Bounds for the heat diffusion through windows of given area [1]

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Abstract

We consider the lowest eigenvalue for the Laplacian in a given Lipschitz domain under mixed boundary conditions: Dirichlet in a subset of the boundary (non–insulated window), Neumann otherwise. This eigenvalue can be interpreted as heat leakage rate due to diffusion.

We give an explicit calculation for a model problem, a rigorous lower bound that depends only on the area, but not on the geometry of the window. This bound confirms the observations from the model problem. Finally, we show that no nontrivial upper bound is possible, i.e., any small area for the window being prescribed, its geometry can be made bad enough to cause heat leak rates arbitrarily close to the ones for no insulation anywhere.

The most important techniques are the Aronszajn–Weinstein method of intermediate variational problems and the Gaussian upper bounds for the heat kernel by E.B. Davies.

0 Introduction

To the author's knowledge the problem discussed in this work has been invented by Walter Craig on the occasion of a cold New England winter day, which produced a draft through an electric outlet. However, the author has removed the draft from the problem, thus avoiding all the difficulties of the Navier Stokes equations. What remains is an intriguing inverse question in geometric spectral theory which is of a type that apparently has not been thoroughly considered before.

Suppose you are in a thermically well insulated room, except for small holes in the insulation, which we will call windows, although in reality, cracks near the frame of a window may be more responsible for the loss of heat. What can be said about the loss of heat through a non-insulated window? How does it depend on the area of the window (e.g. asymptotically for small area), and how does it depend on its geometry? What kind of geometry of the window(s) would be preferable to minimize the heat leak rate (fixing the area)?

Mathematically, we are considering an initial boundary value problem for the heat equation, where we have mixed boundary data:

$$\begin{aligned} u_t(x,t) &= \Delta u(x,t) && \text{for } (x,t) \in \,]0,\infty[\, \times \Omega \\ u(x,0) &= u_0(x) && \\ u(\cdot,t) &= 0 && \text{on } D \subset \partial\Omega \qquad (\forall t) \\ \partial_n u(\cdot,t) &= 0 && \text{on } N := \partial\Omega \setminus D \qquad (\forall t) \end{aligned} \tag{1}$$

[1] AMS Classification: primary: 35P15; secondary: 35J20, 49R15

Here Ω is a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary.

Expanding the solution in terms of an orthonormal eigensystem (λ_k, φ_k) of the Laplacian with respect to the boundary conditions given, one immediately sees that the relative heat loss $\partial_t(\int_\Omega u(x,t)\,dx)/(\int_\Omega u(x,t)\,dx)$ will converge to the first eigenvalue λ_1 for large times, unless the initial temperature distribution u_0 happens to be orthogonal to the first eigenfunction, which we assume not to be the case (this assumption is always true for $0 \neq u_0 \geq 0$ and generically true in any case).

We will define λ_1 by the appropriate variational problem in the Sobolev space $H^1(\Omega)$,

$$\lambda_1(D) := \inf\left\{ \int_\Omega |\nabla u|^2\,dx \;\middle|\; u \in H^1(\Omega)\,,\ \int_\Omega u^2 = 1\,,\ u|_D = 0 \right\}\ . \tag{2}$$

Thus we define the meaning of Neumann boundary conditions on a Lipschitz boundary as natural boundary conditions for the variational problem, which actually coincide with classical Neumann conditions in the case of sufficiently smooth boundary. To assume $\partial\Omega$ as smooth is not necessary in this context, and it would not be natural either (see fig. 1).

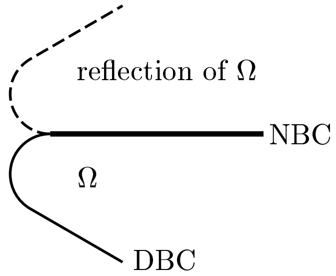


Figure 1: *The reflection principle tells: A jump from DBC to NBC on a smooth boundary is as bad as a re-entrant cusp.*

It is immediate from direct methods of the calculus of variations (eg. [16, I.1] or any modern book on calculus of variations) that the infimum in (2) is actually taken on. One needs to use Rellich's compact embedding theorem $H^1(\Omega) \hookrightarrow L^2(\Omega)$ and the compact trace imbedding theorem $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$, and this is where the Lipschitz condition on the boundary enters (eg. [2] V5.4 or [12] 5.7,5.8,6.4 or [6] 4.3–4.6). For much worse boundary, the spectrum of the Neumann Laplacian may well fail to be pure point (eg. [10],[9] and references therein). In that case the proof of existence of a minimizer for (2) would break down.

We are going to establish the results stated in the abstract. Our proofs are essentially variational and avoid the subtle question of elliptic boundary regularity under mixed boundary conditions most of the time. A natural question connected with the lower bound is whether $\inf\{\lambda_1(D) \mid \mu(D) = A\}$ is actually taken on. The rather non-trivial answer is yes, and it does involve some discussion of sophisticated boundary regularity results; see [5] for this question.

1 A Model Problem; Explicit Examples

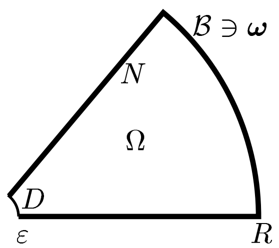
There are not many model examples which can be calculated explicitly. Rectangles with D consisting of two opposite, one or three sides, and their higher dimensional analogs are among them, but not very interesting, because they do not permit to vary the area of D continuously.

A better model in 2 dimensions is an annulus with a (small) varying inner radius and a fixed outer radius. The inner boundary is D, the outer is N. This seems to be a reasonable model for small "round" holes in the insulation, although the domain is not strictly fixed. Sectors of such an annulus and their higher dimensional analogs can be treated in the same way and serve as models for round holes sitting in the middle of some wall or a corner. Products of these domains with intervals can serve as models for thin slits.

