WINDOWS OF GIVEN AREA WITH MINIMAL HEAT DIFFUSION

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Abstract. For a bounded Lipschitz domain Ω, we show the existence of a measurable set $D \subset \partial\Omega$ of given area such that the first eigenvalue of the Laplacian with Dirichlet conditions on D and Neumann conditions on $\partial\Omega \setminus D$ becomes minimal. If Ω is a ball, D will be a spherical cap.

0. Introduction

We are considering the lowest eigenvalue λ_1 of the Laplacian in a bounded Lipschitz domain Ω. The boundary conditions are Dirichlet on one part D of $\partial\Omega$ and Neumann elsewhere:

$$\lambda_1(D) := \inf\left\{ \int_\Omega |\nabla u|^2\,dx \;\middle|\; u \in H^1(\Omega)\,,\ \int_\Omega u^2 = 1\,,\ u|_D = 0 \right\}\,, \tag{0.1}$$

where H^1 denotes the Sobolev space of functions with square integrable first derivatives.

This eigenvalue can be interpreted as the heat diffusion rate through a non-insulated window in an otherwise perfectly insulated room, asymptotically for large time; an interpretation which should not be taken too literally, because no convection is taken into account. However, considering λ_1 as a function of the "window" D is the source for a lot of intriguing geometric inverse spectral problems of a type that has apparently not been considered so far. One key problem is to minimize λ_1 under the constraint of fixed area $\mu(D)$ of D. It is the main purpose of this paper to show that such a minimum (we call it optimal window) actually exists.

The mixed nature of the boundary data is that severe a discontinuity from the elliptic regularity point of view that smoothness of $\partial\Omega$ would be an assumption quite alien to the nature of the problem. Bearing with this opinion makes it necessary to employ recent tools on elliptic problems with minimal regularity as described in [8]. While the author cannot pretend to contribute to this theory, he hopes that this application might contribute to popularizing the results of the theory. The last section contains the proof, kindly supplied by C. Kenig, for the theorem that is needed to carry the existence proof over to nonsmooth boundary. The responsibility for any insufficiency in the exposition of this proof remains with the author. For smooth boundary, the existence proof can also be carried through with more classical tools.

Whereas the shape of optimal windows is not exhibited by our existence proof and leaves open many questions, symmetrization arguments give a complete solution when Ω is a ball. Some heuristic discussion of the shape question as well as quantitative lower bounds for the eigenvalue have been given by the author in [4].

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Moreover, it has been shown there that it is not possible to maximize λ_1 under the area constraint.

1. Existence of optimal windows

In view of (0.1), to find a minimizer for the variational problem

$$\ell(A) := \inf\left\{\lambda_1(D)\;\Big|\;\mu(D) = A\right\} \tag{1.1}$$

is equivalent to find a minimizer of

$$\ell(A) = \inf\left\{I[u,\chi] := \int_\Omega |\nabla u|^2 \;\Big|\; (u,\chi)\in\mathcal{D}(\Omega,A)\right\}\ , \tag{1.2}$$

where

$$\begin{aligned}\mathcal{D}(\Omega,A) := \Big\{ &(u,\chi)\in H^1(\Omega)\times L^\infty(\partial\Omega)\;\Big|\\ &\int_\Omega u^2 = 1\,,\ \chi^2=\chi\,,\ \int_{\partial\Omega}\chi = A\,,\ u\cdot\chi = 0\in L^2(\partial\Omega)\Big\}\end{aligned} \tag{1.3}$$

and the u in $u\cdot\chi$ denotes the trace of $u\in H^1(\Omega)$ on the boundary $\partial\Omega$. (According to [1, 5.22], the restriction map $C^0(\bar\Omega)\to C^0(\partial\Omega)$ extends to a map in particular defined from $H^1(\Omega)$ to $L^2(\partial\Omega)$ and called the trace map.) Obviously, $\chi^2=\chi$ amounts to χ being a characteristic function of some set D; as D is varying, we prefer to include the spurious dependence on $\chi=\chi_D$ into I. We first minimize the same functional in the larger class

$$\begin{aligned}\mathcal{E}(\Omega,A) := \Big\{ &(u,\phi)\in H^1(\Omega)\times L^\infty(\partial\Omega)\;\Big|\\ &\int_\Omega u^2 = 1\,,\ \phi\le 1\,,\ \int_{\partial\Omega}\phi \ge A\,,\ u\cdot\phi = 0\in L^2(\partial\Omega)\Big\}\ ,\end{aligned} \tag{1.4}$$

which has more convenient functional analytic properties, and show

Theorem 1. *For a bounded Lipschitz domain Ω and $A<\mu(\partial\Omega)$, there exists $(u,\phi)\in\mathcal{D}(\Omega,A)$ minimizing $I[u,\phi]$ in $\mathcal{E}(\Omega,A)$. Moreover, all minimizers of I in $\mathcal{E}(\Omega,A)$ actually lie in $\mathcal{D}(\Omega,A)$.*

