A HYPERGEOMETRIC FUNCTION APPROACH TO THE PERSISTENCE PROBLEM OF SINGLE SINE GORDON BREATHERS

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ABSTRACT. It is shown that for an interesting class of perturbation functions, at most one of the continuum of sine Gordon breathers can persist for the perturbed equation. This question is much more subtle than the question of persistence of large portions of the family, because analytic continuation arguments in the amplitude parameter are no longer available.

Instead, an asymptotic analysis of the obstructions to persistence for large Fourier orders is made, and it is connected to the asymptotic behaviour of the Taylor coefficients of the perturbation function by means of an inverse Laplace transform and an integral transform whose kernel involves hypergeometric functions in a way that is degenerate in that asymptotic analysis involves a splitting monkey saddle.

Only first order perturbation theory enters into the argument. The reasoning can in principle be carried over to other perturbation functions than the ones considered here.

1. INTRODUCTION

In this paper, we consider the inhomogeneous linearized Sine Gordon equation subject to boundary conditions:

$$\begin{aligned} &\mathcal{L}v := v_{tt} - v_{xx} + (\cos u^*)\, v = \Delta(u^*)\ , \qquad v \in C^2 \\ (1.1)\qquad &u^*(x,t;m) := 4\,\mathrm{atn}\, \tfrac{m \sin \omega t}{\omega\, \mathrm{ch}\, mx}\ , \quad \omega := \sqrt{1-m^2}\ , \quad 0 < m < 1 \\ &v(x, t + 2\pi/\omega) = v(x,t)\ , \qquad |v| + |v_x| + |v_t| = O(x^{-2}) \ \text{ as } x \to \pm\infty \end{aligned}$$

for a given m and a function $\Delta(\cdot)$ that is analytic in some neighbourhood of 0. This question is suggested from the perturbation theory of the sine Gordon equation $u_{tt} - u_{xx} + \sin u = 0$, for which u^* is a solution, called breather, and in order for it to persist for the perturbed equation

$$u_{tt} - u_{xx} + \sin u = \varepsilon\Delta(u) + O(\varepsilon^2)\ ,$$

it is necessary that a solution v to (1.1) exists. With this application in mind, we call the function $\Delta(\cdot)$ a perturbation, and v, if it exists, a first order breather. Such a persistence question has been resolved in [2], with refinements in [7]; the crucial difference to those results is that we now consider m as fixed, i.e. we do not ask for which perturbations $\Delta(\cdot)$ *all* breathers u^* persist, but for what perturbations *a particular* u^* could possibly persist. The answer to the first question is roughly: "for none". The answer to the second question is more delicate. We give results for the most interesting perturbation functions; it seems beyond reach to treat any

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general perturbation. It should be stressed that the problem is the classical version of a toy problem in quantum field theory. Any result on the problem that is to stay relevant for the *quantized* version, should not make asumptions on a *continuum* of energies (the energy of the breather is $16m$). For more background information, see the introduction of [7] or ask the author for a copy of his thesis.

It is non-existence of solutions to (1.1) that is an interesting result, because it permits to conclude non-persistence of a Sine Gordon breather under the perturbations in question. On the other hand, the existence of solutions to (1.1) may not have interesting consequences.

The main result of the present paper is the following:

Theorem 1. *If* Δ *is a finite trigonometric sum (possibly only quasiperiodic)*

$$\Delta(u) = \sum_{j=1}^{k} s_j \sin \frac{\alpha_j}{2} u + \sum_{j=1}^{k'} c_j \cos \frac{\beta_j}{2} u \;, \tag{1.2}$$

then (1.1) cannot have a solution for two distinct m*, unless* Δ *is a multiple of* $\sin u$ *(in which case (1.1) trivially has solutions for all* m*). Given any single* m*, the space of all finite trigonometric sums permitting solutions to (1.1) is spanned by the family* $\{S^{[\mu]}(u), C^{[\mu]}(u) \mid \mu = 0, 1, 2, \dots\}$ *that can be calculated from theorem 4 below. (See (3.4).)*

This result is indeed a special case of a principle by which other (but not all) classes of perturbation functions can be treated: If a certain sequence (A_p) closely related to the Taylor coefficients of the perturbation function Δ is in the range of the Laplace transform and if one can rescue asymptotic information on this sequence through the inverse Laplace transform (say, by a Tauberian theorem), then an Abel type asymptotic theorem can be used to conclude nonexistence of first order breathers.

For this general scheme, we exhibit the role of the inverse Laplace transform and later prove an Abel theorem for an integral transform involving hypergeometric functions (lemma 18). In particular, this involves proving asymptotic estimates for the hypergeometric function in a special (degenerate) situation that is not covered by the standard tables, to the author's knowledge (see section 5). In order to apply this scheme to the situation given in theorem 1, we prove that the sequences associated to finite trigonometric sums are in the range of the Laplace transform, and verify Tauber conditions that rescue the asymptotic behaviour (section 4). Section 3 identifies and discusses, for given m, the perturbations mentioned above, for which (1.1) is solvable. These three sections are independent of each other. Section 6.2 puts the ingredients together, thus proving theorem 1 (in fact for a broader class of perturbations). Readers bailing out at the technical sections 3–5 may still be interested to read sections 2 and/or 6.3 for a quick access to more information.

2. Preliminary Discussions

Let us first fill in more details.

The range of $\mathcal{L}$ has infinite codimension, thus there are infinitely many obstructions to the solvability of (1.1). In fact, one can show:

Theorem 2. *Let $\Delta(\cdot)$ be an analytic function in some neighbourhood of 0, and let the sequences Δ_p, Γ_p, B_p, A_p be given by*

$$\begin{gathered}\frac{\Delta(4\,\mathrm{atn}\,z)}{1+z^2}=\sum_{p=1}^{\infty}\Delta_p\,z^p\ ,\quad (|z|<\rho)\ ,\\ \Gamma_p:=(-1)^{\lfloor p/2\rfloor}\frac{\Delta_p}{(p+1)^2}\ ,\quad \mathrm{B}_p:=p(\Gamma_p-\Gamma_{p-2})\ ,\quad \mathrm{A}_p:=\mathrm{B}_{p+2}-\mathrm{B}_p\ ,\end{gathered}\tag{2.1}$$

and let $|\frac{m}{\omega}|<\rho$. Then, (1.1) has a solution, if and only if for every positive integer n,

$$R_n\Big(\frac{m}{\omega}\Big):=\sum_{q=0}^{\infty}\frac{\mathrm{A}_{n+2q}}{q!(n+q)!}\left(\frac{-1}{\omega^2}\right)^q\prod_{r=0}^{q}\left[\frac{n^2-1}{4}+r(n+r)m^2\right]=0\ .\tag{2.2}$$

(For $n=1$, this is trivially satisfied.)

Except for the introduction of A_p instead of Δ_p [7], this result is due to Birnir, McKean and Weinstein [2].

We intend to study, for which sequences (A_p) the equations (2.2) are simultaneously satisfied. Calculations in special cases suggested that $\mathrm{A}_p\sim p^\alpha$ (as $p\to\infty$) may lead to $R_n\sim c(\alpha)n^{2\alpha/3}$ (as $n\to\infty$). Given a result of this type (with $c(\alpha)\neq 0$), one can argue that R_n cannot vanish identically, because a leading (non-vanishing) term in the asymptotics of A_p gives rise to such a term for R_n. This is exactly our strategy.

Let us call a distribution G in $\mathcal{D}'(\mathbb{R})$ a Laplace distribution if its support is bounded from the left and there exists $p_0\in\mathbb{R}$ such that $e^{-p_0t}G(t)$ is a tempered distribution. For Laplace distributions, the Laplace transform $\int_{-\infty}^{\infty}e^{-pt}G(t)\,dt$ exists (interpreted as the evaluation of the tempered distribution $e^{-p_0t}G(t)$ on any Schwartz function of the form $e^{(p+p_0)t}\chi(t)$, where $\chi\equiv 1$ on the support of G.

In this paper, the support of G we always be contained in $[0,\infty[$, and functions like t^α will be defined as 0 for negative t.

It is a small step from (2.2) to the appearance of the Laplace transform and hypergeometric functions:

Theorem 3. *Let*

$$\kappa_n:=\sqrt{n^2\omega^2-1}/m\quad\text{and}\quad a_n^\pm:=(n\pm i\kappa_n)/2+1\ ,\tag{2.3}$$

$z_0=-m^2/\omega^2$, and let F_n be the hypergeometric function

$$F_n(z):={}_2F_1\left({a_n^+,a_n^-\atop n+1};z\right)=\sum_{q=0}^{\infty}\frac{(a_n^+)_q\,(a_n^-)_q}{(n+1)_q\,q!}\,z^q\ ,\tag{2.4}$$

where we use the Pochhammer symbol $(a)_q:=\Gamma(a+q)/\Gamma(a)$. If the sequence (A_p) is in the range of the distributional Laplace transform (as explained above), i.e.

$$\mathrm{A}_p=\int_{-\infty}^{\infty}G(t)e^{-pt}\,dt\tag{2.5}$$

for all odd (even) p, then R_n can be expressed in terms of G as follows:

$$R_n=\frac{n^2-1}{4\,n!}\int_{-\infty}^{\infty}G(t)\left(e^{-nt}F_n(e^{-2t}z_0)\right)\,dt\ .\tag{2.6}$$

Noting that the term under the product in (2.2) can be rewritten as $m^2\left((\frac{n}{2}+r)^2+\kappa_n^2/4\right)$, and that $(n+2z\frac{d}{dz})z^q = (n+2q)z^q$, the theorem is in fact a trivial corollary to (2.2).

However, we believe that this reformulation is crucial for understanding the condition (2.2). It leads naturally (though not without hard work) to nonpersistence results for single Sine Gordon breathers, based on first order perturbation theory alone, under certain classes of perturbation functions.

Theorem 4 will show that first order perturbation theory cannot be expected to give nonpersistence for *all* perturbation functions.

3. Perturbations admitting first order breathers

3.1. **Statements of results.** Let us start our discussion by giving the solutions to (2.2). Note that only the A_p with odd (even) subscripts enter, if n is odd (even). Throughout the whole paper, we treat these cases in a parallel way: either p and n (and ℓ below) run over odd numbers only (in which case the symbol 1^* means 1), or else they run over even numbers only (in which case $1^* := 2$).

Theorem 4. *Given any $\frac{m}{\omega} < 1$ (i.e. $m < \sqrt{2}/2$), the following sequences $\boldsymbol{a}^\mu := (\mathrm{A}_p^\mu)$ satisfy all equations (2.2) (in particular, the sums converge):*

$$\mathrm{A}_p^\mu = (p+1)(p-1+4m^2)\sigma_\mu(p) \qquad (\mu = 0,1,2,\dots)\ , \tag{3.1}$$

where the $\sigma_\mu(p)$ are defined by a generating function:

$$\frac{\prod_{\ell=1^*}^{p-2}\left[1-(\ell^2-1)t\right]}{\prod_{\ell=1^*}^{p+2}\left[1+(\ell^2-1)\,t\,m^2/\omega^2\right]} = \sum_{\mu=0}^{\infty}\sigma_\mu(p)t^\mu\ . \tag{3.2}$$

Of course, $\sigma_\mu(p)$ and $\boldsymbol{a}^\mu$ depend also on m, which has been suppressed in the notation. Formulas given for $\sigma_\mu(\cdot)$ will slightly differ for even and for odd p. Proofs of this and the next theorem can be found in sections 3.2–3.4.

Before getting informations on the functions $S^\mu(\cdot), C^\mu(\cdot)$ represented by these sequences, we have to repeat some results from [7]: A sequence (A_p) determines the function $\Delta(u)$ only up to a linear combination of the functions $\sin u$, $u\cos u$, $\cos u$, and $\tilde{\Delta}(u) := 1-4\cos\frac{u}{2}+3\cos u+4\cos u\ln\cos\frac{u}{4}$. These functions satisfy (2.2) identically in m, and the first three of them can be readily interpreted as generators of simple transforms of the Sine Gordon equation (under which the full breather family persists), whereas the last can only be treated by second order perturbation theory [8]. We call the span of these four functions $\Lambda = \Lambda_{\mathrm{o}} \oplus \Lambda_{\mathrm{e}}$, the subscripts o and e denote the odd and the even subspace.

Theorem 5. *$\sigma_\mu(p)$, defined in (3.2), is a polynomial of degree 3μ with respect to p and of degree μ with respect to $Z := \frac{m^2}{\omega^2}$. More precisely,*

$$\begin{aligned}\sigma_\mu(p) = {}& \frac{(-(1+Z)/6)^\mu}{\mu!}\,p^{3\mu} + \\ &+ \frac{(-(1+Z)/6)^{\mu-1}}{(\mu-1)!}\,\frac{(3\mu+2)-(3\mu+12)Z}{10}\,p^{3\mu-1} + O(p^{3\mu-2})\ .\end{aligned} \tag{3.3}$$

For odd (even) p, the functions $S^\mu(\cdot), C^\mu(\cdot)$ that correspond to the sequences (A_p^μ) of (3.1) are, up to elements of Λ_o (Λ_e), trigonometric polynomials of the forms

$$S^\mu(u) = \sum_{j=1}^{3\mu+5} s_j^\mu(Z) \sin\frac{ju}{2} \quad \text{or} \quad C^\mu(u) = \sum_{j=1}^{3\mu+5} c_j^\mu(Z) \cos\frac{ju}{2} \tag{3.4}$$

respectively. Given any $0 < m < 1$, equation (1.1) has a solution for these perturbation functions.

How did we get rid of the condition $\frac{m}{\omega} < 1$ in the last sentence? In [2],[7], the conditions (2.2) are calculated using (2.1) from an equivalent condition in integral from, namely that $\int_{-\infty}^{\infty}\int_0^{2\pi/\omega} \Delta(u^*(x,t))\,\chi_n(x,t)\,dt\,dx = 0$, where χ_n are explicitly known time periodic, bounded functions satisfying $\mathcal{L}\chi_n = 0$. The functions $S^\mu(\cdot), C^\mu(\cdot)$ satisfy this condition for $0 < m < \sqrt{2}/2$ by theorems 2 and 4. The integral arising form inserting these functions is however a real-analytic function of $m \in \,]0,1[$ and must vanish identically by analytic continuation. Therefore, the functions $S^\mu(\cdot), C^\mu(\cdot)$ permit breather solutions to (1.1) for $\sqrt{2}/2 \le m < 1$, too.

