci}'(\delta - it) \end{array} \right\}$$

The derivatives of $\Delta = \langle + \rangle \wedge \langle - \rangle$ will therefore be calculated as elements of the 15-dimensional space $\mathcal{V}^{\wedge 2} := \mathcal{V} \wedge \mathcal{V}$ spanned by $\mathcal{B}^{\wedge 2} := \{u \wedge v \mid u, v \in \mathcal{B}\}$. $\mathcal{V}^{\wedge 2}$ is naturally decomposed into $\mathcal{V}^{\wedge 2}_{\int\int} \oplus \mathcal{V}^{\wedge 2}_{\int\partial} \oplus \mathcal{V}^{\wedge 2}_{\partial\partial}$ with dimensions 1, 8, 6 respectively, where the subscripts denote whether the corresponding elements of $\mathcal{B}$ are integral terms or boundary terms. We are not claiming that the mapping from $\mathcal{V}^{\wedge 2}$ into the subspace of $\mathcal{O}(\mathbb{C}^3)$ generated from $\mathrm{span}\{\Delta\}$ by $\partial_\delta, \partial_t, \partial_\varepsilon$ is injective. Actually it is not, as the last equation of (7.3) below shows.

The following elements of $\mathcal{B}^{\wedge 2}$ and relations therein are noteworthy:

$$\begin{aligned} \Delta &= \langle - \rangle \wedge \langle + \rangle \\ \underset{\infty}{\Delta} &:= [-] \wedge [+] = 2(\mathrm{Ai}_+\,\mathrm{Bi}_- - \mathrm{Bi}_+\,\mathrm{Ai}_-) \\ \underset{\infty}{\acute{\Delta}} &:= [\acute{-}] \wedge [\acute{+}] = 2(\mathrm{Ai}'_+\,\mathrm{Bi}'_- - \mathrm{Bi}'_+\,\mathrm{Ai}'_-) \\ [-] \wedge [\acute{-}] + [+] \wedge [\acute{+}] &= 4W = 4/\pi \\ [+] \wedge [\acute{-}] + [-] \wedge [\acute{+}] &= 0 \end{aligned} \tag{7.3}$$

As there are 10 derivatives of Δ up to order 2 with respect to (δ, t, ε), they must satisfy at least one linear relation with inhomogeneity in $\mathcal{V}^{\wedge 2}_{\partial\partial}$, because $\mathcal{V}^{\wedge 2}_{\int\int} \oplus \mathcal{V}^{\wedge 2}_{\int\partial}$ has dimension 9. The actual calculation shows that even two independent such relations are satisfied. The relevant Mathematica definitions are given in Section 7.3 below. It is advisable to control the successive elimination of basis vectors from $\mathcal{V}^{\wedge 2}_{\int\int} \oplus \mathcal{V}^{\wedge 2}_{\int\partial}$ (in particular the pivot strategy) by hand. Otherwise a blow-up of garbage clutters both the computer and the human. It is straightforward to check the relations thus obtained:

Theorem 7.1

$$\begin{aligned}
&(t^2\partial_\delta^2 + t^2\partial_t^2 + 4\varepsilon^2\partial_\varepsilon^2 - 4\varepsilon t\partial_\varepsilon\partial_t - 2\varepsilon\partial_\varepsilon + 4t\partial_t + 2)\,\Delta(\delta,t,\varepsilon) = 2\underset{\infty}{\Delta}\sinh(8\varepsilon t^3)\\
&(\partial_\varepsilon^2 + 4t^2(4\delta t^2 + 64\varepsilon^2 t^6)\partial_\delta^2 - 64\varepsilon t^4\partial_\varepsilon\partial_\delta + 16t^5\partial_\delta\partial_t + 16t^4\partial_\delta)\,\Delta(\delta,t,\varepsilon) = \\
&\qquad\qquad = 16t^2\sinh(8\varepsilon t^3)(\delta\underset{\infty}{\Delta} + \acute{\underset{\infty}{\Delta}})\ .
\end{aligned} \tag{7.4}$$