To prove the theorem, let $\{(u_n,\phi_n)\}$ be a minimizing sequence, let $v_n := |u_n|$ and let $\chi_n := \chi_{\mathrm{supp}\phi_n}$ be the characteristic function of the support[1] of ϕ_n. As $\phi_n\le 1$, we have $\chi_n\ge\phi_n$, and (v_n,χ_n) is also a minimizing sequence. By Hilbert space theory, Rellich's imbedding theorem, the compact trace map $H^1(\Omega)\to L^2(\partial\Omega)$ (see e.g. [1, thm. 5.4]) and passing to a subsequence, we get the following convergence results:

$$v_n \rightharpoonup: v\ (\text{w-}H^1(\Omega)),\quad v_n\to v\ (\text{s-}L^2(\Omega)\text{ and s-}L^2(\partial\Omega)),\quad \chi_n\rightharpoonup\phi\ (\text{w-}L^2(\partial\Omega)).$$

We have the usual lower semi–continuity of the functional I, and under the above limits we salvage the following constraints: $0\le\phi\le1$, $\int_{\partial\Omega}\phi\ge A$, $v\ge0$, $\int_\Omega v^2=1$, and $\int_{\partial\Omega} v\cdot\phi = 0$. The latter implies $v\cdot\phi=0$ since $v,\phi\ge0$. So we have found a minimizer $(v,\phi)\in\mathcal{E}(\Omega,A)$.

Letting $\chi:=\chi_{\mathrm{supp}\phi}\ge\phi$, we get another minimizer (v,χ), and it holds $v\ge0$ and $\chi^2=\chi$ for this one. We claim that actually $\int_{\partial\Omega}\chi = A$. This will already tell us that the particular minimizer (v,χ) lies indeed in the smaller class $\mathcal{D}(\Omega,A)$.

[1] Here, the support of any $\phi\in L^1(\partial\Omega)$ is the set of all points, where ϕ doesn't vanish. It is only defined modulo null sets, which is enough to make $\chi_{\mathrm{supp}\phi}$ a well–defined element of $L^\infty(\partial\Omega)$. No operation of closure is involved.

As

$$A_1 \le A_2 \Longrightarrow \{u \mid \exists \chi : (u,\chi) \in \mathcal{D}(\Omega, A_1)\} \supset \{u \mid \exists \chi : (u,\chi) \in \mathcal{D}(\Omega, A_2)\} \ ,$$

it is clear that $A \mapsto \ell(A)$ is monotonic nondecreasing. So we can modify (1.1) to

$$\ell(A) = \inf\left\{\lambda_1(D) \;\middle|\; \mu(D) \ge A\right\} = \min\left\{\lambda_1(D) \;\middle|\; \mu(D) \ge A\right\} \ . \tag{1.5}$$

Our claim $\int_{\partial\Omega} \chi = A$ follows immediately from showing that ℓ is actually strictly monotonic.

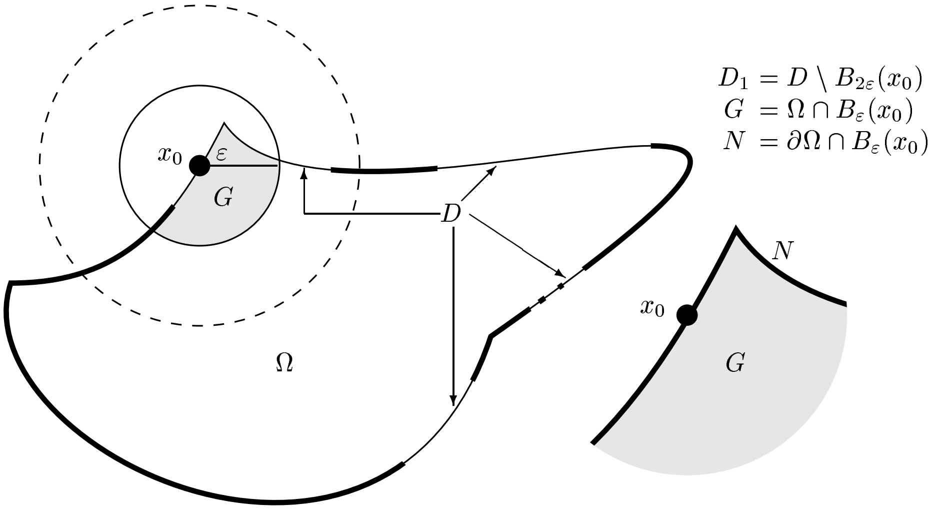

FIGURE 1. *Geometric objects employed in the proof of strict monotonicity of $\ell(A)$*