A closer study of these trigonometrical polynomials may well be interesting; however, we have not succeeded to obtain results in this direction. The readers may wish to glance at figure 1, where some of these functions are plotted, and will certainly agree that an intriguing pattern can be observed.

3.2. **Proof of Theorem 5.** In order to prove theorem 5, we write (3.2) in the form:

$$\sum_{\mu=0}^{\infty} \sigma_\mu(p)\, t^\mu = \exp\left(\sum_{\ell=1^*}^{p-2} \log\left(1-(\ell^2-1)t\right) - \sum_{\ell=1^*}^{p+2} \log\left(1+(\ell^2-1)tZ\right) \right) \tag{3.5}$$

Expanding the logarithms, one sees that inside the big parentheses, t^μ goes with polynomials up to the order $p^{2\mu+1}Z^\mu$, which is not higher than $p^{3\mu}Z^\mu$. After expanding the exponential, the latter is still true. Moreover, if we are not interested in the orders $p^{3\mu-2}$ or lower, we can replace ℓ^2-1 by ℓ^2 in (3.5) and ignore orders t^3 or higher in the expansion of the logarithms, because in both contribute terms of $O(p^{-2})(p^3t)^\mu$. This leaves us with

$$\exp\left[-\left(\sum_{\ell=1^*}^{p-2} \ell^2\right) t - \left(\frac{1}{2}\sum_{\ell=1^*}^{p-2} \ell^4\right) t^2 - \left(\sum_{\ell=1^*}^{p+2} \ell^2\right) Zt + \left(\frac{1}{2}\sum_{\ell=1^*}^{p+2} \ell^4\right) t^2 \right] .$$

A quick way to evaluate the power sums to the desired accuracy is to use $\ell^k = \frac{1}{2}\int_{\ell-1}^{\ell+1} x^k\,dx + O(\ell^{k-2})$. The rest of calculating (3.3) is straightforward. In particular, it follows that the degrees in σ_μ are not smaller than claimed.

Let us next calculate functions $S^\mu(\cdot), C^\mu(\cdot)$ represented by the polynomial sequences (A_p^μ). We drop μ and write $\mathrm{A}(p)$ instead of A_p. In this calculation, only the fact that $\mathrm{A}(p)$ is a polynomial of degree $3\mu+2$ will be used. First consider the case, where p is odd: Translating the formulas (2.1) into generating functions, we obtain

$$-i\frac{\Delta(4\,\mathrm{atn}\, iz)}{1-z^2} = \frac{d}{dz}\, z\, \frac{d}{dz}\, \frac{z}{1-z^2} \left(\frac{d}{dz}\right)^{-1} \frac{z}{1-z^2}\, \mathrm{A}(z\frac{d}{dz}) \frac{z^3}{1-z^2} .$$

Here, initial conditions $\mathrm{B}_3 = \Gamma_1 = 0$ have been chosen (specifying some contribution from Λ_o). $(\frac{d}{dz})^{-1} := \int_0^z \ldots dz$, and all "factors" except the last are operators and act on everything to their right.

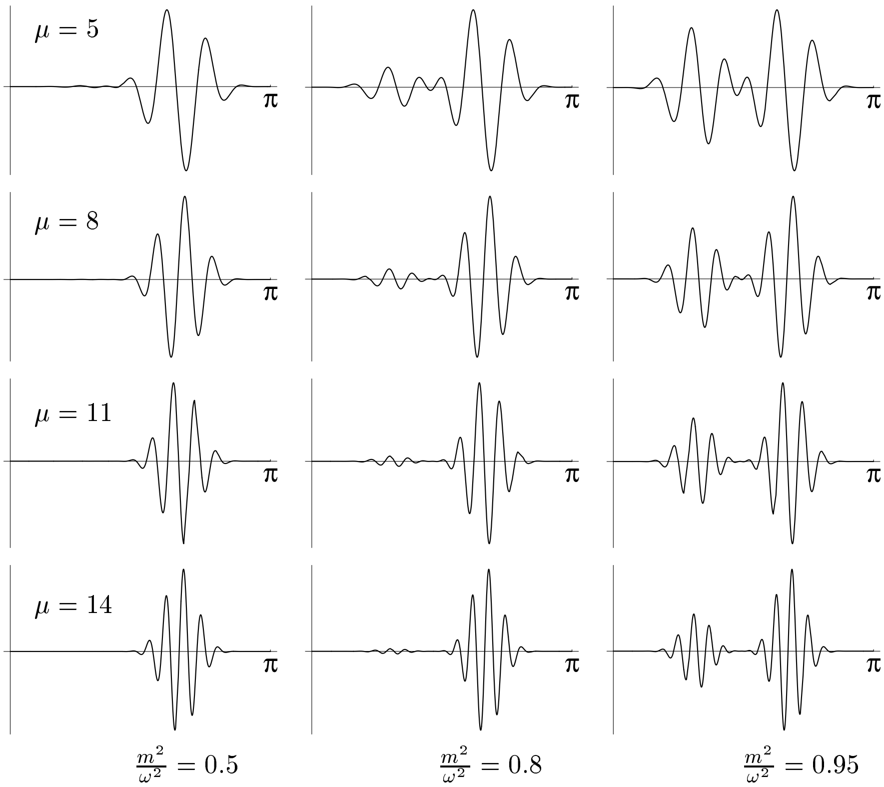

FIGURE 1. *The functions S^μ for $\mu = 5, 8, 11, 14$ and $\frac{m^2}{\omega^2} \in \{0.5, 0.8, 0.95\}$. Plotted by Mathematica. Effective calculation of these polynomial uses formulas (3.1), (3.5) and (3.6) below. The plot shows a half period, the horizontal axis giving $u/2 \in [0, \pi]$. The vertical scale is arbitrary.*

Substituting $z = i \tan \frac{v}{2}$, we get $z\frac{d}{dz} = \sin v \frac{d}{dv}$ and $\frac{z}{1-z^2} = \frac{i}{2} \sin v$ and therefore

$$2\,\Delta(2v) = \left(\frac{d}{dv}\,\sin v\right)^2 \left(\frac{d}{dv}\right)^{-1} \frac{\sin v}{1+\cos v}\,\mathrm{A}\Big(\sin v \frac{d}{dv}\Big)\left(\frac{\sin v}{1+\cos v} - \frac{1}{2}\sin v\right).$$

Now note that (as functions) $\sin v \dfrac{d}{dv} \dfrac{\sin v}{1+\cos v} = \dfrac{\sin v}{1+\cos v}$. Therefore, we get

$$\begin{aligned} 2\,\Delta(2v) = \mathrm{A}(1) \left(\frac{d}{dv}\,\sin v\right)^2 \left(\frac{d}{dv}\right)^{-1} \frac{\sin^2 v}{(1+\cos v)^2} - \\ - \frac{1}{2}\left(\frac{d}{dv}\,\sin v\right)^2 \left(\frac{d}{dv}\right)^{-1} \frac{\sin^2 v}{1+\cos v}\,\mathrm{A}\big(\frac{d}{dv}\,\sin v\big)\,1\;. \end{aligned}$$

The first term can be evaluated to $\frac{1}{2}\mathrm{A}(1)\,(\sin u - u\cos u)$, which is in Λ_o and therefore neglected. In the second term, one has to integrate (with respect to v) the function $(1-\cos v)\,\mathrm{A}(\frac{d}{dv}\,\sin v)\,1$; this latter function evaluates to a linear combination of the terms $\cos jv$ for $j = 0, 1, \ldots, \deg \mathrm{A} + 1$. The integral of the $j = 0$-term again contributes only to Λ_o. The other terms contribute linear combinations of

$\sin jv$ for $j = 1, 2, \ldots \deg \mathrm{A} + 3$. These latter functions are the S^μ referred to in (3.4), and plotted in figure 1. Let us note this formula for easy reference:

$$-4\, S^\mu(2v) = \left(\frac{d}{dv}\sin v\right)^2 \left(\frac{d}{dv}\right)^{-1} \mathrm{Pr}_{1^\perp}\,(1-\cos v)\,\mathrm{A}^\mu\left(\frac{d}{dv}\sin v\right)1\ , \tag{3.6}$$

where $\mathrm{Pr}_{1^\perp}$ denotes the projection operator that just kills the Fourier component $1 = \cos 0v$.

Next, we consider the case of even p. In complete analogy, we get (with initial conditions $\mathrm{B}_2 = \Gamma_0 = 0$):

$$\frac{\Delta(4\,\mathrm{atn}\, iz)}{1-z^2} = \frac{d}{dz}\,z\,\frac{d}{dz}\,\frac{z}{1-z^2}\left(\frac{d}{dz}\right)^{-1}\frac{z}{1-z^2}\,\mathrm{A}\left(z\frac{d}{dz}\right)\frac{z^2}{1-z^2}$$

$$4\,\Delta(2v) = \left(\frac{d}{dv}\sin v\right)^2\left(\frac{d}{dv}\right)^{-1}\frac{\sin v}{1+\cos v}\,\mathrm{A}\Big(\sin v\frac{d}{dv}\Big)\,(1-\cos v)$$

We write $\mathrm{A}(p) =: p\mathrm{A}^*(p) + \mathrm{A}(0)$; the latter contributes only $\frac{1}{2}A(0)(3\cos 3v - \cos v)$ plus an element of Λ_e. The remaining terms can be treated in the same way as the corresponding terms in the odd case. Up to elements of Λ_e, we have

(3.7)

$$4\,C^\mu_{\text{even}}(2v) = \frac{3\mathrm{A}(0)}{2}\cos 3v + \left(\frac{d}{dv}\sin v\right)^2\left(\frac{d}{dv}\right)^{-1}(1-\cos v)\,\mathrm{A}^*\Big(\frac{d}{dv}\sin v\Big)\sin v\ .$$

This ends the proof of theorem 5. We have in fact shown that if A_p is any odd (even) polynomial of degree exactly d, then $\Delta(u)$ is a trigonometric sine (cosine) polynomial of degree exactly $d+3$ in the variable $u/2$. Therefore, given any such trigonometrical polynomial $\Delta(\cdot)$ of degree $\tilde{d}$, there exists a polynomial $\mathrm{A}(p)$ of degree $\tilde{d}-3$ which corresponds to $\Delta(\cdot)$ up to terms $\sin\frac{u}{2}$, $\sin u$ (1, $\cos\frac{u}{2}$, $\cos u$). A straightforward calculation shows the following correspondence:

$\Delta(2v)$	$\sin v$	1	$\cos v$
A_p	$\frac{4}{(p-1)(p+3)}$	$\frac{16[(p+1)^2-2]}{(p+1)(p-1)^2(p+3)^2}$	$\frac{4[(p+1)^2-1]}{(p+1)^2(p-1)(p+3)}$

Hence *all* 4π-periodic trigonometric polynomials $\Delta(\cdot)$ yield rational sequences (A_p), but not arbitrary ones.

3.3. **Formal construction of sequences satisfying the necessary conditions.** Essentially, theorem 4 can be proved by using a *formal* inverse $\mathbf{M}$ to the (infinite) upper triangular matrix $\mathbf{L}$ that represents the map $(\mathrm{A}_p) \mapsto (R_n)$ given in (2.2). We succeed to exhibit sequences of "vectors" (R_n) that tend to zero (in $\mathbb{R}^\mathbb{N}$, equipped with the product topology) but which are mapped by $\mathbf{M}$ to "vectors" (A_p) that tend to the nonzero vectors $\boldsymbol{a}^\mu$. In the next section, we use monotonicity arguments to justify that the limiting process of this construction can be interchanged with $\mathbf{L}$.

Some not very enlightening calculations that need only perseverance and enter into the proofs in this section are omitted here; see [6], pp. 86–89.

Lemma 6. *The mapping* $\mathbf{L} : (\mathrm{A}_p) \mapsto (R_n)$, $R_n = \sum_p \mathbf{L}_{np}\mathrm{A}_p$,

$$\begin{aligned}\mathbf{L}_{n,n+2q} &= \frac{1}{q!(n+q)!}\left(\frac{-1}{\omega^2}\right)^q\prod_{r=0}^{q}\left[\frac{n^2-1}{4}+r(n+r)m^2\right]\\ &= \frac{n^2-1}{4n!}\,\frac{(a_n^+)_q\,(a_n^-)_q}{q!\,(n+1)_q}\left(-\frac{m^2}{\omega^2}\right)^q\ ,\end{aligned} \tag{3.8}$$

restricted to the vector space of finite sequences, is injective, and its inverse map is $\mathbf{M} : (R_n) \mapsto (\mathrm{A}_p)$, $\mathrm{A}_p = \sum_n \mathbf{M}_{pn} R_n$, *is given by*

$$
\begin{aligned}
\mathbf{M}_{p,p+2q} &= \frac{(p+q-1)!}{4\,q!}(p+2q)(p+1)(p-1+4m^2)\left(\frac{1}{\omega^2}\right)^q \times \\
&\times \prod_{s=1}^{q-2}\left[\frac{(p+2q)^2-1}{4} - s(p+2q-s)m^2\right] .
\end{aligned}
\tag{3.9}
$$

For $q \leq 2$, *the last factor in this formula is to be interpreted in the sense* $\prod_1^{q-2} T_s := (\prod_{-2}^{q-2} T_s)/(T_{-2}T_{-1}T_0)$.