In particular, for $\varepsilon = 0$ *and* $\Delta'(\delta,t) := \lim_{\varepsilon\to 0}\Delta(\delta,t,\varepsilon)/\varepsilon$, *it holds*

$$(\partial_\delta^2 + \partial_t^2)\Delta'(\delta,t) = 16t\underset{\infty}{\Delta}\ . \tag{7.5}$$

The right hand side of at least the first equation has been studied reasonably well in Section 5.2.2 on the infinite viscosity case. On the other hand, homogeneous equations of higher order can be obtained right away, if one employs the following second order PDEs for the right hand sides, which can be verified easily:

Lemma 7.2

$$\begin{aligned}
&(\partial_\delta\partial_t + 2t)\,\underset{\infty}{\Delta} = 0\\
&(\partial_\delta^2 - \partial_t^2 - 4\delta)\,\underset{\infty}{\Delta} = 0\\
&(\partial_\delta\partial_t + 2t)\,(\delta\underset{\infty}{\Delta} + \acute{\underset{\infty}{\Delta}}) = 0\\
&(t\partial_\delta^2 - t\partial_t^2 + 4\partial_t - 4\delta t)\,(\delta\underset{\infty}{\Delta} + \acute{\underset{\infty}{\Delta}}) = 0\\
&(\partial_\delta^4 - 4\delta\partial_\delta^2 - 6\partial_\delta - 4t^2)\underset{\infty}{\Delta} = 0\\
&(t\partial_t^4 - \partial_t^3 + 4\delta t\partial_t^2 - 4\delta\partial_t - 4t^3)\underset{\infty}{\Delta} = 0
\end{aligned} \tag{7.6}$$

The fourth order ODEs are immediate by cross differentiation and elimination from the second order PDEs.

For fixed ε, the algorithmic scheme described above can be used to obtain at least one third order PDE with respect to (δ,t) alone. (10 derivatives up to order three in the 9-dimensional space $\mathcal{V}_{\int\partial}^{\wedge 2} \oplus \mathcal{V}_{\int\int}^{\wedge 2}$.) One obtains

Theorem 7.3

$$\Big(-t\partial_t + \partial_\delta^3 + (48\varepsilon^2 t^4 - \delta)\partial_\delta - 2\Big)\Delta(\delta,t,\varepsilon) = \frac{3}{2t^2}\sinh(8\varepsilon t^3)\partial_\delta\underset{\infty}{\Delta}\ . \tag{7.7}$$

In particular, for $\varepsilon = 0$, *this specializes (using also (7.5)) to*

$$\begin{aligned}
&\Big(-t\partial_t + \partial_\delta^3 - \delta\partial_\delta - 2\Big)\Delta'(\delta,t) = 12t\partial_\delta\underset{\infty}{\Delta}\\
&\Big(\partial_\delta^3 - 3\partial_\delta\partial_t^2 - 4t\partial_t - 4\delta\partial_\delta - 8\Big)\Delta' = 0\ .
\end{aligned} \tag{7.8}$$

Under the transformation $\frac{2}{3}(\delta + it)^{3/2} = x + iy$, *the latter reads*

$$\begin{aligned}
\Big(&\frac{3x}{2}\partial_x^3 + \frac{9y}{2}\partial_x^2\partial_y - \frac{9x}{2}\partial_x\partial_y^2 - \frac{3y}{2}\partial_y^3 + 3\partial_x^2 - 3\partial_y^2\\
&\qquad - 6x\partial_x - 6y\partial_y - \frac{2x}{3(x^2+y^2)}\partial_x + \frac{2y}{3(x^2+y^2)}\partial_y - 8\Big)\Delta' = 0\ .
\end{aligned}$$

In principle it is possible to obtain ODEs of comparatively high order by the same method. For $\varepsilon = 0$, we get more relations between the elements of $\mathcal{B}$ and consequently between those of $\mathcal{B}^{\wedge 2}$, and the remaining dimensions of $\mathcal{V}_{\int\int}^{\wedge 2} \oplus \mathcal{V}_{\int\partial}^{\wedge 2} \oplus \mathcal{V}_{\partial\partial}^{\wedge 2}$ are $0 + 4 + 5$. Therefore, Δ' satisfies at least one homogeneous ODE of order at most 9, with respect to the variable t, and the same holds with respect to the variable δ.