Let $\mathcal{B}$ be a domain on the unit sphere $S^{d-1} \subset \mathbb{R}^d$, and let some solution be given for the eigenvalue problem $-\Delta_{\mathcal{B}}\Phi(\boldsymbol{\omega}) = \lambda_\circ \Phi(\boldsymbol{\omega})$ with the Laplace Beltrami operator $\Delta_{\mathcal{B}}$ on $\mathcal{B}$ and with whatever boundary conditions. Then, for any α, μ, and with $\nu := \sqrt{(d/2-1)^2 + \lambda_\circ}$,

$$u(r, \boldsymbol{\omega}) = \Phi(\boldsymbol{\omega}) Z_\nu^{(\alpha)}(\mu r) := \Phi(\boldsymbol{\omega}) \left[(\cos\alpha) J_\nu(\mu r) + (\sin\alpha) Y_\nu(\mu r)\right] (\mu r)^{1-(d/2)} \tag{3}$$

solves $-\Delta u = \mu^2 u$ in the cone $\{r\boldsymbol{\omega} \mid r > 0,\ \boldsymbol{\omega} \in \mathcal{B}\}$.



To get the lowest eigenvalue (characterized by positivity of the eigenfunction) for $\Omega = \{r\boldsymbol{\omega} \mid \varepsilon < r < R,\ \boldsymbol{\omega} \in \mathcal{B}\}$, $D = \{r = \varepsilon\} \cap \bar{\Omega}$, $N = \partial\Omega \setminus D$, we have to choose $\Phi \equiv 1$ ($\lambda_\circ = 0$) and determine α such that $z_k / z'_k = \varepsilon / R$, where z_k is some zero of $Z_\nu^{(\alpha)}$ and z'_k is the next larger zero of $Z_\nu^{(\alpha)\prime}$. Then the eigenvalue $\lambda_1(D)$ equals $\mu^2 = (z'_k)^2 R^{-2}$. Now $\varepsilon = R z_k / z'_k$ can only go to 0, if we take the smallest zeros $k = 1$, small ε/R corresponds to small α, and then the local behaviour of the Bessel functions ([1], 9.1.10,11 and 10.1.3) yields after some calculation

$$\lambda_1(D) = \mu^2 \sim \frac{2R^{-2}}{\ln(R/\varepsilon)} \quad \text{for } d = 2\,, \qquad \lambda_1(D) = \mu^2 \sim R^{-2} \frac{d(d-2)}{4} \left(\frac{\varepsilon}{R}\right)^{d-2} \quad \text{for } d \geq 3\,. \tag{4}$$

Letting $0 < \sigma := \mu(\mathcal{B}) / \mu(S^{d-1}) \leq 1$ be the relative area of $\mathcal{B}$, we get

$$\begin{aligned} \lambda_1(D) = \mu^2 &\sim \frac{c(2)}{\operatorname{vol}(\Omega)} \frac{\sigma}{\ln(1/\mu(D))} && \text{for } d = 2\,, \\ \lambda_1(D) = \mu^2 &\sim \frac{c(d)}{\operatorname{vol}(\Omega)} \sigma^{1/(d-1)} \mu(D)^{1-1/(d-1)} && \text{for } d \geq 3 \end{aligned} \tag{5}$$

with dimensional factors $c(d)$.

Much of the above follows already from dimensional considerations, but not the corner effect described by the parameter σ.

If the window D fits in a small d–dimensional ball of radius ϵ (around 0), then a reasonable upper estimate for the eigenvalue can be obtained from choosing the cut–off fundamental solution $u(r) := \max\{0, \varepsilon^{2-d} - r^{2-d}\}$ in the Rayleigh quotient (for $d = 2$, of course, $u(r) := \max\{0, \ln(r/\varepsilon)\}$ instead). The same type of asymptotics as in (5) is obtained in this case for $\varepsilon \to 0$ and any domain Ω. The lower bounds (independent of the geometry) described below will confirm this order of magnitude that has just been obtained for "round" heat leaks D.

Inasmuch as these model examples are representative for the general case, they suggest the following: For small leaks, the leaking rate goes to 0 more slowly than the area, and this effect is more pronounced the lower the dimension is. Heuristically one may hope that the leaking rates due to small leaks distant from each other just add in good approximation, and then the exponent in the $(\text{area})^{1-1/(n-1)}$–law suggests that the heat leakage is lower for one hole than for several holes of the same total area. Moreover, the leak should preferably be in a corner (small σ). This latter statement can also be understood as placing the heat leak into a place that is less easily accessible to diffusion.

Using the products of the above domains with one or several intervals as models for slits, one comes up with essentially the same formulas again, but the codimension of the slit plays now the role of d in the asymptotic behaviour for small areas. Therefore, slits can be expected to cause a higher leak rate than holes of the same area.

A reasonable conjecture based on these observations seems to be that an optimal window (which exists according to [5]) should be connected and "round" in some sense to be made precise yet. However, the simple example in figure 2 shows that connectedness of an optimal window cannot be concluded without either assuming small area or restricting the global geometry of the room. See figure 2.

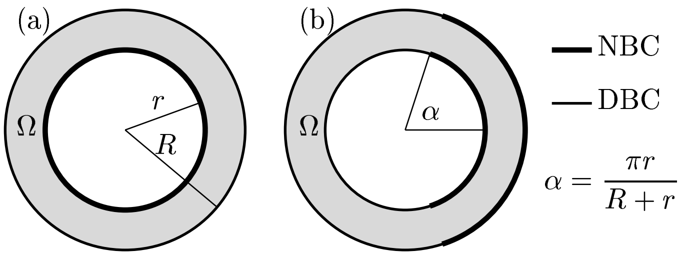


Figure 2: *In an annulus with radii $r < R$, there is only one connected window of size $2\pi R$, (a), but the window on the right (b) can be shown to cause a much smaller heat leak rate.*

The lowest eigenvalue and its eigenfunction for the left case can be evaluated in terms of Bessel functions as above, and for $r/R \nearrow 1$, the eigenvalue is close to $[\pi/2(R-r)]^2$. On the other hand, an upper estimate of the first eigenvalue for the right case of figure 2 can be obtained from the Rayleigh quotient for the function $u(r,\varphi) := \cos[\varphi \cdot (R+r)/2r]$ for $|\varphi| \leq \pi r/(R+r)$ and 0 otherwise, and it will be asymptotic to $1/R^2$ as $r/R \nearrow 1$.