Proof. The proof of this lemma is a messy, but trivial calculation of which we give the skeleton. It suffices to show $\mathbf{L}_{nn}\mathbf{M}_{nn} = 1$ (which is immediate) and $\sum_{q=0}^{Q} \mathbf{L}_{n,n+2q}\mathbf{M}_{n+2q,n+2Q} = 0$ for $Q > 0$. Write $\mathbf{L}_{n,n+2q} = \mathbf{L}_{nn}\prod_{r=1}^{q} a_r$ and $\mathbf{M}_{n+2q,n+2Q} = \mathbf{M}_{n,n+2Q}\prod_{j=1}^{q} b_j$, where a_r and b_j are defined by these equations and depend on m, n, Q as well. Then, one can write

$$\sum_{q=0}^{Q} \mathbf{L}_{n,n+2q}\mathbf{M}_{n+2q,n+2Q} = \mathbf{L}_{n,n}\mathbf{M}_{n,n+2Q}\Big(1 + a_1 b_1\big(1 + a_2 b_2(1 + \cdots a_Q b_Q)\big)\Big) .$$

We calculate from the inside to the outside by letting $X_0 := 1$ and $X_{\lambda+1} := X_\lambda a_{Q-\lambda} b_{Q-\lambda} + 1$ for $\lambda = 0, \ldots, Q-1$. We only have to show that $X_Q = 0$.

This follows from

$$X_\lambda = \frac{(Q-\lambda)(n+Q-\lambda)\Big((n+2Q)^2 - 1 - 4m^2(\lambda-1)(n+2Q+1-\lambda)\Big)}{Q(n+Q)(n+2Q+1-2\lambda)(n+2Q-1-2\lambda+4m^2)} ,$$

which we prove by induction over λ. The induction step from λ to $\lambda+1$ involves expanding many lines, but is otherwise trivial. ☐

With some perseverance, one can rewrite (3.9) for $n \geq p$ as

$$\mathbf{M}_{pn} = M(n)\,(p+1)(p-1+4m^2)\,\frac{\displaystyle\prod_{\ell=1 \text{ or } 2}^{p-2}\left[1 - \frac{\ell^2-1}{n^2-1}\right]}{\displaystyle\prod_{\ell=1 \text{ or } 2}^{p+2}\left[1 + \frac{\ell^2-1}{n^2-1}\,\frac{m^2}{\omega^2}\right]} , \tag{3.10}$$

where $M(n)$ depends only on n and m, but not on p. (Also remember the convention that ℓ, n, p are either all even or all odd!) We denote the quotient of products in (3.10) by $Q^0(p,n)$. For fixed p and $|n^2-1| > ((p+2)^2-1)\frac{m^2}{\omega^2}$, it can be expanded in a Taylor series $\sum_\mu \sigma_\mu(p)(n^2-1)^{-\mu}$, with the σ_μ already defined in (3.2).

If $\boldsymbol{e}_n$ are unit vectors in $\mathbb{R}^{\mathbb{N}}$, then $\boldsymbol{a}^\mu$ is constructed by applying $\mathbf{M}$ to weighted μ^{th} differences $\boldsymbol{b}^{[n],\mu}$ of $\boldsymbol{e}_n$ and then taking the pointwise limit of this as $n \to \infty$. Obviously, these differences of $\boldsymbol{e}_n$ tend to 0 in $\mathbb{R}^{\mathbb{N}}$, equipped with the product topology, but we must then discuss a (better than pointwise) convergence of $\mathbf{M}\boldsymbol{b}^{[n],\mu}$ that insures continuity of $\mathbf{L}$.

We construct vectors $\boldsymbol{a}^\mu \in \ker \mathbf{L}$ for $\mu = 0, 1, 2, \dots$ recursively:

$$\boldsymbol{b}^{[n],0} := M(n)^{-1}\boldsymbol{e}_n$$

$$\boldsymbol{b}^{[n],\mu} := \left(\boldsymbol{b}^{[n],\mu-1} - \boldsymbol{b}^{[n-2],\mu-1}\right)\left(\frac{1}{n^2-1} - \frac{1}{(n-2\mu)^2-1)}\right)^{-1}$$

$$\boldsymbol{a}^\mu := \lim_{n\to\infty} \mathbf{M}\boldsymbol{b}^{[n],\mu}$$

It is convenient to write $(\mathbf{M}\boldsymbol{b}^{[n],\mu})_p = (p+1)(p-1+4m^2)Q^\mu(p,n)$ with Q^0 as mentioned above and

$$Q^\mu(p,n) = \frac{Q^{\mu-1}(p,n) - Q^{\mu-1}(p,n-2)}{t_n - t_{n-2\mu}} \qquad \left(t_n := \frac{1}{n^2-1}\right). \tag{3.11}$$

An easy induction on μ shows that

$$Q^\mu(p,n) = \sum_{k=\mu}^{\infty} \sigma_k(p) \sum_{|\lambda|=k-\mu} t_n^{\lambda_0} t_{n-2}^{\lambda_1} \dots t_{n-2\mu}^{\lambda_\mu}, \tag{3.12}$$

where the inner sum runs over all multi-indices $\lambda = (\lambda_0, \dots, \lambda_\mu)$ of length $|\lambda| := \sum_0^\mu \lambda_s$ as specified. Therefore, $Q^\mu(p,n) \to \sigma_\mu(p)$ as $n \to \infty$, hence a_p^μ is as claimed in (3.1).

3.4. **Convergence of the formal construction; Proof of theorem 4.** For every μ, we have constructed a sequence $\boldsymbol{a}^\mu = (\mathrm{A}_p^\mu)_p$ as a pointwise limit $\mathrm{A}_p^\mu = \lim_{n\to\infty}(\mathbf{L}^{-1}\boldsymbol{b}^{[n],\mu})_p$, and now we want to conclude that indeed $\mathbf{L}\boldsymbol{a}^\mu = 0$. To this end, we are going to show that it holds $|(\mathbf{L}^{-1}\boldsymbol{b}^{[n],\mu})_p| \le |\mathrm{A}_p^\mu|$ for all n; in fact we show that the left hand side is (eventually) monotonous nondecreasing in n. Then Lebesgue's theorem of majorized convergence guarantees that $\boldsymbol{a}^\mu = \lim_{n\to\infty} \mathbf{L}^{-1}\boldsymbol{b}^{[n],\mu}$ in the norm topology of the weighted sequence space ℓ_ρ^2 with the norm $\|\boldsymbol{a}\|_\rho^2 := \sum |a_p|^2\rho^{2p}$ for any $0 < \rho < 1$. Indeed the majorant $\boldsymbol{a}^\mu$ is of polynomial growth and therefore in all of these spaces.

On the other hand, the sequence $((a_n^+)_q\,(a_n^-)_q/q!\,(n+1)_q) \times (m^2/\omega^2)^q$ (appearing in (3.8)) tends to 0 for $|\frac{m}{\omega}| < 1$ and is therefore in $\ell^\infty \subset \ell_{1/\rho}^2 = (\ell_\rho^2)^*$. Hence $\mathbf{L}$ is continuous from ℓ_ρ^2 to $\mathbb{R}^{\mathbb{N}}$, and indeed $\mathbf{L}\boldsymbol{a}^\mu = 0$.

We are only left with showing the claimed monotonicity result. In order to show that for any fixed μ, $|(\mathbf{L}^{-1}\boldsymbol{b}^{[n],\mu})_p| = |(p+1)(p-1+4m^2)Q^\mu(p,n)|$ is eventually monotone nondecreasing in n, it suffices to show that $(-1)^\mu Q^\mu(p,n) \ge 0$ for $n \ge 2\mu + 2$ and ignore smaller n. The monotonicity follows using (3.11).

As $Q^\mu(p,n) = 0$ for $n \le p-2$, we are left with the case $n \ge p$. We simplify notation by dropping p now.

Given that n enters into Q^0 only through t_n, formula (3.11) is a discrete analog to a formula like $Q_\mu(t) = Q'_{\mu-1}(t)/\mu$. We first state the analog of Leibniz' product rule for our situation:

Lemma 7. *Let t_n be any sequence, subject to the condition that $t_n \neq t_{n'}$ for $n \neq n'$. Let $T^0(n) = R^0(n)S^0(n)$ and define T^μ, R^μ, S^μ by*

$$T^\mu(n) = \frac{T^{\mu-1}(n) - T^{\mu-1}(n-2)}{t_n - t_{n-2\mu}}$$

and similar formulas for R and S. Then,

$$T^\mu(n) = R^\mu(n)S^0(n) + R^{\mu-1}(n-2)S^1(n) + \ldots + R^0(n-2\mu)S^\mu(n).$$

This lemma is proved by a straightforward induction on μ that need not be given here. Note that the formula is not symmetric with respect to exchange of R and S. We now consider the single factors appearing in Q^0:

For $R^0_\ell(n) := \big(1 + (\ell^2-1)\frac{m^2}{\omega^2}t_n\big)^{-1}$, one checks by a straightforward induction that

$$R^\mu_\ell(n) = \frac{\Big(-(\ell^2-1)\frac{m^2}{\omega^2}\Big)^\mu}{\big(1+(\ell^2-1)\frac{m^2}{\omega^2}t_n\big)\cdots\big(1+(\ell^2-1)\frac{m^2}{\omega^2}t_{n-2\mu}\big)} .$$

Therefore $(-1)^\mu R^\mu_\ell(n) \geq 0$ for all $n \geq 2\mu+2$.

Considering the numerator,

$$S^0_\ell(n) := 1-(\ell^2-1)t_n\ , \quad S^1_\ell(n) = -(\ell^2-1)\ , \quad S^\mu_\ell(n) = 0 \text{ for } \mu \geq 2\ .$$

Therefore $(-1)^\mu S^\mu_\ell(n) \geq 0$ for $n \geq p \geq \ell$.

Using lemma 7, we first build up the denominator of Q^0: For $R^0(n) := \prod_{\ell=1^*}^{p+2} R^0_\ell(n)$, we get $(-1)^\mu R^\mu(n) \geq 0$ for $n \geq 2\mu+2$.

Now use lemma 7 with R^0 as just defined and $S^0_{1^*}$, the first factor in the numerator of Q^0. For $T^0_{1^*} := R^0 S^0_{1^*}$, we get

$$T^\mu_{1^*}(n) = R^\mu(n)S^0_{1^*}(n) + R^{\mu-1}(n-2)S^1_{1^*}(n)\ .$$

It follows that $(-1)^\mu T^\mu_{1^*}(n) \geq 0$ for $n \geq 2\mu+2 \geq 2$. Repeat this argument for $T^0_{1^*+2} := T^0_{1^*}S^0_{1^*+2}$. We get $(-1)^\mu T^\mu_{1^*+2}(n) \geq 0$ for $n \geq \max(2\mu+2, 4)$. Continuing until one gets $T^\mu_{p-2} = Q^\mu$, one concludes $(-1)^\mu Q^\mu(n) \geq 0$ for $n \geq \max(2\mu+2, p-2+1^*)$. □

Note that we have effectively shown the *complete monotonicity* of the sequence $Q^0(n)$, considered as a sequence not indexed by n, but by t_n. A similar argument will be used for the inverse Laplace transform of the sequence (A_p) corresponding to $\sin \alpha u/2$ in the next section. The author is not aware of a deeper connection indicated by this similarity, in particular the argument has here been applied in the range of $\mathbf{L}$, whereas it will be applied in the domain of $\mathbf{L}$ there. However, it still seems striking, and the challenge to invert (modulo kernel) the integral transform (2.6) might be attacked by similar methods as the inverse Laplace transform. Success in this direction could mean big progress for the persistence problem of single sine Gordon breathers.

4. The sequence associated to $\sin \alpha u/2$

In this section, we let $\alpha > 0$ and $\Delta(u) = \sin \alpha u/2$ or $\Delta(u) = \cos \alpha u/2$, depending on whether the case of odd or even p is considered (see the convention before theorem 4). Having (2.1) in mind, let $D_p := D_p(\alpha) := (-1)^{\lfloor p/2 \rfloor}\Delta_p$ for these functions. We also have to consider the consecutive sequences of differences, namely

$$D^{[0]}_p(\alpha) := D_p(\alpha)\ , \qquad D^{[r]}_p(\alpha) = D^{[r-1]}_{p+2}(\alpha) - D^{[r-1]}_p(\alpha)\ . \tag{4.1}$$

It is immediate from (2.1) that the generating functions of these sequences are

$$g^{[r]}_\pm(z) := \frac{1}{2}\left[\left(\frac{1+z}{1-z}\right)^\alpha \pm \left(\frac{1-z}{1+z}\right)^\alpha\right](1-z^2)^{r-1} = \sum_{p=-\infty}^{\infty} D^{[r]}_{p-2r}(\alpha)z^p\ . \tag{4.2}$$

In this equation, Taylor coefficients of negative powers are defined to be 0; binomial coefficients with negative bottom part will be 0, too. g_+ corresponds to even p, g_- to odd p.

Writing $g_\pm^{[r]}$ in one of the forms $((1+z)^{2\alpha} \pm (1-z)^{2\alpha})/2(1-z^2)^{\alpha+1-r}$ and $\frac{1}{2}[(1+z)^{\alpha+r-1}(1-z)^{-\alpha+r-1} \pm (z \leftrightarrow -z)]$, one immediately gets the formulas

$$D_p^{[r]}(\alpha) = \sum_{j=0}^{\lfloor p/2 \rfloor + r} \binom{2\alpha}{p-2j+2r}\binom{\alpha+j-r}{j} \tag{4.3}$$

$$= \sum_{j=0}^{p+2r} (-1)^j \binom{-\alpha+r-1}{j}\binom{\alpha+r-1}{p+2r-j} . \tag{4.4}$$

For $p \geq 0$, $D_p^{[r]}(\alpha)$ is therefore an odd (even) polynomial of degree $p+2r$ in α, and with zeros at the integers $-(r-1), \dots, (r-1)$. Its leading coefficient is $2^{p+2r}/(p+2r)!$.

We get the following asymptotic series:

Lemma 8. *For $r \geq 0$ and $\alpha > 0$ fixed, the following asymptotic series holds as $p \to \infty$:*

$$D_{p-2r}^{[r]}(\alpha) \sim \sum_{l=0}^{\infty} (-1)^l \binom{\alpha+r-1}{l} 2^{\alpha+r-1-l} \binom{p+\alpha-r-l}{p} \pm (\alpha \leftrightarrow -\alpha) \tag{4.5}$$

The $+$ sign is for even p, the $-$ sign for odd p. The sums are to be understood as the asymptotic series in $p^{\alpha+r-l}$ and $p^{-\alpha+r-l}$ that arise when Stirling's formula is applied to the binomial coefficients depending on p.