So suppose that we have a minimizer (u,χ) for (1.5) with $\mu(\mathrm{supp}\chi) = \int_{\partial\Omega} \chi = A_2$ and $D := \mathrm{supp}\chi$. Let $A_0 < A_2$ and choose $\varepsilon > 0$ small enough such that any ball $B_{2\varepsilon}(x_0)$ of radius 2ε centered in some $x_0 \in \partial\Omega$ intersects $\partial\Omega$ in a set of measure (on the boundary) less than $A_2 - A_0$, and also small enough such that every $B_{2\varepsilon}(x_0)$ lies entirely in one of some finite set of coordinate charts covering the boundary subject to the condition that in each chart, the domain can be described as the epigraph of a Lipschitz function. Choose x_0 such that $B_\varepsilon(x_0)$ intersects D in a set of positive measure. Let $D_1 := D \setminus (B_{2\varepsilon}(x_0) \cap \partial\Omega)$, and let its area be A_1. Then, $A_0 < A_1 < A_2$. Let u_1 be a minimizer for (0.1) with the set D_1. This minimizer will be (Hölder) continuous up to the part $B_\varepsilon(x_0) \cap \partial\Omega$ of the boundary: [10], sections 3.13-14, [17]. (We repeat some details of that argument in section 3.)

We will argue that u_1 cannot vanish anywhere on $B_\varepsilon(x_0) \cap \partial\Omega$ (or at least cannot vanish in a set of positive measure there). In contrast, u does vanish in a set of positive measure in that part of the boundary (namely in D). Hence u is *not* equal to the *unique* nonnegative minimizer u_1 for (0.1) with D_1, and therefore

$$\ell(A_0) \le \ell(A_1) \le \lambda_1(D_1) = \int |\nabla u_1|^2 < \int |\nabla u|^2 = \ell(A_2) \ .$$

This finishes the strict monotonicity of ℓ, and thus the existence of a minimizer in $\mathcal{D}(\Omega, A)$, subject to the claim that $u_1 \neq 0$ in $B_\varepsilon(x_0) \cap \partial\Omega$.

We first establish this latter claim in the case of sufficiently smooth boundary (say $C^{2,\alpha}$ according to [2]). In this case, u_1 satisfies classical homogeneous Neumann conditions. But Hopf's boundary point lemma [6, lemma 3.4], which needs to assume an interior sphere condition (which we have anyway for $C^{2,\alpha}$ boundary), would imply a strictly negative outer normal derivative in any boundary point where u_1 vanishes.

If we actually need to deal with Lipschitz boundary, we cannot use the Hopf boundary point lemma, and so the above reasoning should be redone in a weak setting. The necessary tools are available and due to C. Kenig:

Theorem 2 (C. Kenig)**.** *Suppose $\Delta u \leq 0$, $u \geq 0$ in a bounded Lipschitz domain Ω in $\mathbb{R}^d$. Then, if there exists a subset Σ of $\partial\Omega$ of positive measure on which both u and its normal derivative vanish (in the sense of nontangential limits), then $u \equiv 0$ in Ω.*

This theorem, applied to u_1 on $G := B_\varepsilon(x_0) \cap \Omega$ instead of Ω, can replace the argument with boundary regularity and Hopf's boundary point principle in the above argument, therefore in the case of Lipschitz boundary, we still conclude that $u \neq u_1$.

In section 3, we discuss why and in what sense $\partial_n u_1$ vanishes on $N := \partial\Omega \cap B_\varepsilon(x_0)$. Then we supply Kenig's proof for theorem 2. This will conclude the proof of strict monotonicity of ℓ in the Lipschitz case and therefore the existence of a minimum in $\mathcal{D}(\Omega, A)$.

We still need to show that *any* minimizer $(u, \phi) \in \mathcal{E}(\Omega, A)$ is actually in $\mathcal{D}(\Omega, A)$. But the same argument as before can be repeated. We get $u \geq 0$ (or $u \leq 0$) automatically instead of enforcing it, because the (other) minimizer $|u|$ cannot vanish inside Ω due to the maximum principle. The inequalities

$$\chi := \chi_{\mathrm{supp}\,\phi} \geq \phi \quad \text{and} \quad A = \int \chi \geq \int \phi \geq A$$

show that $\chi = \phi$. Thus every minimizer in $\mathcal{E}(\Omega, A)$ lies in $\mathcal{D}(\Omega, A)$. ■

The theorem is proved, but another consequence for the minimizing sequence (u_n, χ_n) constructed in the proof is worth noting: We have just seen that $\phi :=$ w-lim χ_n is actually a characteristic function itself. Now invoke the simple

Lemma 1. *Take any $1 \leq p < \infty$. Assume a sequence of characteristic functions χ_n on a finite measure space X to converge weakly in $L^2(X)$ to some function ϕ. Then ϕ is a characteristic function if and only if $\chi_n \to \phi$ strongly in $L^p(X)$.*

So we actually get L^2-strong convergence $\chi_n \to \chi$ for the minimizing sequence.