The author does not know whether connectedness of $\partial\Omega$ or small area of D suffice to ensure that optimal windows are connected. An interesting test case for such a conjecture should be something like in figure 3.

It can be shown by rearrangement arguments that for Ω a ball in $\mathbb{R}^d$, a spherical cap of the appropriate $(d-1)$–dimensional measure is an optimal window. This gives more credibility to a conjecture that optimal windows should be "round" and therefore the estimates (5) for round windows give the right order of magnitude for a lower bound.

2 Lower Bounds; an Intermediate Variational Problem

We now establish a quantitative lower bound for $\ell(A) = \inf\{\lambda_1(D) \mid \mu(D) = A\}$. This bound will have the same order of magnitude as the estimates (5) obtained for the model problem above. For this purpose, an intermediate variational problem is introduced, which latter can also be used to show that there is only the trivial upper bound. However, an easier proof is available for this. In special cases (cylindrical Ω), a rearrangement

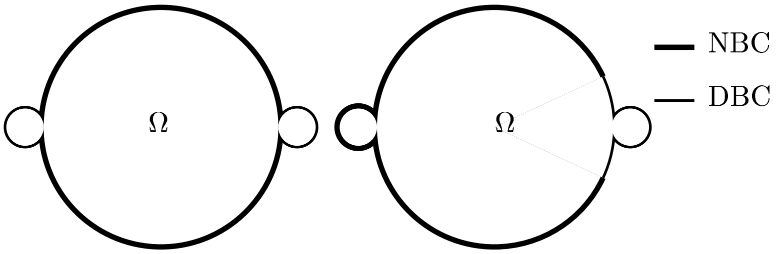


Figure 3: *Contrary to first intuition, it is not clear that the room on the left with two nearly inaccessible windows has less heat leakage than the one on the right, where part of the single window looks very easily accessible: but in this case, most of the "mass" of the first eigenfunction will be in the small compartment rather than in the main hall.*

inequality on the boundary is available to produce a short evaluation of the lower bound. In the general situation, a basic ingredient is the upper estimates of the heat kernel by Davies [4].

Obviously the rooms and passages phenomenon is an obstruction to get good lower bounds, so some assumption on the global geometry of Ω is needed. We assume convexity, but weaker assumptions could be incorporated into the proof for the price of technicalities; convexity is convenient to control certain constants, but not an essential ingredient.

The method of intermediate variational problems goes back to Aronszajn and Weinstein. Its classical paradigma is to give lower bounds for the eigenvalues of a clamped plate in terms of the ones for a hinged plate (which has weaker boundary constraints in the variational problem). A nice introduction into this method is [8].

We start with the Laplace operator with Neumann boundary conditions on Ω. It can be constructed from the symmetric semibounded operator $-\Delta$ defined on $\mathcal{D}_0 := \{u \in C^2(\bar{\Omega}) \mid \partial_n u \equiv 0 \text{ ae. on } \partial\Omega\}$ by the Friedrichs extension process [15, sec. 124] or [17, 5.5]; the quadratic form associated to the operator is $\int_\Omega |\nabla u|^2$, finally defined on $H^1(\Omega)$ (because $\mathcal{D}_0$ is dense in it for Lipschitz domains). Let (ϕ_k) be an orthonormal base of (real) eigenfunctions for this operator, with corresponding eigenvalues ν_k, ie., $-\Delta\phi_k = \nu_k\phi_k$. Then, for any function $u \in H^1(\Omega)$, we have $\int_\Omega |\nabla u|^2 = \sum_k \nu_k \hat{u}_k^2$ where $\hat{u}_k := \int_\Omega u\phi_k$.

It is a non-trivial task of elliptic boundary regularity to conclude that the natural (i.e, not prescribed in the variational problem) boundary conditions are indeed Neumann conditions; in particular for Lipschitz boundary this uses quite recent results [11]. Here, the whole argument rests on the variational problem, so we can avoid the regularity issue; see [5] for more.

With this preparation, we construct the intermediate variational problems

$$\min \left\{ \int_\Omega |\nabla u|^2 \, dx \,\middle|\, u \in H^1(\Omega)\,,\ \int_\Omega u^2 \, dx = 1\,,\ \int_{D_i} u \, dS = 0\,,\ i = 1, \ldots, r \right\}\,, \tag{6}$$

where the D_i are disjoint (up to measure 0) and their union is D. Minimizing the same functional over the smaller set $\{u \in H^1(\Omega) \mid \int u^2 = 1, u|_D = 0\}$ yields the first eigenvalue of the mixed problem(2), so (6) yields a lower bound for the latter. (6) is a finite dimensional perturbation of the Neumann problem.

There exists a solution u^* to (6), due to straightforward direct methods. (Unlike for (2), u^* need not be either unique or nonnegative.) It satisfies

$$\int_\Omega \nabla u^* \cdot \nabla \phi = \vartheta \int_\Omega u^* \phi + \sum_i \mu_i \int_{D_i} \phi \, dS = 0 \quad \forall \phi \in H^1(\Omega) \tag{7}$$

with certain Lagrange multipliers ϑ, μ_i. As (6) without constraints on the boundary would yield the constant function, (μ_i) is not the null vector.

Expand $u^* \in L^2(\Omega)$ in terms of the ϕ_k; the coefficients follow from inserting ϕ_k into (7):

$$u^* = \sum_k \sum_i \mu_i \frac{\int_{D_i} \phi_k \, dS}{\nu_k - \vartheta} \phi_k =: \sum_k \hat{u}_k^* \phi_k \ . \tag{8}$$

A priori, this sum converges in the strong $L^2(\Omega)$ topology (not necessarily absolutely). Actually, the sum converges in the strong $H^1(\Omega)$ topology and we may take the gradient of the sum term by term. This follows from the orthogonality of the $\nabla\phi_k$ in $L^2(\Omega \to \mathbb{R}^n)$ and the fact that $\sum \nu_k (\hat{u}_k^*)^2 = \int |\nabla u^*|^2 < \infty$.