We have done one simple case, which has turned out to be useful, but at the same time lets expect that any more general cases are likely to be far too complicated to be useful:

Lemma 7.4 *For $\delta = 0$, the following ODE with respect to the t variable holds for Δ':*

$$\begin{bmatrix} 1 & t^6 & t^{12} \end{bmatrix} \begin{bmatrix} 2941200 & 76101 & 625 \\ -14823648 & 101676 & 625 \\ -2117664 & 4475 & 0 \\ -1254912 & 1825 & 0 \\ -2862768 & 3125 & 0 \\ -745104 & 625 & 0 \\ 39216 & 8335 & 0 \\ 6536 & 375 & 0 \\ 117648 & 3125 & 0 \\ 39216 & 625 & 0 \end{bmatrix}^T \begin{bmatrix} 8 \\ 4t\ \partial_t \\ -24t^2\partial_t^2 \\ 20t^3\partial_t^3 \\ -3t^4\partial_t^4 \\ 3t^5\partial_t^5 \\ 60\ \partial_t^6 \\ -216t\ \partial_t^7 \\ 3t^2\partial_t^8 \\ -t^3\partial_t^9 \end{bmatrix} \Delta'(0,t) = 0\,. \tag{7.9}$$

7.3 Mathematica Definitions for the Differential Algebraic Relations

In this section, we give the Mathematica definitions for the calculation of derivatives, based on which the PDEs can be found (with work) or checked (easily).

```
basis= {
    INT[ Ci[d+I t z] Cosh[4 eps t^3z] ], (* INT over {z,-1,1} *)
    INT[ Ci[d+I t z] Sinh[4 eps t^3z] ],
    Cisum[d,t],                        (* Ci[d + I t] + Ci[d - I t] *)
    Cidiff[d,t],                       (* Ci[d + I t] - Ci[d - I t] *)
    Ciprimesum[d,t],          (* D[Ci[d + I t], d] + D[Ci[d - I t], d] *)
    Ciprimediff[d,t]          (* D[Ci[d + I t], d] + D[Ci[d - I t], d] *)
}                         (* Ci means the vector {Ai, Bi} of Airy fcts *)

EO[n_Integer,a_,b_] := If[ EvenQ[n], (*then*) a, (*else*) b ]



(* strategy of Integration by parts: *)
(* get rid of derivatives of Ci: *)

INT[ Ci'[d+I t z] Cosh[4 eps t^3z] ] =
    - I/t Cosh[4eps t^3] Cidiff[d,t] \
    + 4 eps t^2 I INT[ Ci[d+I t z] Sinh[4 eps t^3z] ]

INT[ Ci'[d+I t z] Sinh[4 eps t^3z] ] =
    - I/t Sinh[4eps t^3] Cisum[d,t] \
    + 4 eps t^2 I INT[ Ci[d+I t z] Cosh[4 eps t^3z] ]

INT[ z^n_.  Ci'[d+I t z] Cosh[4 eps t^3z] ] :=
    - I/t Cosh[4eps t^3] EO[n, Cidiff[d,t], Cisum[d,t]] \
    + 4 eps t^2 I INT[ z^n Ci[d+I t z] Sinh[4 eps t^3z] ] \
    + I/t n INT[ z^(n-1) Ci[d+I t z] Cosh[4 eps t^3z] ]

INT[ z^n_.  Ci'[d+I t z] Sinh[4 eps t^3z] ] :=
    - I/t Sinh[4eps t^3] EO[n, Cisum[d,t], Cidiff[d,t]] \
    + 4 eps t^2 I INT[ z^n Ci[d+I t z] Cosh[4 eps t^3z] ] \
```

```
    + I/t n INT[ z^(n-1) Ci[d+I t z] Sinh[4 eps t^3z] ]

INT[ Ci''[d+I t z] Cosh[4 eps t^3z] ] =
    - I/t Cosh[4eps t^3] Ciprimediff[d,t] \
    + 4 eps t^2 I INT[ Ci'[d+I t z] Sinh[4 eps t^3z] ]

INT[ Ci''[d+I t z] Sinh[4 eps t^3z] ] =
    - I/t Sinh[4eps t^3] Ciprimesum[d,t] \
    + 4 eps t^2 I INT[ Ci'[d+I t z] Cosh[4 eps t^3z] ]

INT[ z^n_.  Ci''[d+I t z] Cosh[4 eps t^3z] ] :=
    - I/t Cosh[4eps t^3] EO[n, Ciprimediff[d,t], Ciprimesum[d,t]] \