Proof. $g_\pm^{[r]}(z)$ is the odd/even part of $(1-z)^{-\alpha+r-1}(1+z)^{\alpha+r-1}$, and it suffices therefore to consider the coefficients of the latter function.

Write $T_\pm := (1 \pm z)^{\pm\alpha+r-1}$ in the form $S_\pm + R_\pm$ with

$$S_\pm = \sum_{l=0}^{k_\pm} (-1)^l \binom{\pm\alpha+r-1}{l} 2^{\pm\alpha+r-1-l}(1 \mp z)^l$$

and

$$R_\pm = (1 \mp z)^{k_\pm + 1} \varphi_{k_\pm}^\pm(z) ,$$

$\varphi_{k_\pm}^\pm$ being defined by the latter equations. Now, $T_+T_- = T_+S_- + S_+T_- - S_+S_- + R_+R_-$. The first two terms give the sums claimed, up to $l = k_\pm$. The third term does not contribute to the asymptotics of Taylor coefficients at all, because it is a polynomial. The remainder $R_+R_- = ((1-z)^{k_+ +1}\varphi_{k_-}^-)((1+z)^{k_- +1}\varphi_{k_+}^+)$ has ± 1 as its only singularities on the closed unit disc. E.g. the one at $+1$, determined by the first factor, behaves like $(1-z)^{k_+ + r - \alpha}$. Hence derivatives of R_+R_- up to arbitrarily high order $K = K(k_\pm)$ are still integrable over the unit circle. Therefore its Taylor coefficients are of order $o(p^{-K})$. This proves the lemma. $\square$

A formal (term by term) inverse Laplace transform of (4.5) — using that $\binom{p-\beta}{p}$ is the Laplace transform of $\frac{\sin \pi\beta}{\pi}(e^t - 1)^{\beta-1}$ — gives the following result, which will be justified rigorously:

Lemma 9. *For $r > \alpha > 0$ and $p \geq 0$ (even or odd) integer, one has*

$$D_p^{[r]}(\alpha) = (-1)^{r+1} \frac{\sin \pi\alpha}{\pi} \int_0^\infty e^{-(p+2r)t}(e^{2t}-1)^{r-1} \left[\left(\frac{e^t+1}{e^t-1}\right)^\alpha \pm \left(\frac{e^t-1}{e^t+1}\right)^\alpha \right] dt \tag{4.6}$$

(The + sign is for odd p!)

The similarity between (4.6) and the formula for the generating function $g_\pm$ ($z \leftrightarrow e^{-t}$) is intriguing, but probably not significant. After all, there do exist sequences that are not a Laplace transform of anything, so substituting into the generating function cannot in general amount to an inverse Laplace transform of the sequence!

Proof of the lemma. Substitute $x = e^{-t}$, use parity, and write $x = \frac{1}{2}[(1+x)-(1-x)]$ to transform (4.6) into a sum of Beta function integrals $\int_{-1}^{1}(1-x)^a(1+x)^b\,dx$. Then, (4.6) reads

$$D_p^{[r]}(\alpha) = 2^{2r-1}\sum_{j=0}^{p+1}\binom{p+1}{j}\binom{r+\alpha+j-1}{p+2r}\ . \tag{4.7}$$

For the moment, call the right hand side of (4.7) $\bar{D}_p^{[r]}(\alpha)$. We claim that $\bar{D}_p^{[r]}(\alpha) = D_p^{[r]}(\alpha)$ for $r,p \geq 0$. This is an immediate consequence of two observations: Firstly $\bar{D}_{p+2}^{[r-1]}(\alpha) - \bar{D}_p^{[r-1]}(\alpha) = \bar{D}_p^{[r]}(\alpha)$ in analogy to the recursion for $D_p^{[r]}(\alpha)$, and secondly, $\bar{D}_p^{[r]}(\alpha) = D_p^{[r]}(\alpha)$ holds for $p=1$ ($p=0$). Indeed, replace $\binom{p+3}{j}$ in $\bar{D}_{p+2}^{[r-1]}(\alpha)$ by $\binom{p+1}{j-2} + 2\binom{p+1}{j-1} + \binom{p+1}{j}$, shift j in each term to get $\binom{p+1}{j}$, and newly collect the binomial coefficients occurring in the second factor under the sum to establish the first statement. Secondly, $\bar{D}_{0|1}^{[r]}(\alpha)$ shares all properties mentioned for $D_{0|1}^{[r]}(\alpha)$ after (4.3), and we have $D_{0|1}^{[r]}(r) = \bar{D}_{0|1}^{[r]}(r) = 2^{2r-1}$ for $r \geq 1$. This determines the polynomials $\bar{D}_{0|1}^{[r]}(\alpha) = D_{0|1}^{[r]}(\alpha)$ uniquely. $D_{0|1}^{[0]}(\alpha) = \bar{D}_{0|1}^{[0]}(\alpha)$ is checked separately. $\square$

We are now in a position to prove

Lemma 10. *For $\Delta(u) = \sin\alpha u/2$, $\alpha > 0$, the corresponding sequence A_p is of the form*

$$\mathrm{A}_p = p^j \int_0^\infty G(t)e^{-pt}\,dt\ , \tag{4.8}$$

where $j = \lceil\alpha\rceil$ and G is a function that satisfies

$$G(t) \sim \frac{2^{\alpha+3}(\alpha-2)^2}{\Gamma(\alpha+1)\Gamma(j-\alpha+3)}\,t^{j-\alpha+2} \qquad \text{as } t \to 0\ . \tag{4.9}$$

The same holds for $\Delta(u) = \cos\alpha u/2$.

Proof. If α is an integer, then $D_p(\alpha)$ is a polynomial in p according to (4.4), and with(4.13) below, the proof presents no difficulties. So assume that $\alpha \notin \mathbb{Z}$ and thus $\alpha < j < \alpha+1$.

We want to make sense of formula (4.6) without the restriction $r > \alpha$, e.g. for $r = 0$, which amounts to specifying a regularization of the function

$$H_0(t) := -\frac{\sin\pi\alpha}{\pi}(e^{2t}-1)^{-1}\left[\left(\frac{e^t+1}{e^t-1}\right)^\alpha \pm \left(\frac{e^t-1}{e^t+1}\right)^\alpha\right]$$

as a distribution. The singularity at 0 being of order $t^{-(\alpha+1)}$, H_0 is obviously a linear form on the subspace of those test functions that are of order $O(t^j)$ as $t \to 0$; any definition on a basis of some complementary space (j–dimensional) gives rise to a distribution interpretation of H_0.

Assuming p odd, we can find coefficients $a_k(p)$ such that $e^{-pt}+\sum_{k=1}^{j} a_k(p)e^{-(2k-1)t} = O(t^j)$ as $t \to 0$, and these a_k are polynomials of degree $j-1$ in p (a Vandermonde determinant argument). Then we get the regularization

$$\int e^{-pt} H_0(t)\,dt := \int_0^\infty \left[e^{-pt} + \sum_{k=1}^{j} a_k(p)e^{-(2k-1)t}\right] H_0(t)\,dt - \tilde{P}_j(p) \tag{4.10}$$

with a certain polynomial $\tilde{P}_j$ of degree at most $j-1$ defined by the requirement (choice of regularization) that $\int e^{-(2k-1)t} H_0(t)\,dt := D^{[0]}_{2k-1}(\alpha)$ for $k = 1,\ldots,j$. The j^{th} differences of (4.10) are obviously $= D^{[j]}_p(\alpha)$. This, and our choice of regularization guarantee that (4.10) $= D^{[0]}_p(\alpha)$ for all odd p. The case of even p is analogous.

Classical j-fold integration by parts gives now

$$D^{[0]}_p(\alpha) = p^j \int_0^\infty e^{-pt} H_j(t)\,dt + P_j(p)\ ,$$

where

$$H_k(t) := -\int_t^\infty H_{k-1}(t)\,dt \sim (-\alpha)_k \frac{2^{\alpha-1}}{t^{\alpha+1-k}} \quad (t \to 0) \tag{4.11}$$

($k = 1,\ldots,j$), and $P_j(p)$ is some polynomial of degree $\le j-1$ and thus of the form $p^j \int e^{-pt}(\gamma_0 + \ldots + \gamma_{j-1}t^{j-1})\,dt$. The asymptotic information for small t can be carried easily through the recursive definition of H_k in (4.11), because (as long as) it dominates the constant of integration. This gives already the $r = 0$ case of

$$\begin{aligned} D^{[r]}_p(\alpha) &= p^{j-r} \int_0^\infty d_r(t)\, e^{-pt}\,dt \qquad \text{with} \\ d_r(t) &\sim (-\alpha+r)_{j-r} \frac{2^{\alpha-1+r}}{t^{\alpha+1-j}} \quad (t \to 0)\ . \end{aligned} \tag{4.12}$$

For other $r \le j$, this is proved analogously. To get $r > j$, one replaces $-\int_t^\infty$ with $\int_0^t$ in (4.11) for $k > j$.

Now it is straightforward from (2.1) that

$$\frac{\mathrm{A}_p}{p^j} = C_0(p)\,\frac{D^{[0]}_p(\alpha)}{(p+2)^j} - C_1(p)\,\frac{D^{[1]}_p(\alpha)}{p^{j-1}} + C_2(p)\,\frac{D^{[2]}_{p-2}(\alpha)}{(p-2)^{j-2}}\ , \tag{4.13}$$

where the coefficients $C_0 = \frac{16(p^2+2p-1)(p+2)^j}{(p-1)^2(p+1)(p+3)^2p^j}$, $C_1 = \frac{2(3p-1)}{(p-1)^2p(p+1)}$, $C_2 = \frac{(p-2)^{j-2}}{(p-1)^2p^{j-1}}$ are the Laplace transforms of functions asymptotic to $32t^2$, $12t^2$, $2t^2$ respectively. Now $G(t)$ in (4.8) can in principle be evaluated in terms of convolution integrals. But we care for its asymptotic behaviour only and just note that if $c(t) \sim t^\gamma/\Gamma(\gamma+1)$, $d(t) \sim t^\delta/\Gamma(\delta+1)$ as $t \to 0$, then $(c*d)(t) = \int_0^t c(\tau)d(t-\tau)\,d\tau \sim t^{\gamma+\delta+1}/\Gamma(\gamma+\delta+2)$. This immediately implies (4.9). □

5. Asymptotic Expansions for the Hypergeometric Function

We treat both the function F_n already defined in (2.4) and one of its contiguous functions:

$$F_n(z) := {}_2F_1\left({a_n^+, a_n^- \atop n+1}; z\right)\ , \quad F_n^+(z) := {}_2F_1\left({a_n^+, a_n^- \atop n+2}; z\right) \tag{5.1}$$

The latter will be needed for estimates of derivatives of F_n, which will be reduced by contiguous function relations.

5.1. **Saddle Point Asymptotics.**

5.1.1. *Statement of the result.* Section 5.1 is devoted to the proof of the following result:

Theorem 11. *As $n \to \infty$, the following asymptotic expansions are uniform on every compact interval $\eta \in [0, \eta_1]$:*

$$F_n\left(z_0(1-\eta n^{-2/3})\right) = K_n\, C_n(\eta)\, \left(2\,\mathrm{Ai}(\bar\eta) + \frac{\eta^2}{2n^{1/3}}\,\mathrm{Ai}(\bar\eta) + O\left(\frac{1}{n^{2/3}}\right)\right) \tag{5.2}$$

$$\begin{aligned} F_n^+\left(z_0(1-\eta n^{-2/3})\right) &= 2\,K_n\,C_n(\eta)\,\times \\ &\left(2\,\mathrm{Ai}(\bar\eta) + \frac{\eta^2}{2n^{1/3}}\,\mathrm{Ai}(\bar\eta) + \left(\frac{2}{n\omega}\right)^{1/3} 2\omega\,\mathrm{Ai}'(\bar\eta) + O\left(\frac{1}{n^{2/3}}\right)\right) \end{aligned} \tag{5.3}$$

For $\eta = 0$, the remainder term in these formulas is $O(n^{-1})$ instead of $O(n^{-2/3})$. In particular,

$$F_n^+(z_0)/F_n(z_0) = 2 - 2\rho n^{-1/3} + O(n^{-1})\,, \quad \rho = (6\omega^2)^{1/3}\,\Gamma(\tfrac{2}{3})/\Gamma(\tfrac{1}{3})\,. \tag{5.4}$$

In these formulas, we have used the following abbreviations: $z_0 = -m^2/\omega^2$, $\bar\eta := \left(\frac{\omega}{2}\right)^{2/3}\eta$ and K_n and $C_n(\eta)$ are given by

$$\begin{aligned} K_n &:= \frac{\Gamma(n)}{\Gamma(\frac{n+i\kappa}{2})\Gamma(\frac{n-i\kappa}{2})}\,\frac{e^{\pi\kappa/2}}{\mathrm{sh}\,\pi\kappa}\,, \\ C_n(\eta) &:= \pi m\omega^2\,\left(\frac{2}{n\omega}\right)^{1/3}\,\left(\frac{\omega}{m}\right)^n\,\exp\left(\frac{\eta n^{1/3}}{2}\right)\,. \end{aligned} \tag{5.5}$$

5.1.2. *An appropriate integral formula.* As a starting point, we need the following lemma:

Lemma 12.

$$\begin{aligned} F_n(z) &:= {}_2F_1\left({a_n^+, a_n^- \atop n+1}; z\right) = K_n\,\frac{n\,e^{\pi i n/2}}{n-i\kappa}\,\times \\ &\int_{\gamma_1} t^{(i\kappa+n)/2-1}\,(1-t)^{(i\kappa-n)/2}\,(1-z-t)^{-(i\kappa+n)/2-1}\,dt + c.c. \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} F_n^+(z) &:= {}_2F_1\left({a_n^+, a_n^- \atop n+2}; z\right) = K_n\,\frac{-2n(n+1)}{n^2+\kappa^2}\,e^{\pi i n/2}\,\times \\ &\int_{\gamma_1} t^{(i\kappa+n)/2}\,(1-t)^{(i\kappa-n)/2-1}\,(1-z-t)^{-(i\kappa+n)/2-1}\,dt + c.c. \end{aligned} \tag{5.7}$$

Here, $+c.c.$ denotes the operation of adding the complex conjugate of the term in front of the $+$ sign, but complex conjugation refers to the parameters $a_n^\pm$, not to z (nor t). For $\kappa = \kappa_n$ and $a_n^\pm$, see (2.3) in theorem 3, the path γ_1 is described in figure 2 below. K_n is given by (5.5).