Using the continuous (in fact: compact) trace embedding $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$, we may evaluate the constraints $\int_{D_i} u^* \, dS = 0$ on the boundary term by term, and we get $\sum_j W_{ij}(\vartheta)\mu_j = 0$ with W_{ij} as in the following (9). Consequently ϑ must satisfy the condition thta the Weinstein determinant vanishes:

$$W(\vartheta) := \det W_{ij}(\vartheta) := \det \sum_k \frac{(\int_{D_i} \phi_k \, dS)(\int_{D_j} \phi_k \, dS)}{\nu_k - \vartheta} = 0 \ . \tag{9}$$

In particular, this series converges for all $\vartheta \in \mathbb{C} \setminus \{\nu_k\}$. Sufficiently high (depending on the dimension) termwise derivatives of the series converge absolutely (as $\nu_k \sim c\, k^{2/n}$ for $k \to \infty$), and locally uniformly so, thus representing an analytic function (except for simple poles at the ν_k). $W^{[r]}$ is therefore meromorphic. By the way, each of the possible poles of $W^{[r]}$ at ν_k cannot have a higher multiplicity than the multiplicity of ν_k as an eigenvalue. This is because the terms contributing the poles in $W_{ij}^{[r]}$ are all rank one matrices.

The smallest positive solution ϑ_1 to (9) will be a lower bound for the first eigenvalue of the mixed problem. Taking a sequence $r \to \infty$ and subsequent refinements of the decomposition $D = \bigcup_i D_i^{[r]}$, we get a sequence of minimizers $\overset{*}{u}{}^{[r]}$ (with corresponding minima $\vartheta_1^{[r]} \nearrow: \vartheta_1^{[\infty]}$ that converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ and $L^2(\partial\Omega)$ to some function $\overset{*}{u}{}^{[\infty]}$ satisfying each constraint involved in all but finitely many steps of the process, i.e., all the constraints. They enforce Dirichlet boundary conditions on all of D, provided we have arranged the refinements such that the mesh tends to 0. Using lower semicontinuity of $\int |\nabla u|^2$, we have

$$\lambda_1(D) \geq \vartheta^{[\infty]} = \lim_{r\to\infty} \vartheta^{[r]} = \lim_{r\to\infty} \int_\Omega |\nabla \overset{*}{u}{}^{[r]}|^2 \geq \lim_{r\to\infty} \int_\Omega |\nabla \overset{*}{u}{}^{[\infty]}|^2 \geq \lambda_1(D) \ .$$

So the lower bounds are *effective*, i.e., they approximate the eigenvalue of the mixed problem arbitrarily well.

We need to use only the case $r = 1$ ($D_1 = D$) of (6), and we state it for reference:

Theorem 1 *Let (ν_k, ϕ_k) be an orthonormal eigensystem (beginning with $\nu_0 = 0$) for the Neumann Laplacian in a bounded Lipschitz domain Ω. Let D be a measurable subset of $\partial\Omega$. Then the first zero ϑ_1 of*

$$W(\vartheta) := \sum_{k=0}^{\infty} \frac{(\int_D \phi_k \, dS)^2}{\nu_k - \vartheta} = -\frac{\mu\,(D)^2}{\vartheta \operatorname{vol}(\Omega)} + \int_0^\infty e^{\vartheta t} \int_D \int_D \tilde{E}(x, y; t)\, dS(x)\, dS(y)\, dt \qquad (10)$$

(which lies in $]0, \nu_1[$) is a lower bound for $\lambda_1(D)$. Here, $\tilde{E}(x, y; t) := E(x, y; t) - 1/\operatorname{vol}(\Omega)$ is the reduced heat kernel with Neumann boundary conditions in Ω, and the integral representation is valid for $\vartheta < \nu_1$ only.

PROOF:
$\lim_{\vartheta \searrow 0} W(\vartheta) = -\infty$ and $\lim_{\vartheta \nearrow \nu_1} W(\vartheta) = +\infty$, so clearly $\vartheta_1 \in\,]0, \nu_1[$. From $E(x, y; t) = \sum_k \phi_k(x)\phi_k(y)\exp(-\nu_k t)$, the integral representation follows immediately. Everything else has already been discussed above. ∎

Next, we make the bound from the theorem more explicit. From (10) it is clear that if $\tilde{E}$ is replaced by something bigger, then the first zero of the whole expression will decrease. So this is what we need to do in order to estimate ϑ_1 and thus $\lambda_1(D)$ from below. For this purpose, we will use

Lemma 2 *The Neumann heat kernel $E(x, y; t)$ for a Lipschitz domain $\Omega \subset \mathbb{R}^d$ satisfies*

$$E(x, y; t) \leq c_0(T, \Omega) t^{-d/2} \exp\left[-|x - y|^2/5t\right] \qquad \text{for} \quad 0 < t \leq T\,.$$

The constant $c_0 = c_0(T, \Omega)$ depends on T only through a bound for T, i.e., can be chosen uniform in T as long as T remains bounded.

Bounded Lipschitz domains Ω satisfy a cone condition, i.e. there exist $r > 0$ and $\alpha \in\,]0, \pi/2[$ such that for all $x_0 \in \partial\Omega$ there is a cone

$$C_{r,\alpha,v} := \left\{ x \in \mathbb{R}^d \;\middle|\; |x| \leq r\,,\ \langle x, v\rangle \geq (\cos\alpha)|x|\,|v| \right\} \setminus \{0\}$$

satisfying $x + C_{r,\alpha,v} \subset \Omega$.

The dependence of c_0 on Ω enters only through r and α.