    + 4 eps t^2 I INT[ z^n Ci'[d+I t z] Sinh[4 eps t^3z] ] \
    + I/t n INT[ z^(n-1) Ci'[d+I t z] Cosh[4 eps t^3z] ]

INT[ z^n_.  Ci''[d+I t z] Sinh[4 eps t^3z] ] :=
    - I/t Sinh[4eps t^3] EO[n, Ciprimesum[d,t], Ciprimediff[d,t]] \
    + 4 eps t^2 I INT[ z^n Ci'[d+I t z] Cosh[4 eps t^3z] ] \
    + I/t n INT[ z^(n-1) Ci'[d+I t z] Sinh[4 eps t^3z] ]

(* and once the derivatives are gone, get rid of powers of z using Airy ODE *)

INT[ z^n_.  Ci[d+I t z] Cosh[4 eps t^3z] ] :=
    -I/t INT[ z^(n-1) Ci''[d+I t z] Cosh[4 eps t^3z] ] \
    + I d/t INT[ z^(n-1)  Ci[d+I t z] Cosh[4 eps t^3z] ]

INT[ z^n_.  Ci[d+I t z] Sinh[4 eps t^3z] ] :=
    -I/t INT[ z^(n-1) Ci''[d+I t z] Sinh[4 eps t^3z] ] \
    + I d/t INT[ z^(n-1)  Ci[d+I t z] Sinh[4 eps t^3z] ]


(* Definition of spectral determinant: *)
f[0,0,0] = INT[ Ci[d+I t z] Cosh[4 eps t^3z] ] **
           INT[ Ci[d+I t z] Sinh[4 eps t^3z] ]
(* The ** is to be anticommutative, represents the det of two 2-vectors *)



Unprotect[NonCommutativeMultiply]  (* ** is bilinear and anticommutative *)
                                   (*      and expands automatically      *)
a_Plus ** b_ := (# ** b &) /@ a
b_ ** a_Plus := (b ** # &) /@ a

(a_ b_) ** c_ := a (b ** c) /; NumberQ[a]
b_ ** (a_ c_) := a (b ** c) /; NumberQ[a]

(d^a_.  b_) ** c_ := d^a (b ** c)
b_ ** (d^a_.  c_) := d^a (b ** c)
(t^a_.  b_) ** c_ := t^a (b ** c)
b_ ** (t^a_.  c_) := t^a (b ** c)
(eps^a_.  b_) ** c_ := eps^a (b ** c)
b_ ** (eps^a_.  c_) := eps^a (b ** c)
```

```
(Cosh[a_] b_) ** c_ := Cosh[a] (b ** c)
b_ ** (Cosh[a_] c_) := Cosh[a] (b ** c)
(Sinh[a_] b_) ** c_ := Sinh[a] (b ** c)
b_ ** (Sinh[a_] c_) := Sinh[a] (b ** c)

a_ ** 0 := 0
0 ** a_ := 0

a_ ** b_ := -(b ** a) /; Not[OrderedQ[{a,b}]]
a_ ** a_ := 0

Protect[NonCommutativeMultiply]



(* Derivatives of the spectral determinant *)
(*
  f[a,b,c] means a-fold derivative wrt d
             b-fold derivative wrt t
             c-fold derivative wrt eps
*)

f[a_Integer,b_Integer,c_Integer] := (
    f[a,b,c] = derivdelta[ f[a-1,b,c] ]//Expand ) /; a>0

f[a_Integer,b_Integer,c_Integer] := (
    f[a,b,c] = derivt[ f[a,b-1,c] ]//Expand ) /; b>0

f[a_Integer,b_Integer,c_Integer] := (
    f[a,b,c] = deriveps[ f[a,b,c-1] ]//Expand ) /; c>0



(* Derivation rules for derivdelta etc.: *)

derivdelta[0]=0
derivdelta[a_Plus] := derivdelta /@ a
derivdelta[a_ b_NonCommutativeMultiply] := D[a,d] b + a derivdelta[b]
derivdelta[a_ ** b_] := derivdelta[a] ** b + a ** derivdelta[b]
derivdelta[INT[a_]] := INT[D[a,d]]

derivdelta[ Cisum[d,t] ]  = Ciprimesum[d,t]
derivdelta[ Cidiff[d,t] ] = Ciprimediff[d,t]
derivdelta[ Ciprimesum[d,t] ]  = d Cisum[d,t] + I t Cidiff[d,t]
derivdelta[ Ciprimediff[d,t] ] = d Cidiff[d,t] + I t Cisum[d,t]

derivt[0]=0
derivt[a_Plus] := derivt /@ a
derivt[a_ b_NonCommutativeMultiply] := D[a,t] b + a derivt[b]