The proof of the lemma is as follows: If $\phi^2 = \phi$, we get

$$\begin{aligned} \|(\chi_n - 1)\phi\|_{L^1} &= \textstyle\int \phi(1 - \chi_n) = \int \phi(\phi - \chi_n) \to 0 \\ \|\chi_n(1 - \phi)\|_{L^1} &= \textstyle\int \chi_n(1 - \phi) = \int (\chi_n - \phi)(1 - \phi) \to 0 \end{aligned}$$

and therefore $\chi_n - \phi = \chi_n(1-\phi) + (\chi_n - 1)\phi \to 0$ strongly in L^1. Using $\|\chi_n - \phi\|_{L^\infty} \leq 1$ and the Hölder inequality, the same conclusion follows for L^p-convergence.

On the other hand, if $\chi_n \to \phi$ strongly in L^2, we conclude $\int(\phi - \phi^2) = 0$, and with $\phi - \phi^2 \geq 0$, this implies ϕ is a characteristic function. ■

It is a noteworthy feature of theorem 1 that, contrary to common practice, D is not assumed to be closed, not even $\bar{\mathring{D}} = \mathring{D}$. Stampacchia ([17], thm 4.1) has shown minimizers (critical points) u to be continuous up to the boundary under reasonably weak assumptions on the fixed closed set D. Whenever such a continuity result holds, the set $\{u = 0\}$ will be closed automatically. However, to the author's knowledge, continuity up to the boundary has not been proved anywhere under the general assumptions on D made here, most likely it will not even be true. It seems a reasonable conjecture, however, that optimal windows have a nice geometry and eigenfunctions continuous up to the boundary.

The strict monotonicity of ℓ prevents minimizing sequences from smearing out (i.e, converging weakly without converging strongly). Conversely, maximizing sequences typically exhibit weak convergence that is not strong. This behaviour is responsible for the fact that $\sup\{\lambda_1(D) \mid \mu(D) = A\} = \lambda_1(\partial\Omega)$, an easy proof of which has been given in [4]. Indeed, it was the sudden increase in effective window area for sequences D_n exhibiting the smearing out phenomenon, which motivated the key idea of the proof, namely that broadening the class $\mathcal{D}(\Omega, A)$ to $\mathcal{E}(\Omega, A)$ should not change the problem.

Another observation is that minimization over the intermediate set with the constraints $1 \geq \phi \geq 0$, $\int_{\partial\Omega} \phi = A$ would have given the same result again. The characteristic functions χ are the extremal points of that subset of $L^\infty(\partial\Omega)$, so theorem 1 is of the type "minima are taken on only on extremal points of the domain of definition". This suggests that some yet undiscovered convexity structure may be behind the problem. It should be pointed out that a convexity argument does enter into the existence theorem for minimizers of the intermediate variational problem used in [4] to establish quantitative lower bounds for $\lambda_1(D)$.

2. The Optimal Window in a Ball

No results on how the (an) optimal window will look has been given in section 1. Heuristic arguments (as given in [4]) suggest that it should look "round" rather than like a slit and probably be connected. In the special case of a ball, this can be made rigorous. The argument supplies an independent existence proof along with the explicit shape of the optimal window.

Theorem 3. *Let Ω be a d-dimensional (unit) ball and $A < \mu(\partial\Omega)$. Then D is an optimal window, i.e. $\lambda_1(D) = \ell(\mu(D))$, if and only if D is a spherical cap (up to a null set).*

The proof is by spherical symmetrization: Given a measurable set $K \subset \mathbb{R}^d$, the distinguished origin $0 \in \mathbb{R}^d$, and a distinguished direction (defining the north pole on all spheres centered at 0), the spherical symmetrization K^* of K is constructed as follows: For each r, take $K \cap \partial B_r(0)$ and replace it by the spherical cap of the same area and centered at the north pole of $\partial B_r(0)$. This can be done for ae. r. The union of these caps is K^*. The spherical symmetrization u^* of a measurable function $u \geq 0$ is constructed by symmetrizing the superlevel sets: $\forall t : \{u \geq t\}^* = \{u^* \geq t\}$. See e.g. [16] for more details.

Symmetrization arguments have first been used by Steiner and Schwarz in the isoperimetric problem and then by Faber and Krahn in order to show that for given area the minimal first Dirichlet eigenvalue is taken on for a disk. The classical reference for these types of results is [11].

Similar as for Steiner and Schwarz symmetrization, one has the following

Theorem 4. *For $u \in C^\infty(B_r(0))$, $u \geq 0$, it holds*

$$\int_{B_r(0)} |\nabla u^*|^2 \leq \int_{B_r(0)} |\nabla u|^2 \quad , \qquad \int_{B_r(0)} |u^*|^2 = \int_{B_r(0)} |u|^2 \ .$$

For equality to hold in the gradient estimate, it is necessary that on each sphere $S_r(0)$, u^* and u coincide up to a rotation (which may depend on r).

The estimate, actually in more generality, is due to Sperner [16]. He does not discuss equality, and we give a proof of that part below. It is definitely not true that $\int_{B_r(0)} |\nabla u^*|^2 = \int_{B_r(0)} |\nabla u|^2$ would imply $u \circ R = u^*$ for some rotation R: e.g, if u happens to be constant on some sphere $S_r(0)$, a counterexample can be given by taking different rotations inside and outside that sphere.