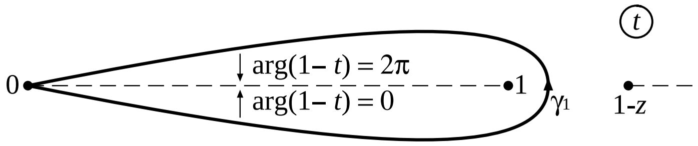

FIGURE 2. *The loop γ_1. Where γ_1 intersects the real axis, one has* $\arg t = 0$.

Proof. Employ formula 15.3.8 from [1], namely

$$(5.8)\qquad \begin{aligned} {}_2F_1\left({a,b \atop c};z\right) &= (1-z)^{-a}\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}{}_2F_1\left({a,c-b \atop a-b+1};\frac{1}{1-z}\right) \\ &+ (1-z)^{-b}\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}{}_2F_1\left({b,c-a \atop b-a+1};\frac{1}{1-z}\right), \end{aligned}$$

and use formula 15.3.1 of [1] (for $z<0$) on the right hand side of it, namely

$$(5.9)\quad {}_2F_1\left({a,c-b \atop a-b+1};\zeta\right) = \frac{\Gamma(a-b+1)}{\Gamma(a-c+1)\,\Gamma(c-b)}\int_0^1 t^{c-b-1}(1-t)^{a-c}(1-t\zeta)^{-a}\,dt$$

(and a corresponding formula with a and b exchanged).

For those parameters for which this and the other integral is defined, namely $a,b\in\,]c-1,c[$, replace $\int_0^1$ in (5.9) by $(1-e^{2\pi i(a-c)})^{-1}\int_{\gamma_1}$. Make an analytic continuation in the parameters to cover the values needed in our case, and simplify the Γ–functions. □

5.1.3. *Saddle point analysis and asymptotic expansions.* In order to obtain the asymptotic estimates needed, we use the Riemann saddle point method, which is lucidly explained in [15]. However, in our case, we have to deal with a degeneracy, namely a monkey saddle, and we want to cover its splitting into two ordinary saddles under change of a parameter, too. This situation has been considered by [5],[10]. The monkey saddle method (but not the splitting into ordinary saddle points) is already included in [14]. Readers may also wish to consider [3], which has a similar flavour. This section gives a general outline of the method in this case; the next one will apply it to the particular case of hypergeometric functions given in theorem 11.

Consider $\int_\gamma \exp\left[nf_0(t)+f_1(t)+O(\frac{1}{n})\right]\,dt$, where f_0 and f_1 are analytic in some domain containing the path of integration and depend also on a parameter τ suppressed in the notation. Assume that, when $\tau=0$, $\operatorname{Re} f(\cdot)$ has a monkey saddle point in $t^*\in\gamma$, i.e. $f_0'(t_*)=f_0''(t_*)=0$, $f_0'''(t_*)\neq 0$, and that for every neighbourhood $\mathcal{U}$ of t_*, there exists a $\delta>0$ such that $\operatorname{Re} f_0(t)\le \operatorname{Re} f_0(t_*)-\delta$ for all $t\in\gamma\setminus\mathcal{U}$, and that G is bounded and γ has finite length.

Moreover assume that for $\tau>0$ the monkey saddle splits into two ordinary saddles at t_+ and t_-. We want to introduce a change of coordinates $t\leftrightarrow v$ in a neighbourhood of t_* that makes the exponent $f_0(t)$ a function of v that is as easy as possible while still showing the splitting of the monkey saddle.

Lemma 13. *Suppose f_0 is holomorphic in a domain $\mathcal{U}$ with exactly two critical points there, $f_0'(t_+)=0=f_0'(t_-)$, but $f_0''(t_+)\neq 0\neq f_0''(t_-)$. Let $f_0(t_+)-f_0(t_-)=$*

$4b^3 > 0$. *Then, there exists a biholomorphic mapping* $t \mapsto v = \varphi(t)$, $\mathcal{U} \to \varphi(\mathcal{U})$ *such that*

$$f_0(t) - f_0(t_+) = -3bv^2 - v^3 \,, \qquad \varphi(t_+) = 0 \,, \quad \varphi(t_-) = -2b \,. \tag{5.10}$$

Proof. This is theorem 1 of [5]; or see page 287 of [10]. □

We are going to write

$$f_0(t) - f_0(t_+) = -C^2(t - t_+)^2 \; (1 + a_1(t - t_+) + \dots) \;. \tag{5.11}$$

For τ sufficiently small, t_- is close to t_+, and $\mathcal{U}$ can be chosen to be a disc about t_+ that also contains t_-; then the coordinate change of lemma 13 can be described by a Taylor series that converges in $\mathcal{U}$. From (5.11), one gets

$$t - t_+ = \frac{\sqrt{3b}}{C}\, v \left(1 - \Big(\frac{a_1}{2} - \frac{C}{2(3b)^{3/2}} \Big) \frac{\sqrt{3b}}{C}\, v + \dots \right) \tag{5.12}$$

in $\mathcal{U}$. (When $t_+ - t_- \to 0$ for $\tau \to 0$, there must be large cancellations in the coefficients of this power series, because any small error in b has the effect of radically shrinking the disk of convergence such that t_- will not be inside it any more.)

The contribution to our integral from outside $\mathcal{U}$ can be estimated by $e^{nf_0(t_+)} O(e^{-n\delta})$, which will be negligible. Carrying out the change of coordinates in $\mathcal{U}$ one obtains:

$$\begin{aligned} \int_\gamma e^{nf_0(t) + f_1(t) + O(1/n)}\, dt = \frac{\sqrt{3b}}{C} e^{nf_0(t_+) + f_1(t_+)} \int_{\varphi(\gamma \cap \mathcal{U})} e^{-n(3bv^2 + v^3)} \times \\ \left(1 + \Big(f_1'(t_+) - a_1 + \frac{C}{(3b)^{3/2}} \Big) \frac{\sqrt{3b}}{C}\, v + O\Big(\frac{3bv^2}{C^2}\Big) + O\Big(\frac{1}{n}\Big) \right) dv \\ + e^{nf_0(t_+)} O(e^{-n\delta}) \end{aligned} \tag{5.13}$$

By deforming γ, we can assume with no loss of generality that γ goes along gradient curves of $\operatorname{Re} f_0$ inside $\mathcal{U}$. Then $\varphi(\gamma \cap \mathcal{U})$ lies on curves, where $-3bv^2 - v^3$ is real. Continuing these curves to infinity at both ends, we obtain a path that will be called γ'. By a slight deformation, one can also make γ' the union of two rays among $\left\{ re^{i\varphi} \;\middle|\; r \geq 0 \,,\; \varphi \equiv 0 \bmod 2\pi/3 \right\}$.

We restrict ourselves to the case where τ is small (dependent on n) such that $\beta := b(\tau) n^{1/3}$ stays bounded as $n \to \infty$. For larger τ, the cutoff term would not be of lower order than the calculated terms, in accordance with the fact that then the asymptotic behaviour is determined by one of the two saddles alone. We substitute $w = vn^{1/3}$ and $\beta = bn^{1/3}$ inside $\mathcal{U}$ and then replace the integration in (5.13) by integration over γ'.

A standard estimate as in Seifert [15] (p.94) guarantees that extending the integration from $\varphi(\gamma \cap \mathcal{U})$ to γ' introduces only an error that is negligibly small, namely $\exp[n \operatorname{Re} f_0(t_+)]\, O(n^{-a})$ for arbitrarily large a.

For the evaluation of the integrals, one can use the following lemma:

Lemma 14. *Let* $\gamma' = e^{i\pi/3}\,]-\infty, 0] \cup [0, \infty[$, *oriented like this:* . *Then*

$$\int_{\gamma'} \exp[-3\beta w^2 - w^3]\, w^j \, dw = \exp(-2\beta^3) \left(\frac{d}{dy} - \beta \right)^j \operatorname{Yi}(3^{-1/3} y) \Big|_{y \,=\, 3\beta^2} \,, \tag{5.14}$$

where $\mathrm{Yi}(x) := \pi 3^{-1/3}(\mathrm{Bi}(x) - i\,\mathrm{Ai}(x))$ *and* $\mathrm{Ai}(\cdot)$, $\mathrm{Bi}(\cdot)$ *are the usual Airy functions ([1], 10.4).*

Proof. We show that

$$\int_{\gamma'} \exp[-s^3 + ys](s-\beta)^j\, ds = \left(\frac{d}{dy} - \beta\right)^j \mathrm{Yi}(3^{-1/3}y)\ . \tag{5.15}$$

For $y = 3\beta^2$, this reduces to (5.14) under the substitution $w = s - \beta$. The case of arbitrary j follows from $j = 0$ by differentiation with respect to y. For $j = 0$, we only need to note that both sides of (5.15) satisfy the Airy differential equation $u''(y) = 3^{-2/3} y\, u(y)$ with the same initial conditions $u(0)$, $u'(0)$. Indeed, the integral for $y = 0$ is equivalent to Euler's integral for the Γ function. □

5.1.4. *Saddle points and level sets for the hypergeometric function; Proof of theorem 11.* Write the integrand of (5.6) in the form $\exp\left[-\frac{\pi i n}{2} + n\left(f_0(t) + \frac{1}{n} f_1(t) + O(\frac{1}{n^2})\right)\right]$ where

$$f_0(t) = \frac{m+i\omega}{2m}\ln t - \frac{m-i\omega}{2m}\ln(1-t) - \frac{m+i\omega}{2m}\ln(T-t) + \frac{i\pi}{2} \tag{5.16}$$

and where we have already used the first of the following abbreviations, which serve to streamline the calculations:

$$\begin{array}{c|c} \begin{aligned} T &:= 1 - z \\ \tau &:= 1 + z/\tfrac{m^2}{\omega^2} \\ T &= (1 - m^2\tau)/\omega^2 \end{aligned} & \begin{aligned} \sqrt{D} = \sqrt{D(\tau)} &:= \sqrt{T - T^2\omega^2} \\ \Omega_\pm = \Omega_\pm(\tau) &:= -i\omega T \pm \sqrt{D} \\ M_\pm = M_\pm(\tau) &:= -mT \pm \sqrt{D} \\ \sqrt{D} &= \tfrac{m}{\omega}\sqrt{\tau\,(1 - m^2\tau)} \end{aligned} \end{array} \tag{5.17}$$

Of course, f_0 and f_1 depend on τ, but we usually suppress this in the notation.

We are in particular interested in the cases $z := -\frac{m^2}{\omega^2}$ and $z \in \left]-\frac{m^2}{\omega^2}, 0\right[$.

f_0 has two critical points, $f_0'(t_\pm) = 0$, which coalesce to the single point t_*, if $\tau = 0$, namely

$$\begin{aligned} t_\pm = t_\pm(\tau) &= (m+i\omega)\,\Omega_\pm & t_* &= 1 - i\tfrac{m}{\omega} \\ T - t_\pm &= -(m+i\omega)\,M_\pm & \tfrac{1}{\omega^2} - t_* &= \tfrac{m}{\omega^2}(m+i\omega) \\ 1 - t_\pm &= \Omega_\pm\, M_\pm/T & 1 - t_* &= i\tfrac{m}{\omega} \end{aligned} \tag{5.18}$$

At the saddle points, f_0 is real for $T \le 1/\omega^2$. In fact,

$$\begin{aligned} 2m\, f_0(t_\pm) &= -m\ln\left(M_\pm^2/T\right) \mp 2\omega\,\mathrm{atn}(\sqrt{D}/T\omega) \\ &= -2m\left[\ln\tfrac{m}{\omega} - \tfrac{\tau}{2} \mp \tfrac{\omega}{3}\tau^{3/2} - \tfrac{1}{4}\tau^2 + O(\tau^{5/2})\right] \\ f_0(t_*)\big|_{\tau=0} &= -\ln\tfrac{m}{\omega} \end{aligned} \tag{5.19}$$

Let primes denote derivatives with respect to t (τ held fixed). We shall need the second derivative at the critical points:

$$\begin{aligned} f_0''(t_\pm) &= \pm\frac{T}{m}\sqrt{D}\left(\frac{m-i\omega}{\Omega_\pm M_\pm}\right)^2 \\ &= \mp\left(\frac{m-i\omega}{m}\right)^2 \omega^3\tau^{1/2}\left(1 \pm 2(\omega - im)\tau^{1/2} + O(\tau)\right) \end{aligned} \tag{5.20}$$

and also

$$f_0'''(t_*)\Big|_{\tau=0} = -\omega\left(\frac{\omega}{m}\right)^3 (m - i\omega)^3\ . \tag{5.21}$$

Restrict t to $\mathbb{C}_{\mathrm{cut}}$, the complex plane with branch cuts as in figure 3. It is important for us to understand the level set of $\operatorname{Re} f_0$ at the critical value(s). From (5.16), one gets that $2 \operatorname{Re} f_0 = O(1) - \ln|(1-t)(T-t)/t|$, so the level set is bounded, and it is a union of one dimensional manifolds that intersect at critical points only. They do not bound a subdomain of $\mathbb{C}_{\mathrm{cut}}$ because of the maximum principle. These manifolds are therefore lines issuing from critical points and ending at $\partial\mathbb{C}_{\mathrm{cut}}$, the branch cuts. We discuss them for $\tau = 0$. In this case, one has for *real* t

$$\operatorname{Re} f_0(t - i0) = -\frac{1}{2} \ln \left| \frac{(1-t)(1/\omega^2 - t)}{t} \right| - \frac{\pi\omega}{2m} \times \begin{cases} 1 & \text{if} \quad 1 < t < 1/\omega^2 \\ -1 & \text{if} \quad t < 0 \\ 0 & \text{else} \end{cases}$$

and

$$\operatorname{Re} f_0(t + i0) = \operatorname{Re} f_0(t - i0) - \begin{cases} \pi\omega/m & \text{if} \quad 0 < t < 1 \text{ or } t > 1/\omega^2 \\ 2\pi\omega/m & \text{if} \quad t < 0 \end{cases} .$$

So, the intermediate value theorem and elementary calculations imply that the level set $\operatorname{Re} f_0(t) = f_0(t_*)$ meets the real axis twice from below in $]-\infty, 0[$ and both from below and above in each of the intervals $]0,1[$ and $]1/\omega^2, \infty[$. So topologically, the level set looks like shown in figure 3.