PROOF:
The estimate is a version of a result by Davies [4, thm 3.2.9]. Going through his proof (thm. 2.4.4 → cor 2.4.3 → lemma 2.2.6), one sees that the dependence on Ω enters through the constant ins Sobolev's imbedding theorem

$$\|f\|_{2d/(d-2)}^2 \leq c(\|\nabla f\|_2^2 + \|f\|_2^2)\,,$$

which we need only for $d \geq 3$ even if $\Omega \subset \mathbb{R}^2$. "Lipschitz $\Longrightarrow$ cone condition" is easy and well–known. Consult lemma 5.10 of Adams [2] for the dependence of the Sobolev constant on Ω. ∎

We need to complement this estimate for small t with one for large t:

Lemma 3 *For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, the reduced Neumann heat kernel satisfies*

$$|\tilde{E}(x,y;t)| \le c_1(T,\Omega)\exp[-\nu_1 t] \quad \text{for} \quad t \ge T\ ,$$

where ν_1 is the first Neumann eigenvalue. c_1 can be taken decreasing in T, and its dependence on Ω enters only through a lower bound for ν_1 and the cone data r, α that already entered in lemma 2.

PROOF:
Start with thm. 2.4.4 from [4], i.e., lemma 2 of this work, and, using the maximum principle, get immediately

$$0 \le E(x,y;t) = \sum_{k=0}^{\infty} \phi_k(x)\phi_k(y)e^{-\nu_k t} \le \frac{c_0(T,\Omega)}{\min(t,T)^{d/2}}\ , \tag{11}$$

where c_0 depends only on the cone parameters r, α of Ω as in lemma 2. We may and do assume with no loss of generality that $c_0(T,\Omega)$ is increasing as a function of T. (Otherwise, replace $c_0(T)$ by $\sup_{0\le t\le T} c_0(t)$.) From this, as explained in [4], lemma 2.1.2 and ex. 2.1.8, we conclude that

$$\|\phi_k\|_\infty \le \frac{e^{\nu_k t} c_0(\tau)^{1/2}}{\min(\tau,t)^{d/4}}$$

for every t; choosing $t = \min\{\tau, d/4\nu_k\}$, we get

$$\|\phi_k\|_\infty \le g(\nu_k) := c_0(\tau)^{1/2} \begin{cases} (4e/d)^{d/4}\,\nu_k^{d/4} & \text{if } \nu_k \ge d/(4\tau) \\ e^{\nu_k\tau}\,\tau^{-d/4} & \text{if } \nu_k \le d/(4\tau) \end{cases} .$$

Therefore,

$$\left|\sum_{k=1}^{\infty} \phi_k(x)\phi_k(y)e^{-\nu_k t}\right| \le \sum_{k=1}^{\infty} \|\phi_k\|_\infty^2 e^{-\nu_k t} \le e^{-\nu_1 t}\sum_{k=1}^{\infty} g(\nu_k)^2 e^{-(\nu_k-\nu_1)t}\ . \tag{12}$$

In order to get lower estimates for the ν_k, we let $N(x) := \#\{k \mid \nu_k \le x\}$ and again conclude from the trace of (11):

$$\begin{aligned} \frac{c_0(\tau)}{\min(\tau,t)^{d/2}} &\ge \sum_{k=0}^{\infty} e^{-\nu_k t} = \int_0^\infty e^{-tx}\,dN(x) = \int_0^\infty te^{-tx}N(x)\,dx \ge \\ &\ge N(1/\tau)\int_{1/\tau}^\infty te^{-tx}\,dx = N(1/\tau)e^{-t/\tau}\ , \end{aligned}$$

and therefore with $t := \tau$, we obtain $N(x) \le e\,c_0(1/x)x^{d/2}$, which implies $N(x) \le e\,c_0(1/\nu_1)x^{d/2}$. Consequently,

$$\nu_k = \inf\{x \mid N(x) \ge k\} \ge \inf\left\{x \,\middle|\, e\,c_0(1/\nu_1)x^{d/2} \ge k\right\} = (e\,c_0(1/\nu_1)/k)^{-2/d}\ .$$

So, for $k \ge (2\nu_1)^{d/2}\,e\,c_0(1/\nu_1) =: c_2(\nu_1)$, we get $\nu_k - \nu_1 \ge \nu_k/2 \ge c_3(\nu_1)k^{2/d}$.

On the other hand, bounding the Neumann eigenvalues from above by the Dirichlet eigenvalues and using domain monotonicity for the latter, we get $\nu_k \leq c_4(r,\alpha)k^{2/d}$. With this, (12) implies

$$e^{\nu_1 t}\,|\tilde{E}(x,y;t)| \leq c_2\, g(c_4\, c_2^{2/d})^2 + \sum_{k\geq c_2} g(c_4 k^{2/d})^2 \,\exp\left[-c_3 k^{2/d} t\right] .$$

The lemma follows immediately. The choice of τ implicit in g is arbitrary and could be optimized in a more explicit context. ■

Finally a geometric lemma will be used:

Lemma 4 *If $K \subset L \subset \mathbb{R}^d$, K convex, L bounded, ∂L rectifiable, then $\mu\,(\partial K) \leq \mu\,(\partial L)$. In particular, if Ω is convex and $x \in \partial\Omega$, then $\mu\,(\partial\Omega \cap B_\ell(x)) \leq h_{d-1}\ell^{d-1}$, where h_{d-1} is the surface area of half a unit ball in $\mathbb{R}^d$.*

Proof:
Among several equivalent (thm. 3.2.26 of [7]) possibilities, we interpret $\mu\,(\partial K)$ as the $d-1$-dimensional Hausdorff measure (2.10.2 of [7]). The map $p : \mathbb{R}^d \to \bar{K}$, $p(x) = y :\Longleftrightarrow |y-x| = \mathrm{dist}(\bar{K},x)$ is well-defined and satisfies $|p(x_1)-p(x_2)| \leq |x_1-x_2|$. Moreover $p(\partial L) = \partial K$. With this, if $\{S_j\}$ covers ∂L, then $\{p(S_j)\}$ covers $p(\partial L) = \partial K$, and the lemma follows, because $\mathrm{diam}\, p(S_j) \leq \mathrm{diam}\, S_j$.