Sperner uses this theorem for a different variational problem, where the symmetrization changes (and optimizes) Ω. However, in our setting it is crucial to start with a ball Ω so that the symmetrization only changes the window, but not the room Ω.

Choosing the lowest eigenfunction $u \geq 0$ for any given window D, u^* will be a candidate for (0.1) with window D^*, a spherical cap, and hence $\lambda_1(D^*) \leq \lambda_1(D)$. Hence spherical caps are optimal windows.

We now discuss the case of equality in order to show that only spherical caps can be optimal windows. Let $n := x/|x|$ be the radial unit vector and decompose $\nabla u = n\langle n, \nabla u\rangle + \nabla_t u$. It is an immediate corollary (functions independent of the radial variable) of theorem 4 or else can be shown independently by more or less the same method [15] that for functions u on a *sphere* $S_r(0)$, it holds

$$\int_{S_r(0)} |\nabla_t u^*|^2 \leq \int_{S_r(0)} |\nabla_t u|^2 \ . \tag{2.1}$$

On the other hand, we will argue that

$$\int_{S_r(0)} \langle n, \nabla u^*\rangle^2 \leq \int_{S_r(0)} \langle n, \nabla u\rangle^2 \ . \tag{2.2}$$

(Clearly, (2.2), unlike (2.1), depends on the values of u in a neighbourhood of the sphere.) If equality holds in theorem 4, it holds in both (2.1) and (2.2) for almost every r. Going through the proof of (2.1) in [15], and denoting by σ the $(d-2)$–dimensional Hausdorff measure on the $(d-1)$–sphere, $\int_{S_r(0)} |\nabla_t u^*|^2 = \int_{S_r(0)} |\nabla_t u|^2$ implies

$$\int_{S_r(0)} |\nabla_t u^*| = \int_0^\infty \sigma((u^*|_{S_r(0)})^{-1}\{z\})\, dz = \int_0^\infty \sigma((u|_{S_r(0)})^{-1}\{z\})\, dz = \int_{S_r(0)} |\nabla_t u| \tag{2.3}$$

and

$$|\nabla_t u(x)| = f(u|_{S_r(0)}(x)) \tag{2.4}$$

for some function f which may depend on r, whereas

$$\begin{aligned} \sigma((u^*|_{S_r(0)})^{-1}\{z\}) &= \sigma(\partial(u^*|_{S_r(0)})^{-1}[z,\infty[) \leq \\ &\leq \sigma(\partial(u|_{S_r(0)})^{-1}[z,\infty[) = \sigma((u|_{S_r(0)})^{-1}\{z\}) \end{aligned} \tag{2.5}$$

for a.e. z. But the middle inequality of (2.5) is just the isoperimetric inequality on the $(d-1)$–dimensional sphere, and it must become an equality in the present situation. So, $(u|_{S_r(0)})^{-1}[z,\infty[$ must be a disk (i.e. spherical cap) first for a.e. and then for every z. The sharp isoperimetric inequality on the sphere is discussed in [13, II, p. 231]. Then (2.4) implies the caps are concentric, i.e., $u|_{S_r(0)} = u^*|_{S_r(0)} \circ R_r$ for some rotations R_r.

We still need to justify (2.2): Expressing the derivative as a limit of difference quotients, this is an immediate consequence of $\int_{S_r(0)} u^*v^* \geq \int_{S_r(0)} uv$. This latter result can actually be seen as a result about Schwarz symmetrization in $\mathbb{R}^{d-1}$, using a *measure* preserving variant of the stereographic projection: the radial variable ρ in $\mathbb{R}^{d-1}$ relates to the polar distance ϑ on the sphere by $\rho^{d-2}\,d\rho = (r\sin\vartheta)^{d-2}\,d\vartheta$. It is a simple special case of the higher dimensional version of Riesz's inequality [12] $\int\int u^*(x)v^*(y)k^*(x-y)\,dx\,dy \geq \int\int u(x)v(y)k(x-y)\,dx\,dy$. Let $k = k^*$ converge to a Dirac distribution for that purpose. The higher dimensional version was introduced by Sobolev [14], but without discussion of the convergence of an infinite sequence of symmetrizations used in the inductive step along the dimension. A complete and concise proof of an even more general result can be found in [3].