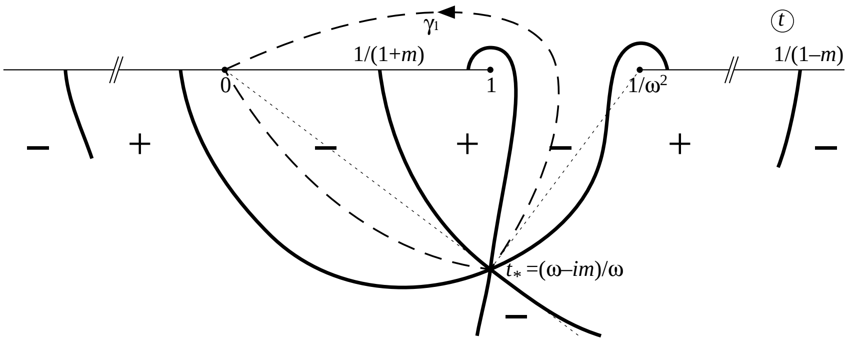

FIGURE 3. *The level set of the monkey saddle for* $\operatorname{Re} f_0$. *The line* $0t_*$ *is tangent to the level set at* t_*. $+$ *and* $-$ *denote regions, where* $\operatorname{Re} f(t)$ *is larger or smaller than* $\operatorname{Re} f(t_*)$ *respectively.*

It follows from (5.21) that the set of directions of the level lines in t_* is equal to $\{(\omega - im)e^{i\pi k/3} \mid k = 1, \dots, 6\}$. However, in order to identify, which of these directions corresponds to which region, we need some global information: on the segment that joins 0 to t_*, $\operatorname{Re} f_0(t)$ is increasing, because

$$2\frac{d}{d\lambda} \operatorname{Re} f_0(\lambda t_*) = \frac{\omega^2\,(1-\lambda)^3(1+\lambda)}{\lambda\,(\lambda^2 + \omega^2 - 2\lambda\omega^2)\,(1 + \lambda^2\omega^2 - 2\lambda\omega^2)} > 0$$

for $0 < \lambda < 1$. Therefore the level set does not meet this segment except at t_*, and the geometry is as shown in figure 3. Another argument that serves the same purpose is that the semicircle with diameter $[\frac{1}{1+m}, \frac{1}{1-m}]$ is part of the level set.

Figure 3 also shows the path γ_1, given already in figure 2, but deformed so that it satisfies the assumptions made in section 5.1.3. Figure 4 shows the splitting of the monkey saddle for small $\tau > 0$, as it is now obvious, using (5.18), (5.19).

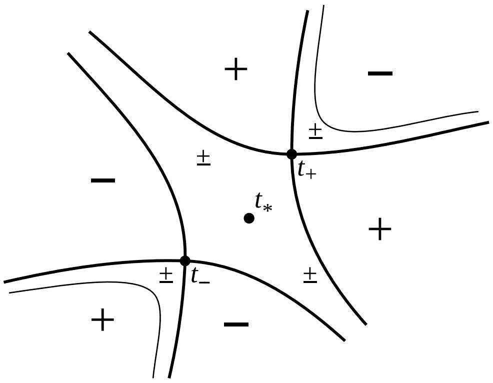

FIGURE 4. *The + (−) sign denotes regions where* $\operatorname{Re} f(t) > \operatorname{Re} f(t_+)$ *(*$\operatorname{Re} f(t) < \operatorname{Re} f(t_-)$*), the ± sign those, where* $\operatorname{Re} f(t_-) < \operatorname{Re} f(t) < \operatorname{Re} f(t_+)$.

We can now prove (5.2) and (5.3). Pull the factor $\exp(-\pi i n/2)$ out of the integral (5.6). Insert (5.19) into $4b^3 = f_0(t_+) - f_0(t_-)$ and (5.20) into $C^2 = -\frac{1}{2} f_0''(t_+)$; conclude $a_1 = f_0'''(t_+)/3f_0''(t_+)$ from (5.11). One gets:

$$
\begin{aligned}
3b &= 3\left(\tfrac{\omega}{6}\right)^{1/3} \tau^{1/2}\,(1+O(\tau)) \\
C &= (\omega^3/2)^{1/2}(m-i\omega)m^{-1}\tau^{1/4}\left(1+(\omega-im)\tau^{1/2}+O(\tau)\right) \\
\sqrt{3b}/C &= \left(\tfrac{6}{\omega}\right)^{1/3}\tfrac{m}{\omega}(m+i\omega)\left(1-(\omega-im)\tau^{1/2}+O(\tau)\right) \\
a_1 &= \tfrac{\omega}{3m}(m-i\omega)\tau^{-1/2} - \tfrac{4i\omega}{3m} + O(\tau^{1/2})
\end{aligned}
\tag{5.22}
$$

The sign of C has been determined by the condition that for $\tau \to 0+$, real positive v should correspond to t satisfying $\arg(t-t_+) \to \arg(m+i\omega)$, see (5.12) and figure 3.

Let $\tau =: \eta n^{-2/3}$ and remember that $\beta = bn^{1/3}$. According to lemma 12, we have $f_1(t) = -\ln t - \ln(T-t)$ for F_n and $f_1^+(t) = -\ln(1-t) - \ln(T-t)$ for F_n^+. So we get uniformly on compact sets of η :

$$
\begin{aligned}
\exp[nf_0(t_+) - 2\beta^3] &= \left(\tfrac{\omega}{m}\right)^n \exp[\tfrac{\eta}{2}n^{1/3}]\left(1+\tfrac{\eta^2}{4}n^{-1/3}+O(n^{-2/3})\right) \\
\exp[f_1(t_+)] &= i(m-i\omega)^2\omega^3/m\,\left(1+(\omega-im)\eta^{1/2}n^{-1/3}+O(n^{-2/3})\right) \\
\exp[f_1^+(t_+)] &= -i(m-i\omega)\omega^3/m^2\,\left(1+(2\omega-im)\eta^{1/2}n^{-1/3}+O(n^{-2/3})\right) \\
f_1'(t_+) &= -i\omega/m + O(n^{-1/3}) \\
f_1'(t_+) - a_1 + \tfrac{C}{(3b)^{3/2}} &= O(n^{-1/3}) \\
{f_1^+}'(t_+) &= -i\omega/m + (m-i\omega)\omega^2/m + O(n^{-1/3}) \\
{f_1^+}'(t_+) - a_1 + \tfrac{C}{(3b)^{3/2}} &= (m-i\omega)\omega^2/m + O(n^{-1/3})
\end{aligned}
\tag{5.23}
$$

Collecting these results in order to evaluate (5.13) with the help of lemma 14, formulas (5.2),(5.3) follow immediately from lemma 12.

The $O(n^{-2/3})$ term, which has not been evaluated explicitly, contains as a factor the integral (5.14) with $j = 2$. When η and hence β vanish, this integral also vanishes. Therefore, the error term is even $O(n^{-1})$ in this case. (5.4) follows immediately.

5.2. **Uniform Asymptotic Expansions.**

5.2.1. *Statement of the estimates.* For use in connection with (2.6), we claim

Lemma 15. *Let*

$$
v(\tau) := \left(\frac{3\omega}{4}\int_0^\tau \sqrt{\frac{\tau}{1-\tau m^2}}\,\frac{d\tau}{1-\tau}\right)^{2/3} = \left(\frac{\omega}{2}\right)^{2/3}\tau + O(\tau^2)
\tag{5.24}
$$

and

$$\tilde{K}_n(\tau) := \frac{F_n((1-\tau)z_0)}{F_n(z_0)}\,(1-\tau)^{n/2}\ . \tag{5.25}$$

Define $\tilde{K}_n^j(\tau)$ *and* $\tilde{K}_n^+(\tau)$ *by the same equation but with* $(2z\frac{d}{dz}+n)^j F_n(z)$ *or* $F_n^+(z)$ *in the numerator instead of* $F_n(z)$*. (Not in the denominator!) Then, the following estimate holds uniformly for* $0 \le \tau \le 1$ *(but the* O*-symbol might of course depend on* m*):*

$$\begin{aligned} \tilde{K}_n(\tau) &= A(\tau)\left(1+O(n^{-1})\right)\ , \\ \tilde{K}_n^+(\tau) &= 2\frac{1-m^2\tau}{1-\tau}\left(A(\tau)+\frac{\omega}{n^{1/3}}B(\tau)\right)\left(1+O(n^{-1})\right)\ , \end{aligned} \tag{5.26}$$

where

$$\begin{aligned} A(\tau) &= (1-\tau m^2)^{-3/4}\left(\frac{v(\tau)}{\tau}\right)^{1/4}\left(\frac{2}{\omega}\right)^{1/6}\frac{\mathrm{Ai}(n^{2/3}v(\tau))}{\mathrm{Ai}(0)}\ , \\ B(\tau) &= (1-\tau m^2)^{-5/4}\left(\frac{\tau}{v(\tau)}\right)^{1/4}\left(\frac{2}{\omega}\right)^{1/6}\frac{\mathrm{Ai}'(n^{2/3}v(\tau))}{\mathrm{Ai}(0)}\ . \end{aligned} \tag{5.27}$$

Moreover, for every j*, there exist* a_0*,* n_0*,* c_j *such that for* $n \ge n_0$*,* $a \ge a_0$ *and* $\tau \in [an^{-2/3}, 1]$*, one has the uniform estimates*

$$|\tilde{K}_n^j(\tau)| \le c_j\, n^{2j/3}\, \exp\left(-\frac{\omega}{3}a^{3/2}\right)\left(\frac{1-\tau}{1-an^{-2/3}}\right)^{n^{2/3}a^{1/2}\omega/2}\ . \tag{5.28}$$

c_j *can be taken independent of* a*, provided the latter is bounded away from 0.* $\tilde{K}_n^+(\tau)$ *satisfies also estimate (5.28) for* $j=0$*.*

The method employed to obtain the basic asymptotic expansion (5.26) is described in Cherry [4]. Therefore, we will skip straightforward calculations in the proof; the reader can insert them easily or consult Cherry's paper.

The results of the following lemma will follow mainly from differential inequalities and the maximum principle.

Lemma 16. *On the interval* $[z_0, 0]$*, all of* F_n*,* F_n^+*,* F_n'*,* ${F_n^+}'$ *are positive functions. For* n *sufficiently large,* F_n^+/F_n *is decreasing in this interval; its range is contained in* $[1, 2[$*.*

5.2.2. *Proof of lemma 15 for* $j = 0$*.* Using the hypergeometric differential equation satisfied by F_n, namely

$$z(1-z)F_n''(z) + (n(1-z)+(1-3z))\,F_n'(z) - \left(\left(\tfrac{n+2}{2}\right)^2 + \tfrac{n^2\omega^2-1}{4m^2}\right)F_n(z) = 0\ , \tag{5.29}$$

we conclude that $H_n(z) := (1-z)\,(-z)^{(n+1)/2}\,F_n(z)$ satisfies

$$H_n''(z) - \left(\frac{n^2}{4}\,\frac{1-z/z_0}{z^2(1-z)} - \frac{1-z/z_0}{4z^2(1-z)}\right)H_n(z) = 0\ , \tag{5.30}$$

where $z_0 = -m^2/\omega^2$. The factor $(-z)^{(n+1)/2}(1-z)$ killed the first derivative ([4],p.277). The part $(-z)^{n/2}$ of it is desirable anyway, if we have the lemma in mind.

Similarly, F_n^+ satisfies

(5.31)
$$z(1-z)F_n^{+\prime\prime}(z) + (n(1-z) + (2-3z))\,F_n^{+\prime}(z) - \left((\tfrac{n+2}{2})^2 + \tfrac{n^2\omega^2-1}{4m^2}\right) F_n^+(z) = 0$$

and then $H_n^+(z) := (1-z)^{1/2}\,(-z)^{(n+2)/2}\,F_n^+(z)$ satisfies

$$H_n^{+\prime\prime}(z) - \left(\frac{n^2}{4}\,\frac{1-z/z_0}{z^2(1-z)} + \frac{n}{2}\,\frac{1}{z^2(1-z)} + \frac{\omega^2 z - 1}{4m^2 z(1-z)^2}\right) H_n^+(z) = 0\ . \tag{5.32}$$

The term containing $n/2$ is not covered by Cherry's general framework, and we are lucky that the coefficient of H_n does not contain such a term. The following argument depends on this fact. The equations for F_n^+ and H_n^+ will be used only later in the proof of lemma 16.