derivt[a_ ** b_] := derivt[a] ** b + a ** derivt[b]
derivt[INT[a_]] := INT[D[a,t]//Expand]
```

```
derivt[ Cisum[d,t] ]  = I Ciprimediff[d,t]
derivt[ Cidiff[d,t] ] = I Ciprimesum[d,t]
derivt[ Ciprimesum[d,t] ]  = I d Cidiff[d,t] - t Cisum[d,t]
derivt[ Ciprimediff[d,t] ] = I d Cisum[d,t] - t Cidiff[d,t]

deriveps[0]=0
deriveps[a_Plus] := deriveps /@ a
deriveps[a_ b_NonCommutativeMultiply] := D[a,eps] b + a deriveps[b]
deriveps[a_ ** b_] := deriveps[a] ** b + a ** deriveps[b]
deriveps[INT[a_]] := INT[D[a,eps]]

deriveps[ Cisum[d,t] ]  = 0
deriveps[ Cidiff[d,t] ] = 0
deriveps[ Ciprimesum[d,t] ]  = 0
deriveps[ Ciprimediff[d,t] ] = 0



(* Linearity of INT: *)

INT[a_Plus] := INT /@ a
INT[a_ b_] := a INT[b] /; NumberQ[a]
INT[t^a_.  b_] := t^a INT[b]
INT[eps^a_.  b_] := eps^a INT[b]

INT[d^a_.  b_] := d^a INT[b]
      (* this will not occur, but I want things logically simple *)



wedgebasis= Drop[ Union[ Flatten[
       Outer[NonCommutativeMultiply, basis, basis] /.
       - a_NonCommutativeMultiply :> a
       ] ], 1 (* drop the zero element *) ]

wdgbastomatrix= Thread[ wedgebasis -> IdentityMatrix[15] ]

Null

(* *******************************************************************
  We try to get simple equations for the 6 no integral-terms first.
  Then equations for the one integral-terms with inhomogeneity expressed
  by the former. Our first task is to select (by hand) a reasonable
  ordering of the terms in wedgebasis and express the derivatives explicitly
  as matrices.
********************************************************************** *)

sortwbas= wedgebasis[[{3,1,10,2,7,6,
       4,5,13,14,8,9,11,12,
       15}]]
```

```
sortwbasshort= {CmCp, CmCm1, -CpCp1, CmCp1, -CpCm1, Cm1Cp1,
       CmCH, CmSH, CpCH, CpSH, Cm1CH, Cm1SH, Cp1CH, Cp1SH,
       CHSH}
(* where the mnemotechnic abbreviations should be clear *)
(* defined a posteriori of course, after inspection of Mma's sort order *)

sgnchng= sortwbasshort/ Abs[sortwbasshort] /.  Abs-> (#&)

swbtoshort=  Thread[ sortwbas -> sortwbasshort ]
swbfromshort= (* Reverse /@ swbtoshort *) \
              (* does not work because of minus signs *) \
     Thread[ sortwbasshort * sgnchng -> sortwbas * sgnchng ]

swbtomatrix= Thread[ sortwbasshort * sgnchng -> IdentityMatrix[15] * sgnchng ]
swbfrommatrix[any_List] := any .  sortwbasshort


Ddeltamat= (derivdelta /@ sortwbas) /.  swbtoshort /.  swbtomatrix
Dtmat=     (derivt /@ sortwbas)     /.  swbtoshort /.  swbtomatrix
Depsmat=   Table[Table[0,{15}],{6}] ~ Join~
           (deriveps /@ sortwbas[[Range[7,15]]]) /.  swbtoshort /.  swbtomatrix
  (* the first 6 entries would otherwise yield scalar 0 instead of vector 0 *)


sparsity[0]= 0
sparsity[a_List] := sparsity/@a
sparsity[_]= 1
(* helps pivoting by hand *)
```