This finishes the proof of theorem 4. ∎

We now finish the uniqueness part of theorem 3. Let $u \geq 0$ be the eigenfunction for an optimal window D. For this u, equality holds in theorem 4. u satisfies $-\Delta u = \lambda u$ in $B_1(0)$ and is real analytic there. The restriction u_r of u to the sphere $S_r(0)$ of radius r will be a monotonic nonincreasing function of the distance (on the sphere) from some x_r alone, according to theorem 4. (One can conclude strict monotonicity from analyticity and the fact that the Dirichlet problem for $\Delta + \lambda$ is uniquely solvable in $B_r(0)$, but we do not need this.) Now, as $r \to 1$, u_r will converge to the trace of u on the boundary in the strong L^2 sense (by compactness of the trace map). On the other hand, by compactness, x_r will converge to some x_1 as $r \to 1$ on an appropriate subsequence. By Helly's theorem on monotonic functions, the u_r will converge a.e. to some function u_1, which is a nonincreasing function of the distance from x_1 alone. The L^2 and *a.e.* limits must coincide, therefore the trace of u has the mentioned monotonicity property. So it vanishes exactly on some spherical cap of $\partial B_1(0)$ whose measure equals $\mu(D^*)$. D is a subset of this cap, but $\mu(D) = \mu(D^*)$. Therefore $D = D^*$ up to rotation. ∎

The result as well as the method of proof suggest some relation between the shape of optimal windows and isoperimetrical problems. However, it is not clear yet how such a connection could be formulated for general rooms in detail.

Also note that for any set D that is sandwiched between spherical caps C_- and C_+ of radii $r-\varepsilon$ and $r+\varepsilon$ respectively, it will be true that $\lambda_1(C_-) \leq \lambda_1(D) \leq \lambda_1(C_+)$, in particular, $\lambda_1(D)$ will be arbitrarily close to $\lambda_1(C) = \ell(\mathrm{vol}(C))$ (with C the cap of radius r), provided ε is small enough. But such sets D with $\mu(D) = \mu(C)$ exist with arbitrarily large perimeter. In other words, nearly optimal windows from the heat leak point of view need not be nearly optimal in the isoperimetric sense.

Similarly, given any $x_0 \in \partial\Omega$ and any δ, one can find an extra window of sufficiently small but positive area and located near x_0 that doesn't increase λ_1 by more than δ. In particular, also the diameter of nearly optimal windows cannot be estimated in any nontrivial way.

Therefore any possible result along these lines cannot be a result on quasi–minimizers, but must be a result on minimizers, and some version of Euler equations should be employed. At present, even a differentiable structure on a reasonable space of windows is lacking.

3. Results on (Low) Elliptic Boundary Regularity for Mixed Conditions and Lipschitz Boundary

In this section, we give some technicalities of elliptical regularity theory needed in the proof of theorem 1. The "hard" theorems are proved in the referred literature, but a detailed reference to the single ingredients and how they combine seems appropriate to make the proof clearly reproducible.

First we give some details of the proof that the minimizer u of (0.1) is Hölder up to the boundary near points which are away from D. Due to the possibly bad geometry of D, we need to point out one subtlety not explicitly discussed in the standard references. Otherwise, the result is well-known.

The first step (following section 3.13 of [10]) is to use the weak Euler equations

$$\int_\Omega \nabla u \cdot \nabla\phi\, dx = \lambda \int_\Omega u\phi\, dx \quad (\forall \phi \in H^1(\Omega) \text{ such that } \phi|_D = 0) \tag{3.1}$$

with the function $\phi := \zeta^2\,(u-k)_+$, ζ a smooth cutoff function with support in $B_\rho(x_0)$, and $k \geq 0$, $x_0 \in \Omega$ fixed (this does satisfy the boundary conditions). We let $\Omega_\rho := \Omega \cap B_\rho(x_0)$, $A_{k,\rho} := \{x \in \Omega_\rho \mid u(x) \geq k\}$. As explained in [10], this implies

$$\int_{A_{k,\rho'}} |\nabla u|^2 \leq \frac{c}{(\rho-\rho')^2} \int_{A_{k,\rho}} (u-k)_+^2 + c\,k^2 \operatorname{vol}(A_{k,\rho}) \tag{3.2}$$

for $\rho' < \rho$, where c depends only on Ω and we have used the trivial upper bound for λ, namely the first Dirichlet eigenvalue. This inequality implies an upper bound for u (cf. 2.(5.12) in [10]). The lower bound $u \geq 0$ happens to be trivial in our case. It is crucial that no geometric assumption on D enters into the L^∞ estimate. However, without such a geometric assumption, the next step, namely Hölder estimates up to the boundary cannot be carried through. This difficulty is natural, because for Dirichlet conditions, Hölderness of the boundary data must be used in the proof, whereas for Neumann conditions, the equation itself must give the result. If both types of conditions accumulate in one point, neither the proof for Dirichlet not the one for Neumann conditions applies there.

A careful local discussion which under moderate assumptions on the geometry of D also works near the interface where D and $\partial\Omega \setminus D$ meet can be found in [17]. However, in open subsets of the boundary that carry pure Neumann or pure Dirichlet conditions, the simpler thm 2.7.2 of [10] applies already, as discussed in section 3.14 there. But in any case the global L^∞ estimate from the first step enters, and we could not have obtained it from local information alone. This already proves the first part of the following lemma the second part of which needs more recent techniques.