Even though the construction of the map p is standard, we sketch it to have things self-contained:

The $\inf\{|y-x| \mid y \in \bar{K}\}$ is taken on, because $\bar{K}$ is compact. The minimizer y is unique, because for minimizers y_1, y_2, we have $|\frac{y_1+y_2}{2} - x| < \frac{|y_1-x|}{2} + \frac{|y_2-x|}{2}$ unless $y_1 - x$ and $y_2 - x$ are parallel with the same (not opposite) direction. But in this case $|y_1 - x| = |y_2 - x|$ implies $y_1 = y_2$. To find preimages $p^{-1}(y)$ of $y \in \partial K$, take the normal to any supporting plane of $\bar{K}$ in y. Finally, from $|x - [p(x) + t(p(y)-p(x))]|^2 \geq |x-p(x)|^2$ (for $0 \leq t \leq 1$) we conclude $\langle x-p(x)|p(y)-p(x)\rangle \leq 0$ (and similarly with x and y exchanged). Therefore $|p(x)-p(y)|^2 - |x-y|^2 = 2\langle x-p(x)|p(y)-p(x)\rangle + 2\langle y-p(y)|p(x)-p(y)\rangle - |p(x)-p(y)-x+y|^2 \leq 0$. ■

We are ready to resume estimating ϑ_1 from theorem 1:

$$\begin{aligned}\frac{\mu\,(D)^2}{\vartheta\,\mathrm{vol}\,(\Omega)} &= \int_0^\infty e^{\vartheta t}\int_D\int_D \tilde{E}(x,y;t)\,dS(x)\,dS(y)\,dt \\ &\leq \int_0^T e^{\vartheta t}\int_D\int_D E(x,y;t)\,dS(x)\,dS(y)\,dt \;+\; \mu\,(D)^2\,c_1(T)\int_T^\infty e^{(\vartheta-\nu_1)t}\,dt\end{aligned} \tag{13}$$

$$
\begin{aligned}
\mu(D)^2 & \left[\frac{1}{\vartheta \operatorname{vol}(\Omega)} - \frac{c_1(T) e^{-(\nu_1-\vartheta)T}}{\nu_1 - \vartheta}\right] \leq \\
& \leq c_0(T) \int_0^T e^{\vartheta t} \int_D \int_D t^{-d/2} e^{-|x-y|^2/5t} \, dS(x) \, dS(y) \, dt \\
& \leq c_0(T) e^{\nu_1 T} \int_0^\infty t^{-d/2} \int_0^1 \mu\left(\left\{(x,y) \in D \times D \,\middle|\, e^{-|x-y|^2/5t} \geq h\right\}\right) dh \, dt \\
& = c_0(T) e^{\nu_1 T} \int_0^\infty t^{-d/2} \int_0^\infty \mu\left(\left\{(x,y) \in D \times D \,\middle|\, |x-y| \leq \ell\right\}\right) \frac{2\ell}{5t} e^{-\ell^2/5t} \, d\ell \, dt
\end{aligned}
\tag{14}
$$

But

$$
\begin{aligned}
\mu\left(\left\{(x,y) \in D \times D \,\middle|\, |x-y| \leq \ell\right\}\right) & = \int_D \mu\left(\left\{y \in D \,\middle|\, |x-y| \leq \ell\right\}\right) dS(x) \\
& = \int_D \mu\left(B_\ell(x) \cap D\right) dS(x) \\
& \leq \mu(D) \min\left\{\mu(D), \sup_x \mu\left(B_\ell(x) \cap \partial\Omega\right)\right\} \\
& \leq \mu(D) \, h_{d-1} \min(\ell, \ell_D)^{d-1} .
\end{aligned}
\tag{15}
$$

In this last inequality, ℓ_D is defined by $h_{d-1}\ell_D^{d-1} = \mu(D)$, and we have used lemma 4. Unless ϑ_1 happens to be $\geq \nu_1/2$ (in which case we already have a lower estimate), we may restrict to $\vartheta \leq \nu_1/2$, let T be large enough for $c_1(T)e^{-(\nu_1/2)T} \leq 1/(2\operatorname{vol}(\Omega))$; the latter is possible, because $c_1(T)$ remains bounded as $T \to \infty$. Then, inserting (15) into (14), we get

$$
\begin{aligned}
\frac{\mu(D)}{2\vartheta \operatorname{vol}(\Omega)} & \leq \frac{2}{5} c_0(T) e^{\nu_1 T} h_{d-1} \Big[\int_0^{\ell_D^2/5} t^{-d/2-1} \int_0^\infty \ell^d e^{-\ell^2/5t} \, d\ell \, dt \\
& \qquad + \int_{\ell_D^2/5}^\infty t^{-d/2-1} \ell_D^{d-1} \int_0^\infty \ell e^{-\ell^2/5t} \, d\ell \, dt \Big] \\
& \leq \frac{2}{5} c_0(T) e^{\nu_1 T} h_{d-1} c_d \ell_D = \tilde{c}_1(T,d) \mu(D)^{1/(d-1)}
\end{aligned}
\tag{16}
$$

for $d > 2$. So we have shown

Theorem 5 *Let $\Omega \subset \mathbb{R}^d$ be a bounded convex domain, $d \geq 3$. Then for any window $D \subset \partial\Omega$ of area $\mu(D) = A$, one has the estimate*

$$
\lambda_1(D) \geq \min\left\{ c_2(\Omega) \frac{A^{(d-2)/(d-1)}}{\operatorname{vol}(\Omega)}, \frac{\nu_1}{2} \right\} \qquad (d \geq 3) .
$$

In this estimate, $c_2(\Omega)$ depends only on a lower bound for the first Neumann eigenvalue ν_1 of Ω, and the data r, α in an interior cone condition for Ω.