Continuing towards a proof of (5.26), the next transformation will also simplify the coefficient of H_n (cf.[4],(2.3)). Let $z = z_0(1-\tau)$ and then apply the substitution

$$\tau = \phi(u)\ , \quad const\, H_n\big(z_0(1-\tau)\big) = h_n(u)\,\sqrt{\phi'(u)} \tag{5.33}$$

(the constant being determined later) given by

(5.34)
$$\begin{aligned} u &= \frac{\omega}{2}\int_0^\tau \sqrt{\frac{\tau}{1-\tau m^2}}\,\frac{d\tau}{1-\tau} \\ &= -\frac{\omega}{2m}\,\mathrm{atn}\,\frac{2m\sqrt{\tau(1-\tau m^2)}}{1-2\tau m^2} + \frac{1}{2}\ln\left(1 + \frac{2\omega}{1-\tau}\left(\omega\tau + \sqrt{\tau(1-\tau m^2)}\right)\right) \\ &= \frac{\omega}{3}\tau^{3/2}\left(1 + \frac{3(2+m^2)}{10}\tau + O(\tau^2)\right)\ . \end{aligned}$$

Then we obtain

$$h_n''(u) + (-n^2 + f(u))h_n(u) = 0\ , \tag{5.35}$$

$$f(\phi^{-1}(\tau)) = \frac{(1-\tau)^2\,[5 + 4\tau - 4m^2\tau(2+\tau)]}{4\tau^3(1-m^2\tau)\omega^2}\ . \tag{5.36}$$

A side effect of substitution (5.33) is that the regular point $\tau = 0$ is transformed into a regular singular point $u = 0$, but what is essential is that the leading term in n has simplified as much as possible. Moreover, a further substitution $v = (\frac{3}{2}u)^{2/3}$, $h_n(u) = \tilde{h}_n(v)v^{1/4}$ will transform $u = 0$ into a regular point $v = 0$ again ([4],(2.4)). The general scheme is made for uniform asymptotics as $u \to \infty$ (i.e. $z \to 0$), which is how far we want to integrate in (2.6). But as we want uniformity also up to $u = 0$, i.e. $z = z_0$, the singularity of f at $u = 0$ may become disturbing. Luckily, its leading order $5/36u^2$ is even desirable, and its lower orders are not too bad for our purpose. Indeed, according to lemma 2 in [4], if w, w_1 and w_2 solve

$$w'' + (-n^2 + \tfrac{5}{36u^2})\,w = 0\ , \tag{5.37}$$

and w_1 and w_2 are linearly independent, then any solution h of the integral equation

(5.38)
$$h(u) = w(u) + \frac{1}{W[w_1; w_2]}\int_u^\infty [w_2(u)w_1(t) - w_1(u)w_2(t)]\Big(f(t) - \frac{5}{36t^2}\Big)h(t)\,dt$$

solves equation (5.35). (A trivial sign error is in [4].) Here, $W[w_1, w_2] := w_1(u)w_2'(u) - w_2(u)w_1'(u)$ is the Wronskian (a non-zero constant), and as linearly independent solutions to (5.37), we choose $w_1(u) := K_*(nu) := (nu)^{1/2}K_{1/3}(nu)$, and $w_2(u) := I_*(nu) := (nu)^{1/2}I_{1/3}(nu)$. The Bessel functions $K_{1/3}$ and $I_{1/3}$ are discussed e.g. in section 9.6 of [1]. Their Wronskian $W[w_1, w_2] = n$. A further choice, $w(u) := w_1(u)$, selects a particular solution to (5.35), which will only later be exhibited to be the one we are looking for.

Letting $g(u) := f(u) - 5/36u^2$ and

$$\mathcal{K}(x,y) := K_*(x)I_*(x) - K_*(x)^2 \frac{I_*(y)}{K_*(y)} \qquad (y \le x) , \tag{5.39}$$

we get for this particular solution the integral equation

$$\frac{h_n(u)}{K_*(nu)} = 1 + \frac{1}{n}\int_u^\infty \mathcal{K}(nt,nu)\, g(t)\, \frac{h_n(t)}{K_*(nt)}\, dt . \tag{5.40}$$

Careful calculations with power series in τ and u yield from (5.34) and (5.36) that $g(u) = O(u^{-2/3})$ as $u \to 0$. On the other hand, $g(u) = O(u^{-2})$ as $u \to \infty$, because $f(u)$ decays exponentially fast as $u \to \infty$ (again from (5.34) and (5.36)). Now, for $0 \le y \le x$,

$$0 = \mathcal{K}(x,x) \le \mathcal{K}(x,y) \le \mathcal{K}(x,0) \to \frac{1}{2} \text{ as } x \to \infty , \quad \mathcal{K}(x,0) \to 0 \text{ as } x \to 0 .$$

Therefore, $\mathcal{K}(\cdot, nu)$ is uniformly bounded in the space $BC^0[0,\infty]$ and $g \in L^1[0,\infty[$. Hence the integral equation (5.40) can be solved by fixed point iteration in $BC^0[0,\infty]$, with the factor n^{-1} being responsible for the contraction, provided n is sufficiently large.

The unique solution obtained from (5.40) is a multiple of the particular solution h_n defined by (5.33), because it satisfies $h_n(u) = K_*(nu)(1+O(n^{-1}))$ uniformly on $[0,\infty[$ and is therefore bounded as $u \to \infty$, whereas every linearly independent solution to (5.35) is not. The constant factor can be calculated at $u = 0$.

This, together with an evaluation of the Bessel function in terms of the Airy function, namely $K_*(t) = \pi(12t)^{1/6}\,\mathrm{Ai}\big((\frac{3}{2}t)^{2/3}\big)$ according to 10.4.14 of [1], immediately implies (5.26) for $\tilde{K}_n(\tau)$ in lemma 15.

The corresponding estimate for $\tilde{K}_n^+(\tau)$ follows from

$$F_n^+(z) = z^{-(n+1)}(n+1)\int_0^z z^n F_n(z)\, dz ,$$

hence

$$\tilde{K}_n^+(\tau) = (1-\tau)^{-(n/2+1)}(n+1)\int_\tau^1 (1-\sigma)^{n/2}\tilde{K}_n(\sigma)\, d\sigma . \tag{5.41}$$

We know that $\tilde{K}_n(\sigma) = A(\sigma)(1+O(n^{-1}))$. For the purpose of integration by parts, use that

$$\begin{aligned} A'(\tau) &= \frac{O(1)}{1-\tau}A(\tau) + \frac{\omega n^{2/3}}{2(1-\tau)}B(\tau) , \\ B'(\tau) &= \frac{O(1)}{1-\tau}B(\tau) + \frac{\omega n^{4/3}}{2(1-\tau)}\,\frac{\tau}{1-\tau m^2}A(\tau) . \end{aligned} \tag{5.42}$$

Let us drop the n and τ, σ where possible. Insert (5.26) into (5.41) and note that the factor $1 + O(n^{-1})$ can be taken in front of the integral, because $A(\tau)$ does not change sign. Integrate by parts and absorb everything possible in the $O(n^{-1})$ symbol. This yields

$$\tilde{K}^+ = 2(1+O(n^{-1}))\left(A + \frac{\omega n^{2/3}}{2}(1-\tau)^{-n/2-1}\int_\tau^1 B(1-\sigma)^{n/2}\,d\sigma\right) . \tag{5.43}$$

In particular, from $B < 0 < A$ and $\tilde{K}^+ > 0$, we get $|\tilde{K}^+| \le O(1)\,A$. Let the second term inside the big parentheses of (5.43) be abbreviated as B^+. Again integrating by parts and simplifying, we get

(5.44)

$$B^+ = \frac{2}{n}(1+O(n^{-1}))\left(B + \frac{\omega n^{4/3}}{2}(1-\tau)^{-n/2-1}\int_\tau^1 (1-\sigma)^{n/2} A\frac{\sigma}{1-\sigma m^2}\,d\sigma\right) .$$

In this latter integral, a different integration by parts (integrate everything but $t/(1-\sigma m^2)$) introduces $\tilde{K}^+$ again. We get from (5.43) and (5.44):

(5.45)

$$\begin{aligned}\tilde{K}^+ = 2(1+O(n^{-1}))\Bigg[A &+ \frac{\omega}{n^{1/3}}B + \frac{\omega^2 n}{2}\frac{1}{n+1}\times\\ &\left(\frac{\tau(1+O(n^{-1}))}{(1-\tau m^2)}\tilde{K}^+ + (1-\tau)^{-n/2-1}\int_\tau^1 \frac{(1-\sigma)^{n/2+1}}{(1-\sigma m^2)^2}\tilde{K}^+\,d\sigma\right)\Bigg] .\end{aligned}$$

In the latter term, one can estimate $(1-\sigma)\tilde{K}_n^+(\sigma) \le O(1)(1-\tau)A(\sigma)$. Therefore we get

$$\tilde{K}^+\frac{1-\tau}{1-\tau m^2} = 2(1+O(n^{-1}))\left[A + \frac{\omega}{n^{1/3}}B + O(n^{-1})(1-\tau)\tilde{K}^+\right] ,$$

hence the desired estimate (5.26). Estimate (5.28) is proved in section 5.2.4.

5.2.3. *Proof of lemma 16.* In this proof, all statements on functions refer to the interval $[z_0, 0]$. We also suppress the subscript n.

The fact that F and F^+ are positive, is a consequence of the estimates just proved for $\tilde{K}$ and $\tilde{K}^+$, provided n is large enough. Alternatively, (5.30) implies that if the global minimum of H on $[z_0, 0]$ is in the interior, then its value is nonnegative. But $H(0) = 0$, hence it is taken on at the boundary. Similarly, the global maximum is taken on at the boundary. Depending on knowledge of the sign of $H(z_0)$, we can either conclude that H is nonnegative or nonpositive. The first alternative holds. The sign is known either from (5.26) or from local analysis as $z \to 0-$. In fact, H is strictly positive except at 0 due to the uniqueness theorem for ODEs as applied to an assumed interior point where H and H' vanish.

Therefore, F is strictly positive. The analogous argument works for F^+.

But $(1-z)F' = F + (n-1)F^+/(4m^2)$ according to (5.48) for $j = 1$; this implies that $F' > 0$.

From (5.29) and (5.31), one gets, letting $W := {F^+}'F - F^+F'$,

$$z(1-z)W' + (n(1-z) + (2-3z))\,W + F^+F' = 0 .$$

This implies that if we had $W(z^*) = 0$, then $W'(z^*) > 0$. As $W(0) < 0$, we would therefore get some positive local maximum z^{**} between z^* and 0, which contradicts the differential equation. Therefore, W does not change sign and stays negative. This in turn implies that $(F^+/F)' < 0$. In particular, F^+/F takes its values between those at 0 (namely 1) and at z_0 (namely the one given by (5.4), which is less than 2).

Using (5.48) again, we now find ${F^+}' = (n+1)(F - F^+)/z > 0$

5.2.4. *Uniform asymptotic estimates for the derivatives; proof of lemma 15 for* $j > 0$. In this section, we let $x := -z/((1-z)m^2)$, $z\frac{d}{dz} = (x - m^2x^2)\frac{d}{dx}$, and primes will denote derivatives with respect to x. Note that $x = 1$ corresponds to $z = z_0 = -m^2/\omega^2$, and to $\tau = 0$ in lemma 15.

We are going to use the following simple result:

Lemma 17. *Let $F_n(z)$, $F_n^+(z)$ be as in (5.1). Let $\mathbf{P}_0(x) :\equiv \mathbf{E}$ be the unit matrix and*

$$\mathbf{D}(x) = \begin{bmatrix} n - 2m^2x & -\frac{n-1}{2}x \\ 2(n+1) & -(n+2) \end{bmatrix} . \tag{5.46}$$

Define

$$\mathbf{P}_j(x) := \mathbf{P}_{j-1}(x) \cdot \mathbf{D}(x) + f(x)\mathbf{P}'_{j-1}(x) , \quad f(x) := 2(x - m^2x^2) . \tag{5.47}$$

Then,

$$\left(2z\frac{d}{dz} + n\right)^j \begin{bmatrix} F_n(z) \\ F_n^+(z) \end{bmatrix} = \mathbf{P}_j(x) \begin{bmatrix} F_n(z) \\ F_n^+(z) \end{bmatrix} . \tag{5.48}$$

Proof. For $j = 1$, this immediately follows from the formulas for derivatives of hypergeometric functions and the contiguous function relations (see [1], 15.2). The general case follows from a straightforward induction. $\square$

After these preparations, we are now ready to complete the proof of (5.28) for $j > 0$.

It is immediate from (5.47) that the entries of $\mathbf{P}_j(x)$ are polynomials in n of degree at most j. However it is crucial that many leading orders will cancel for $x = 1$, which is why (5.28) contains a factor $n^{2j/3}$ rather than n^j.

Let us Taylor–expand $\mathbf{P}_j$: With respect to x, $\mathbf{P}_j$ is also a polynomial of degree $\leq j$.

$$\mathbf{P}_j(x) = \mathbf{P}_j(1) + \mathbf{P}'_j(1)(x-1) + \cdots + \frac{1}{j!}\mathbf{P}_j^{(j)}(1)(x-1)^j \tag{5.49}$$

We need to know the highest power of n in $\mathbf{P}_j^{(k)}(1)$. Each $\mathbf{P}_j^{(k)}(x)$ can be written as a linear combination of products of matrices $\mathbf{D}(x)$ and $\mathbf{D}'(x)$, the coefficients being certain polynomials in x. Every such product can be written in the form

$$\mathbf{D}'^{\lambda_0}\mathbf{U}_{\mu_1}\mathbf{D}'^{\lambda_1} \ldots \mathbf{U}_{\mu_r}\mathbf{D}'^{\lambda_r} , \quad \text{where } \mathbf{U}_\mu(x) := (\mathbf{D}(x)\mathbf{D}'(x))^\mu \, \mathbf{D}(x) \tag{5.50}$$

and $\lambda_0, \ldots, \lambda_r, \mu_1, \ldots, \mu_r \geq 0$. We can assume that $\lambda_1, \ldots, \lambda_{r-1}$ are different from 1. (Else, replace $\mathbf{U}_\mu\mathbf{D}'\mathbf{U}_\nu$ by $\mathbf{U}_{\mu+\nu+1}$.)

For such a product to appear in $\mathbf{P}_j^{(k)}(x)$, it is necessary that

$$2\lambda_0 + \cdots + 2\lambda_r + 3\mu_1 + 1 + \cdots + 3\mu_r + 1 \leq j + k , \tag{5.51}$$

as one can see by induction. The condition means that in each step of either using (5.47) or taking a derivative, we get at most either one extra matrix $\mathbf{D}(x)$ in the product or else one prime attached to such a matrix. For terms that contain derivatives od f, (5.51) is a strict inequality.