Lemma 2. *Any minimizer u for (0.1) for whatever measurable set D is bounded in Ω and Hölder continuous up to any part of the boundary that has positive distance from either D or $\partial\Omega \setminus D$. For almost every point $x_0 \in \partial\Omega \setminus D$, it holds $\langle \nabla u(x), \vec{n}(x_0)\rangle \to 0$ as $x \to x_0$ nontangentially.*

We now need to (give a meaning to and) establish the claim that $\partial_n u = 0$ on $(\partial\Omega \setminus D)^\circ$. A priori, $u \in H^1(\Omega)$, so its gradient does not have a trace on the boundary; on the other hand, u will *not* be in $H^2(\Omega)$, if the domain has re-entrant corners.

We say that a function u defined in the ball $B_1(0)$ converges nontangentially almost everywhere (ntae), if for every $\delta > 0$ and a.e. $x_0 \in \partial B_1(0)$, the limit $\lim_{x\to x_0} u(x)$ exists under the constraint that $\langle x - x_0, x\rangle/|x||x-x_0| > \delta$. In this case, the limit will not depend on δ.

In the case of a Lipschitz domain Ω, neighbourhoods of sufficiently small parts of $\partial\Omega$ can be mapped into neighbourhoods of parts of the boundary of a unit ball by a bilipschitzian mapping. This defines convergence ntae in any Lipschitz domain (independent of the particular choice of the bilipschitzian mapping).

In order to localize near any compact subset N_0 of $(\partial\Omega \setminus D)^\circ$, let $\varphi \geq 0$ be a C^2 function that is identically 1 in a neighbourhood of N_0, with support disjoint from $\bar{D}$, and define $\tilde{u} := u\varphi$. Then we have $\Delta\tilde{u} = -\tilde{f}$, $\partial_n\tilde{u} = \tilde{\psi}$ in the variational sense, i.e.:

$$\int_\Omega \nabla\tilde{u}\nabla\phi = \int_\Omega \tilde{f}\phi + \int_{\partial\Omega}\tilde{\psi}\phi \qquad (\forall\phi \in H^1(\Omega)) \tag{3.3}$$

with

$$\tilde{f} = \lambda u\varphi - 2\nabla u\cdot\nabla\varphi - u\Delta\varphi \in L^2(\Omega) \hookrightarrow L^2(\mathbb{R}^d)\ , \quad \tilde{\psi} = u(\partial_n\varphi) \in L^\infty(\partial\Omega)$$

where we have used the boundedness of u (on the support of φ). $\tilde{f}$ is defined to vanish outside Ω. $\tilde{u}$ satisfies a pure Neumann boundary problem on all of Ω. The outside normal vector $\vec{n}$ implicit in ∂_n is defined ae. on $\partial\Omega$.

We let $\bar{u}(x) := -(\Gamma * \tilde{f})(x) := -\int_{\mathbb{R}^d}\Gamma(x-y)\tilde{f}(y)\,dy$ where Γ is the fundamental solution [6, (2.12)] in $\mathbb{R}^d$. Then, $\Delta\bar{u} = -\tilde{f}$ and, according to the Calderón–Zygmund estimates [6, thm. 9.9], $\bar{u} \in H^2(\Omega)$. As such, its gradient has a trace in $H^{1/2}(\partial\Omega)$, in particular $\partial_n\bar{u} \in L^2(\partial\Omega)$, and we have

$$\int_\Omega \nabla\bar{u}\nabla\phi = \int_\Omega \tilde{f}\phi + \int_{\partial\Omega}(\partial_n\bar{u})\phi \qquad (\forall\phi \in H^1(\Omega)) \tag{3.4}$$

Actually, near N_0 (ie., where $\varphi \equiv 1$), $\tilde{f}$ is in L^p for all $p \leq \infty$, and therefore $\bar{u}$ will be a $W^{2,p}$ function there, for all $p < \infty$, due to the Calderón–Zygmund estimate again. By Sobolev's imbedding, $\nabla\bar{u} \in C^\beta$ in that neighbourhood of N_0 for all $\beta < 1$. In particular, $\nabla\bar{u}$ converges uniformly to its boundary values, and therefore $\langle\nabla\bar{u}(x), \vec{n}(x_0)\rangle \to \langle\nabla\bar{u}(x_0), \vec{n}(x_0)\rangle =: \partial_n\bar{u}(x_0)$ as $x \to x_0$ for ae. x_0.

Because of (3.3), (3.4), $h := \tilde{u} - \bar{u}$ is a harmonic function with Neumann boundary data $\psi := \tilde{\psi} - (\partial_n\bar{u}) \in L^2(\partial\Omega)$. In particular, $\int_{\partial\Omega}\psi = 0$. For the Laplacian in Lipschitz domains, this implies that for ae. x_0, $\langle\nabla h(x), \vec{n}(x_0)\rangle$ converges to $\psi(x_0)$ as $x \to x_0$ nontangentially. This is the content of cor. 2.1.11 combined with remark 2.1.18 in [8].