The same result holds more generally for Lipschitz domains without the convexity assumption. In this case, however, $c_2(\Omega)$ will also depend on a constant h such that $\mu(\partial\Omega \cap B_\ell(x)) \leq h\ell^{d-1}$ for all balls $B_\ell(x)$ centered anywhere on $\partial\Omega$.

For small A, this universal lower bound has the same order of magnitude as the estimate given in the model problem (5).

The argument in the case $d = 2$ differs only insignificantly: in that case, the t-integral diverges as $t \to \infty$. However, we can omit the simplification from $\int^T \dots dt$ to $\int^\infty \dots dt$ already in (14). Then a formula similar to (5) arises from (16) also in the case $d = 2$.

An interesting modification is available in the special situation where $\Omega = B \times [0, L] \subset \mathbb{R}^{d-1} \times \mathbb{R}$ is a cylinder, $d \geq 2$, B a bounded Lipschitz domain, and $D \subset B \times \{0\} \subset \partial\Omega$. Then instead of using lemma 4, one can estimate in (13) as follows:

$$\begin{aligned}\int_D \int_D E(x, y; t)\, dS(x)\, dS(y) &\leq \frac{c}{t^{(d-1)/2}} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} e^{-|x-y|^2/4t} \chi_D(x) \chi_D(y)\, dx\, dy \\ &\leq \frac{c}{t^{(d-1)/2}} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} e^{-|x-y|^2/4t} \chi_{D^*}(x) \chi_{D^*}(y)\, dx\, dy\ ,\end{aligned}$$

where D^* is the Schwarz symmetrization of D (constructed in $\mathbb{R}^{d-1}$) and we have used Riesz's inequality ([14],10.14 in one dimension, [3] thm. 1.2 for arbitrary dimension with even more generality). This reasoning makes visible at least a loose connection between round windows and small heat leak rates. However, this concerns a calculation for the simplest version ($r = 1$) of the intermediate variational problem (6) only, and it does not carry over to better approximants ($r > 1$) or even the original problem (2), because the monotonic relation between estimates of $\tilde{E}$ and estimates of the zero $\vartheta_1(D)$ is not preserved in these cases. Moreover no symmetrization is available on general $\partial\Omega$. So this track of argument does not appear to lead further.

We conclude this section with a proof that there exist optimal windows for the modified problem (6), i.e.

Theorem 6

$$\check{\ell}(A) := \inf\{\vartheta_1(D) \mid \mu\,(D) = A\} \tag{17}$$

is taken on.

Although this kind of result about an auxiliary problem may be of minor importance for its own sake, its merits lie in the comparison with the original problem (2). A similar result holds there [5], and its proof seems to suggest that some kind of convexity is hidden in the problem. We do have an obvious convexity argument here. On the other hand, it is interesting to remark that the supremum of $\vartheta_1(D)$ is *not* taken on under the area constraint, and neither is $\sup \lambda_1(D)$. See the next section.

PROOF:
In order to show that the infimum in (17) is taken on, i.e., that there exists a D with $\vartheta_1(D) = \check{\ell}(A)$ and $\mu\,(D) = A$, we need to find a solution χ_D of

$$\frac{A^2}{\vartheta \operatorname{vol}(\Omega)} = \sum_{k=1}^{\infty} \frac{\left(\int_{\partial\Omega} \chi_D \phi_k\right)^2}{\nu_k - \vartheta} \tag{18}$$

for $\vartheta := \check{\ell}(A)$, knowing that for $\vartheta > \check{\ell}(A)$, there exists such a solution $\chi_{D(\vartheta)}$, whereas for $\vartheta < \check{\ell}(A)$, a $>$ sign will replace the $=$ sign in (18) in any case. For this purpose we try to maximize the right hand side of (18) as a function of D, with $\vartheta := \check{\ell}(A)$ fixed. If we have such a maximizer $\hat{D}$, then for $\vartheta > \check{\ell}(A)$ we have

$$\frac{A^2}{\check{\ell}(A)\,\mathrm{vol}\,(\Omega)} \geq \sum_{k=1}^{\infty} \frac{\left(\int_{\partial\Omega} \chi_{\hat{D}} \phi_k\right)^2}{\nu_k - \check{\ell}(A)} \geq \sum_{k=1}^{\infty} \frac{\left(\int_{\partial\Omega} \chi_{D(\vartheta)} \phi_k\right)^2}{\nu_k - \check{\ell}(A)} \geq$$
$$\geq \sum_{k=1}^{\infty} \frac{\left(\int_{\partial\Omega} \chi_{D(\vartheta)} \phi_k\right)^2}{\nu_k - \vartheta} = \frac{A^2}{\vartheta\,\mathrm{vol}\,(\Omega)} \longrightarrow \frac{A^2}{\check{\ell}(A)\,\mathrm{vol}\,(\Omega)} \;.$$

So $\hat{D}$ is a solution to (18). We are left with establishing the existence of such a $\hat{D}$ for given ϑ.

Now $J : \varphi \mapsto \sum_{k=1}^{\infty} \frac{(\int \varphi\phi_k)^2}{\nu_k - \vartheta}$ is a positive semidefinite continuous quadratic form on $L^2(\partial\Omega)$, bounded on the closed convex set $K := \{\varphi \subset L^\infty(\partial\Omega) \mid 0 \leq \varphi \leq 1\,,\ \int \varphi = A\} \subset L^2(\partial\Omega)$. From a maximizing sequence φ_i, we can select a subsequence such that $\varphi_i \rightharpoonup: \varphi$ in $L^2(\partial\Omega)$ and $\varphi_i \otimes \varphi_i \rightharpoonup: \psi$ in $L^2(\partial\Omega \times \partial\Omega)$, and then $\psi = \varphi \otimes \varphi$, because in the dual L^2-spaces, the (algebraic) tensor product $L^2(\partial\Omega) \otimes L^2(\partial\Omega)$ is dense in $L^2(\partial\Omega \times \partial\Omega)$. Using lemma 3 for $\tilde{E}$ and Lebesgue's convergence theorem, we see that J is continuous under this limit. So we get a maximizer in K.