Index of Notations

This index does not contain every single symbol; the author hopes to have found a decent compromise between completeness and clarity. Various local purposes use their own symbols, so that the symbol may have different meanings in different contexts (which is why chapters etc. are given). However, ambiguities should not arise.

Symbols from the Latin alphabet:

symbol	explanation
A, a	$A = 64\varepsilon t^6$ in Sec. 4.2; $a = \sqrt{A}$
A	in Chapter 6: see (6.25)
$\mathbf{A}$	in Ch. 1,2: linearized Navier Stokes operator
$\mathbf{A}_0$	in Ch. 2: $\lambda_0 - \mathbf{A}$
a_{-n}, a'_{-n}	zeros of Ai, Ai$'$ respectively; cf. Fig. 5.1, page 54
Ai, Bi	Airy functions
$\mathrm{Ai}_\pm, \mathrm{Bi}_\pm$	$\mathrm{Ai}(\delta \pm it)$, $\mathrm{Bi}(\delta \pm it)$
B	in Chapter 6: see (6.25)
b_n	zeros of Bi, see page 55
b, $\tilde{b}$	in Theorems 5.14, 6.8 and the relating estimates parameters that delimit the "wing" and "body" domains respectively
$\tilde{C}_n$	complex branch: see Figures 5.4 (page 65) and 6.2–6.4, 6.6–6.7; Theorems 5.15, 6.10
$c_j^{k,r}$, $\tilde{c}_j^{k,r}$	coefficients in formula for generalized Wronskian, see Lemma 4.2
c_k	series coefficients: (3.12)
Ci	any (possibly vector valued) solution to Airy's ODE
$\boldsymbol{c}_n$	branch point on $\tilde{C}_n$; Chapters 5 and 6
d_k	series coefficients: (3.12)
$\mathcal{D}$	domain of definition of an operator, Section 2.5
D	$\mathrm{Bi} / \mathrm{Ai}$
$\boldsymbol{f}$	generic element of function space in Section 2.5
g_k	series coefficients: (3.25)
H^k	Sobolev space $W^{k,2}$; also with subscript s for divergence free vector fields; see Section 2.4
j	in Chapter 4 the order of the expansion for large δ / small t in Chapter 6: $2j$ and $2j-1$ are generic indices for branches
k_1, k_3	wave numbers, Fourier dual to x_1, x_3: Ch. 1
k	generic integer or positive integer
$\boldsymbol{k}_{2j}$, $\boldsymbol{k}_{2j-1}$	branch point on $\tilde{O}_{2j}$; Chapter 6
L^p	Banach space of p-integrable functions or vector fields; see Section 2.4