Hence $\langle\nabla\tilde{u}, \vec{n}(x_0)\rangle$ converges (in the same sense) to its boundary values $\tilde{\psi}(x_0)$, which vanish on N. Using $\tilde{u} \equiv u$ near N, we have therefore completed the proof of lemma 2. ∎

Proof of theorem 2:
This proof is due to Kenig. I owe it to private communication by J. Pipher and him. The result is closely related with the unique continuation problem. However, there is no published reference for the theorem in the version needed here, so I detail out the proof, also attempting to make the ideas easily accessible to non-specialists.

References to the original papers that coalesced into a theory for divergence form operators with nonsmooth coefficients or in nonsmooth domains can be found in Kenig's recent book [8], which collects these results. As a recent culmination, we mention [9]; the (smooth) Laplacian on Lipschitz domains is somewhat simpler and admits stronger results. It has been treated in [7].

In a first step we prove the theorem under the extra assumptions that $\Delta u \equiv 0$ and u vanishes on an open (in $\partial\Omega$) set $\tilde{\Sigma} \supset \Sigma$. In this case, for any other nonnegative harmonic function G vanishing on $\tilde{\Sigma}$, the comparison principle [8, lemma 1.3.7] will imply that there exists a constant c and a neighbourhood T of $\tilde{\Sigma}$ in $\bar{\Omega}$ such that $G \leq cu$ there, unless u vanishes somewhere in $\mathring{T}$ and hence identically. Then, for $x_0 \in \Sigma$,

$$0 \leq \frac{G(x) - G(x_0)}{|x - x_0|} \leq c\frac{u(x) - u(x_0)}{|x - x_0|} \to 0 \tag{3.5}$$

as $x \to x_0$ "radially", i.e $x - x_0 \parallel \vec{n}(x_0)$. We conclude $\partial_n G(x_0) = 0$. The expressions in (3.5) make sense as u, G will be Hölder up to $\tilde{\Sigma}$ in that case. The argument holds in particular for Green's function G [8, thm. 1.2.8]. However, the normal derivative of Green's function is the density of the harmonic measure with respect to the Lebesgue measure, and in the case of a Lipschitz domain, the harmonic measure cannot vanish on a set of positive Lebesgue measure, because both are absolutely continuous with respect to each other [8, thm 2.1.5]. (The classical proof by Green's integral formula carries over to $\partial_n G$ for Lipschitz boundary, if we test the harmonic measure on sufficiently regular boundary data and use section 2.1 of [8].)

This proves the first step, and next we drop the assumption that u vanishes in $\tilde{\Sigma} \supset \Sigma$. Assume only that it vanishes in the set Σ of positive measure (but still $\Delta u \equiv 0$). Take open sets $\tilde{\Sigma}_k \supset \Sigma$ and let u_k be the harmonic function with boundary values 0 on $\tilde{\Sigma}_k$ and u on $\partial\Omega \setminus \tilde{\Sigma}_k$. These boundary data are in $L^2(\partial\Omega)$, and the existence and uniqueness of u_k is guaranteed by thm 1.7.7 and cor. 2.1.6 of [8]. As $0 \leq u_k \leq u$ from the maximum principle, the same argument that showed $\partial_n G = 0$ above now shows $\partial_n u_k = 0$ on Σ. Therefore, by the first step, $u_k \equiv 0$. This time, $u(x_0) = 0$ is taken on in the sense of convergence ntae, which is still good enough for (3.5) to apply a.e.

Now, we can take $\tilde{\Sigma}_k$ such that $\|u_k - u\|_{L^2(\partial\Omega)} \to 0$, and this implies (see 1.7.3&7 of [8]) $\|u_k^* - u^*\|_{L^2(\partial\Omega)} \to 0$, where u^*, u_k^* is the nontangential maximal function [8, p. 13]. (Essentially, $u^*(x_0) := \sup\{|u(x)| : x \in$ an inner cone at $x_0\}$.) Therefore $u^* \equiv 0$ and hence $u \equiv 0$.

In the last step we treat the general case. Denoting by h the harmonic function with the same boundary values as u, we get $0 \leq h \leq u$, and $h = 0$, $\partial_n h = 0$ on Σ. Hence $h \equiv 0$ as before. So we are left with $\Delta u =: f \leq 0$ where u vanishes on the boundary ($u \in \mathring{H}^1(\Omega)$). This implies $u(x) = \int_\Omega G(x,y)f(y)\,dy$ with G being Green's function again. For any compact $K \subset\subset \Omega$, let $u^{[K]}(x) := \int_K G(x,y)f(y)\,dy$. From $0 \leq u^{[K]} \leq u$, we get $\partial_n u^{[K]} = 0$ wherever $\partial_n u = 0$. But $u^{[K]}$ is harmonic near $\partial\Omega$ (namely in $\Omega \setminus K$), so $u^{[K]} \equiv 0$ there. But now, due to the strong maximum principle for superharmonic functions [6, Thm. 3.5], this means that $u^{[K]} \equiv 0$ in all of Ω. Letting $K \nearrow \Omega$, we get $u^{[K]} \nearrow u$ and therefore $u \equiv 0$. ∎

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