Actually, the quadratic form J is positive definite on $\{\varphi \in L^2(\partial\Omega) \mid \int \varphi = 0\}$, hence *strictly* convex on K, and therefore maximizers can only occur in extremal points of K, namely in characteristic functions. To see this positive definiteness, we only need to show that $\int_{\partial\Omega} \varphi\phi_k = 0$ $(\forall k \geq 0)$ implies $\varphi = 0$. This is the content of the following lemma, which is interesting for its own sake:

Lemma 7 *The traces $\phi_k \in L^2(\partial\Omega)$ of an orthonormal basis $\phi_k \in L^2(\Omega)$ of eigenfunctions of the Neumann Laplacian on a bounded Lipschitz domain Ω span a dense subset of $L^2(\partial\Omega)$.*

PROOF:
Let $\int_{\partial\Omega} f = 0$ and $\int_{\partial\Omega} f\phi_k = 0$ for $k \geq 1$. Let g be a harmonic function in Ω satisfying the boundary conditions $\partial_n g = f$, i.e., a solution to the variational problem

$$\min \left\{ \frac{1}{2} \int_\Omega |\nabla g|^2 - \int_{\partial\Omega} fg \;\Big|\; g \in H^1(\Omega) \right\} \;. \tag{19}$$

This function g (whose existence is guaranteed by direct methods) satisfies the Euler-Lagrange equation $\int_\Omega \nabla g \nabla \phi = \int_{\partial\Omega} f\phi$ for all $\phi \in H^1(\Omega)$, in particular for the ϕ_k. The ϕ_k satisfy $\int_\Omega \nabla\phi_k \nabla u = \nu_k \int_\Omega \phi_k u$ for all $u \in H^1(\Omega)$, in particular for $u = g$. We conclude that $\int_\Omega \phi_k g = \nu_k^{-1} \int_{\partial\Omega} \phi_k f = 0$, i.e., g is a constant. Therefore the minimum (19) equals 0 and we get

$$\int_{\partial\Omega} fh \leq \frac{1}{2} \int_\Omega |\nabla h|^2 \quad \forall h \in H^1(\Omega) \;.$$

But this implies $\int_{\partial\Omega} fh = 0$ for all $h \in \mathrm{image}(H^1(\Omega) \hookrightarrow L^2(\partial\Omega))$, and this latter image is dense in $L^2(\partial\Omega)$. Hence $f = 0$. ∎

3 The Supremum of the Leak Rate

We show the

Theorem 8 *Let Ω be a bounded Lipschitz domain. For every $A \in]0, \mu(\partial\Omega)[$,*

$$\sup\{\lambda_1(D) \mid \mu(D) = A\} = \lambda_1(\partial\Omega) .$$

A maximizing sequence of windows D_n can be given in such a way that the sequence of the characteristic functions $\chi_n := \chi_{D_n}$ converges weakly in $L^2(\partial\Omega)$ to a constant.

PROOF:
Take any sequence D_n as specified in the theorem. We first explain why such sequences exist, then that they have the claimed properties.

Decompose $\partial\Omega$ into finitely many pieces S_j of diameter $< 1/n$, then select a portion of relative measure $A/\mu(\partial\Omega)$ from each of the pieces. Such portions exist according to Lyapunov's convexity theorem [13], because the measure on $\partial\Omega$ is free of atoms. Let D_n consist of the selected portions of each piece. For any $g \in C^0(\partial\Omega)$, one has $\int \chi_n g \to \int (A/\mu(\partial\Omega))g$ (using uniform continuity and approximating g by a step function constant on each of the S_j). The same holds for $g \in L^2(\partial\Omega)$, because $C^0(\partial\Omega)$ is dense in $L^2(\partial\Omega)$.

Let $u_n \geq 0$ be a minimizer for (2) with D_n, the minimum being $\lambda_1(D_n)$.

Obviously $\lambda_1(D_n) \leq \lambda_1(\partial\Omega)$ by comparison of the admissible sets of functions, so we get a uniform H^1 bound for u_n. By Rellich's imbedding theorem and the compact trace imbedding, we may pass to a subsequence and thus ensure that $u_n \to: u^* \geq 0$ strongly in both $L^2(\Omega)$ and $L^2(\partial\Omega)$.

u^* is an admissible function for the variational problem for $\lambda_1(\partial\Omega)$. This follows by writing the boundary condition as $(1-\chi_n)u_n = 0$, an equation of functions in $L^2(\partial\Omega)$, and passing to the limit: $\int_{\partial\Omega}(1 - A/\mu(\partial\Omega))u^* = 0$. As the term under the integral doesn't change sign, we get $(1 - A/\mu(\partial\Omega))u^* = 0$, hence $u^* = 0$ on the boundary. Now lower semi-continuity ensures that

$$\lambda_1(\partial\Omega) \geq \liminf_{\chi_n \rightharpoonup A/\mu(\partial\Omega)} \lambda_1(D_n) \geq \int_\Omega |\nabla u^*|^2 \, dx \geq \lambda_1(\partial\Omega) .$$

■

A similar result can be proved for the intermediate variational problems considered in section 2. The weak limit can immediately be carried out in (9), and the factor $(\mu(D)/\mu(\partial\Omega))^r$ cancels from the equation. Using an appropriate sequence of intermediate variational problems in order to approach $\lambda_1(D)$ from below, one can get another proof of theorem 8. Actually this was the author's first proof.

A heuristic argument to explain the theorem would be that the vanishing of the minimizers u_n in subsets of the boundary which in the limit become dense there must imply vanishing of the limit everywhere. However, there cannot be regularity theorems giving continuity estimates up to the boundary *independent* of the geometry of the vanishing set D. Otherwise, one could prove that (weak) Dirichlet conditions in a single point (seen as the limit of an interval) on the boundary are obeyed classically, which is not true.

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