Index of Notations

symbol	explanation
m, $\boldsymbol{m}$	in Chapter 4: $m = k + r$; series index for powers of $-\delta t^2$. $\boldsymbol{m}$ corresponding operator in Sec 4.2
n , $\boldsymbol{n}$	in Chapter 4: series index for powers of $\varepsilon^2 t^6$. $\boldsymbol{n}$ corresponding operator in Sec 4.2 otherwise n is generic positive integer
$\tilde{O}_{2j}$	compact complex branch; see Figures 6.3, 6.4, 6.7, Theorem 6.10
O, o	Landau symbol
$\mathbf{P}$	projection operator onto solenoidal vector fields, Section 2.5
$P_N(\delta)$	polynomials, see page 30
$P_j^{\mathrm{odd}}(k), p_j^{2r}$ etc.	other polynomials, see Lemma 4.1
$Q_N(\delta)$	polynomials, see page 30
$Q_j^{\mathrm{odd}}(k), q_j^{2r}$ etc.	other polynomials, see Lemma 4.1
r	occasionally index of summation or integer (and then obvious from the context). Otherwise, in particular in Chapters 5 and 6: real part of δ.
R	<u>not</u> the Reynolds number (which is ε^{-1}: one day I may regret this choice myself...)
$\tilde{R}_n$	real branch: see Figures 5.4 (page 65) and 6.2, 6.4, 6.6; Theorems 5.15, 6.10
$\boldsymbol{r}_n$	branch point on $\tilde{R}_n$; Chapters 5 and 6
t	very rarely the time (p. 11, Sec. 2.5), otherwise $\frac{1}{2}(\alpha/\varepsilon)^{1/3}$: see (1.12)
$U = U(y)$	velocity profile of plane parallel Navier Stokes flow
$\boldsymbol{u}$	velocity in Navier–Stokes equations
$\boldsymbol{w}$	velocity in linearized Navier–Stokes equations
$\hat{\boldsymbol{w}}$	Fourier transform of $\boldsymbol{w}$ (with respect to x_1, x_3)
w	variable in OS equation; the 2-component of $\hat{\boldsymbol{w}}(k_1, \cdot, k_2)$ $w = \sqrt{x}$ in Ch. 4
W	Wronskian of Airy functions
$W^{k,p}$	Sobolev space; see Section 2.4
$\boldsymbol{x}$, x_1, x_2, x_3	space coordinates
x	$x = 4\delta t^2$ in Sec. 4.2
z	only occasionally (and obviously) generic complex variable; otherwise $z \in [-1, 1]$ parametrizes the distance between the plates and becomes a variable of integration with this semantics.
$\mathcal{Z}_\infty$, $\mathcal{Z}_\varepsilon$, $\mathcal{Z}_0$	solution set to $\Delta = 0$; exact def. see (5.1), (5.9), (6.4)

Greek and non-alphabetic symbols:

symbol	explanation
α	in Ch. 0, 1, 2: wave number; in Sec 2.5 power of an operator; in Ch. 5, 6: auxiliary parameter, see Lemma 5.13
Γ	Gamma function; occasionally path of integration
Δ	$=\Delta(\delta,t,\varepsilon)$, see (2.1)
Δ'	$=\Delta'(\delta,t)$, for $\varepsilon=0$, see (6.3)
$\tilde{\Delta}$	up to boring factors Δ
Δ^*	leading part of Δ'
$\underset{\infty}{\Delta}$	See (7.3) and (6.5)
δ	transformed eigenvalue parameter for OS eqn; see (1.12)
$\delta_\pm$	short for $\delta \pm it$
ε	inverse of Reynolds number; only very rarely, in obvious cases, the generic small quantity
$\tilde{\varepsilon}$	see page 14
ζ	generically, $\frac{2}{3}z^{3/2}$ (in Sec. 3.3)
$\zeta_\pm$	$\zeta_\pm = \frac{2}{3}(\delta \pm it)^{3/2}$
Θ, $\boldsymbol{\Theta}$	see Lemma 4.2 and (4.33)
λ	eigenvalue in OS eqn; for $\tilde{\lambda}$ see page 14
μ	generic eigenvalue parameter in Section 2.5
ρ_n	branch points of $\mathcal{Z}_\infty$, interpreted as complex numbers
σ	various local definitions; in Ch. 6 $\sigma = \operatorname{Im}\delta$
τ	occasional local definitions; in Ch. 6, $\tau = t - \sqrt{3}\operatorname{Re}\delta$
$[\![\cdot\vert\cdot]\!]$	higher order Wronskian: (4.10)
$\langle + \rangle$, $[+]$ etc.	see Sec 7.2
$\uparrow$, $\downarrow$	$x^{\uparrow n} := x(x+1)\cdots(x+n-1)$; $x^{\downarrow n} := x(x-1)\cdots(x-n+1)$; see (4.11)

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