Next, write $\mathbf{D}(1) =: n\mathbf{D}_1 + \mathbf{D}_0$, $\mathbf{D}'(1) =: n\mathbf{D}'_1 + \mathbf{D}'_0$ with

$$\mathbf{D}_1 = \begin{bmatrix} 1 & -1/2 \\ 2 & -1 \end{bmatrix} , \quad \mathbf{D}'_1 = \begin{bmatrix} 0 & -1/2 \\ 0 & 0 \end{bmatrix} , \quad {\mathbf{D}_1}^2 = {\mathbf{D}'_1}^2 = 0 . \tag{5.52}$$

$j+k$ mod 3	d	contributions to $\mathbf{P}_j^{(k)}(1)$ only from $r, \{\lambda_i\}$	contribute to $(2z\frac{d}{dz}+n)^j F_n$
1	$\frac{2(j+k)+1}{3}$	2 {0,0}	$(\mathbf{D}_1\mathbf{D}_1')^{(j+k-1)/3}\mathbf{D}_1$ $N_1(-2\omega^2)^{(j+k-1)/3}(F_n - F_n^+/2)$
0	$\frac{2(j+k)}{3}$	2 {0,1}	$(\mathbf{D}_1\mathbf{D}_1')^{(j+k)/3}$ and $(\mathbf{D}_1'\mathbf{D}_1)^{(j+k)/3}$ $N_2(-2\omega^2)^{(j+k)/3}F_n^+$ and $N_3(-2\omega^2)^{(j+k)/3}(F_n - F_n^+/2)$
-1	$\frac{2(j+k)-1}{3}$	don't care; d is too small	

TABLE 1. *Leading order in n of $\mathbf{P}_j^{(k)}(1)$. The N_i are nonnegative integers (combinatorial coefficients).*

$\mathbf{U}_\mu$ can contribute at most order $n^{2\mu+1}$ to $\mathbf{P}_j^{(k)}(1)$, whereas $\mathbf{D'}^\lambda$ can at most contribute $n^{\lceil\lambda/2\rceil}$, because ${\mathbf{D}_1'}^2 = 0$. Moreover, if one of $\lambda_1, \dots, \lambda_{r-1}$ vanishes, we get a factor $\mathbf{D}^2(1)$. In this case, the degree of at least one adjacent factor $\mathbf{U}_\mu$ has been overestimated by 1, because $\mathbf{D}_1^2 = 0$. Let therefore $s := \#\{i \mid 1 \le i \le r-1;\ \lambda_i = 0\}$. We have to maximize the degree, which is bounded by

$$d := 2\mu_1 + 1 + \cdots + 2\mu_r + 1 + \left\lceil \frac{\lambda_0}{2} \right\rceil + \cdots + \left\lceil \frac{\lambda_r}{2} \right\rceil - \left\lceil \frac{s}{2} \right\rceil , \tag{5.53}$$

subject to the condition (5.51). From (5.53) and (5.51), one has

$$3d - 2(j+k) + 1 \le \sum_{i=0}^{r} \left(3\left\lceil \frac{\lambda_i}{2} \right\rceil - 4\lambda_i + 1 \right) - 3\left\lceil \frac{s}{2} \right\rceil . \tag{5.54}$$

One can now easily enumerate that the maximal degree d in n is as given in table 1. In particular, note that according to (5.47) a factor $f(1) = 2\omega^2$ goes with every $\mathbf{D}_1'$, and that products going with $f'(1)$ can only affect $O(n^{(2(j+k)-1)/3})$. Using this result with (5.48) and (5.49), one finds

$$\begin{aligned} \left(2z\frac{d}{dz} + n\right)^j F_n(z) = n^{2j/3} \Bigg(& P_0 F_n(z) + Q_0 F_n^+(z) + \\ & P_1 n^{1/3} \left(F_n(z) - \frac{1}{2}F_n^+(z) \right) + n^{-1/3} \left(P_{-1}F_n(z) + Q_{-1}F_n^+(z) \right) \Bigg) , \end{aligned} \tag{5.55}$$

where the P_i and Q_i depend on j and are polynomials in the variables $n^{2/3}(x-1) = -\omega^2 n^{2/3}\tau/(1-m^2\tau)$ and $n^{-1/3}$. Their degree in the first variable is at most j, which is the maximum of all k. All of them can therefore be bounded by $c\,[(n^{2/3}\tau)^j + 1]$. Consequently, using (5.55) and (5.26),

$$|\tilde{K}_n^j(\tau)| \le c_j\, n^{2j/3}[(n^{2/3}\tau)^j + 1](1-\tau)^{n/2-1} \Big(|A(\tau)|(1 + n^{1/3}\tau) + |B(\tau)| \Big)$$

uniformly in $\tau \in [0,1]$. ($\tilde{K}_n^+$ is also covered by $j = 0$.) Using (5.27) and the asymptotic expansions $\mathrm{Ai}(z) \sim (4\pi)^{-1/2} z^{-1/4} \exp(-\frac{2}{3}z^{3/2})$, $\mathrm{Ai}'(z) \sim -(4\pi)^{-1/2} z^{1/4} \exp(-\frac{2}{3}z^{3/2})$, as $z \to +\infty$ ([1],10.4.59,61), we get uniformly on

$\tau \in [an^{-2/3}, 1]$:

$$|\tilde{K}_n^j(\tau)| \le c_j\, n^{2j/3}[(n^{2/3}\tau)^j \exp(-\tau n/2)](n^{2/3}\tau)\exp[-2nv(\tau)^{3/2}/3]\ ,$$

and using the estimate

$$\int_0^\tau \sqrt{\frac{\tau}{1-\tau m^2}}\,\frac{d\tau}{1-\tau} \ge \int_0^\delta \tau^{1/2} d\tau + \sqrt{\delta}\int_\delta^\tau \frac{d\tau}{1-\tau}$$

for $\tau \ge \delta := n^{-2/3}\tau$, we conclude (5.28)

6. Nonpersistence from the Asymptotic Behaviour of the Obstruction

6.1. **Abel type asymptotics.**

Lemma 18. *Suppose the Laplace distribution $G(t)$ is in fact in $L^1_{\mathrm{loc}}[0,\infty[$, $G(t) \sim r\,t^{\alpha-1}$ as $t \to 0$, and $\alpha > 0$, $r \neq 0$. Then*

$$\int_0^\infty G(t)\left(-\frac{d}{dt}\right)^j \left(\frac{e^{-nt}F_n(e^{-2t}z_0)}{F_n(z_0)}\right) dt \sim r\,n^{-2(\alpha-j)/3}\,g(\alpha-j)\,\Gamma(\alpha) \tag{6.1}$$

with

$$g(\alpha) = \Big(\frac{1}{6\omega^2}\Big)^{\alpha/3}\,\frac{\Gamma(2/3)}{\Gamma\big((\alpha+2)/3\big)}\ . \tag{6.2}$$

Note: Let R_n, be the obstruction corresponding to $G^{(j)}(t)$ according to (2.6), and R_n^1 the one corresponding to $G(t) = \delta(t)$. Then, (6.1) is just R_n/R_n^1.

Proof of the lemma. With no loss of generality let $r = 2^{\alpha-1}$. Substituting $e^{-2t} = 1-\tau$ in (6.1), we find

$$n^{2(\alpha-j)/3}\frac{R_n}{R_n^1} = \frac{1}{2}n^{2(\alpha-j)/3}\int_0^1 \tau^{\alpha-1}\tilde{G}(\tau)\,\tilde{K}_n^j(\tau)\,d\tau\ , \tag{6.3}$$

where $\tilde{G}(\tau) := \tau^{1-\alpha}G(-\frac{1}{2}\ln(1-\tau))/(1-\tau) \to 1$ as $\tau \to 0$ and $\tilde{K}_n^j(\tau)$ is defined and estimated in lemma 15.

We will show below that for all $\varepsilon > 0$, there exists an a such that

$$\limsup_{n\to\infty} n^{2(\alpha-j)/3}\left|\frac{1}{2}\int_{n^{-2/3}a}^1 \tau^{\alpha-1}\tilde{G}(\tau)\tilde{K}_n^j(\tau)\,d\tau\right| \le \varepsilon\ . \tag{6.4}$$

In the remaining part of (6.3) substitute $\eta := n^{2/3}\tau$ to get

$$\frac{1}{2}\int_0^a \eta^{\alpha-1}\tilde{G}(n^{-2/3}\eta)n^{-2j/3}\tilde{K}_n^j(n^{-2/3}\eta)\,d\eta\ . \tag{6.5}$$

According to lemma 15, we have the limits (as $n \to \infty$)

$$\tilde{K}_n^+(n^{-2/3}\eta) \to 2\,\mathrm{Ai}\left((\tfrac{\omega}{2})^{2/3}\eta\right)/\,\mathrm{Ai}(0)$$
$$n^{1/3}\big(\tilde{K}_n(n^{-2/3}\eta) - \tfrac{1}{2}\tilde{K}_n^+(n^{-2/3}\eta)\big) \to -(2\omega^2)^{1/3}\,\mathrm{Ai}'\left((\tfrac{\omega}{2})^{2/3}\eta\right)/\,\mathrm{Ai}(0)$$

uniformly in $\eta \in [0,a]$, and also $\tilde{G}(n^{-2/3}\eta) \to 1$ in $L^\infty[0,a]$.

Then, with (5.48) and table 1, we can let $n \to \infty$ in (6.5) and then $\varepsilon \to 0$ and find that the expression in (6.3) converges to

$$\left(\frac{2}{\omega}\right)^{2\alpha/3}\left[C_1(j,\omega)\int_0^\infty w^{\alpha-1}\,\mathrm{Ai}(w)\,dw + C_2(j,\omega)\int_0^\infty w^{\alpha-1}\,\mathrm{Ai}'(w)\,dw\right]/\,\mathrm{Ai}(0)\ ,$$

where $C_1(0,\omega) = 1/2$ and $C_2(0,\omega) = 0$, as can be traced through the argument easily. The coefficients for $j > 0$ are hidden behind the combinatorial factors in table 1. But we certainly get (6.1) with some $g(\alpha, j)$ instead of $g(\alpha - j)$, and this $g(\alpha, j)$ is analytic in α.

Using a consequence of 6.561.16 in [11] and the triplication formula for the Γ function, namely

$$\int_0^\infty w^\alpha \frac{\mathrm{Ai}(w)}{\mathrm{Ai}(0)}\, dw = 3^{(2-\alpha)/3}\Gamma\left(\frac{2}{3}\right) \frac{\Gamma(\alpha)}{\Gamma(\alpha/3)}\,, \tag{6.6}$$

we get $g(\alpha, 0) = g(\alpha)$ as given in (6.2).

We can avoid a more detailed calculation for $g(\alpha, j)$: Use the lemma for $j = 0$ with $G(t) = rt^{\alpha-1}$, $\alpha = \beta - j_0 > 0$, $r = (\beta-1)\cdots(\beta-j_0)$. The result must be the same as for $G(t) = t^{\beta-1}$, $r = 1$, $j = j_0$. This implies $g(\beta,0)\Gamma(\beta) = (\beta-1)\cdots(\beta-j_0)g(\beta-j_0,j_0)\Gamma(\beta-j_0)$, hence $g(\beta,j) = g(\beta-j)$ for $\beta > j$. Analytic continuation shows the same for $\beta > 0$.

We still have to show (6.4). To this end, estimate $\tilde{G}(\tau)$ by $c\,(1-\tau)^{-A}$, and $(1-\tau)/(1-an^{-2/3})$ in (5.28) by $\exp[-(\tau - an^{-2/3})/(1-an^{-2/3})]$. An upper estimate for the term in (6.4) is therefore

$$c\,(1-an^{-2/3})^{-A}\exp[-\omega a^{3/2}/3]\int_a^\infty \eta^{\alpha-1}\exp[(a-\eta)(a^{1/2}\omega/2 - An^{-2/3})]\, d\eta\,,$$

which immediately gives the result. □

6.2. **Proof of theorem 1.** Let Δ be of the form (1.2) and m fixed. Considering e.g. odd n (and thus the odd part of Δ), let α be the biggest of all α_j occurring and $s = s_j \neq 0$ the corresponding coefficient in (1.2). With lemma 10, A_p is in the range of the distributional Laplace transform, and we get a nonvanishing leading term in the asymptotic behaviour of of $G(t)$ as $t \to 0$. Lemma 18 can be applied, showing that the asymptotic behaviour of R_n as $n \to \infty$ is

$$\frac{R_n}{R_n^1} \sim \frac{-3s\sin\frac{\pi(\alpha+1)}{3}}{\pi}\,\frac{\Gamma(\frac{2}{3})\Gamma(\frac{\alpha+1}{3})}{\Gamma(\alpha+1)}\,(6\omega^2)^{\alpha/3-1}2^{\alpha+1}(\alpha-2)\,n^{2\alpha/3-2}\,.$$

Unless $\alpha \equiv 2 \bmod 3$, the coefficient is nonzero, so that R_n cannot vanish identically. If however $\alpha = 3\mu + 2$, we subtract from Δ an appropriate multiple of the trigonometric polynomial S^μ, given in (3.4), which does not change R_n. Repeating the argument with the new Δ, one concludes that the odd part of Δ must be a linear combination of the S^μ, if $R_n = 0$ for odd n and one fixed m. This already accounts for the second part of theorem 1; for the first part, we only must show that the spaces spanned by the sequences $\sigma_\mu(p)$ for different m have zero intersection (separately in the even and odd case). For the functions $\Delta(u)$ this leaves $\sin u$ and $\cos u$ in the intersection (and two functions that are not trigonometric polynomials), but for $\cos u$, (1.1) cannot be satisfied as $x \to \pm\infty$. According to (3.3), the highest three powers in p of any polynomial that is a linear combination of $\sigma_\mu(p)$ depend only on the single contribution of the highest μ, and the relation of the coefficients of $p^{3\mu}$ and $p^{3\mu-1}$ in σ_μ uniquely determines $Z = \frac{m^2}{\omega^2}$ and thus m. This completes the proof. □

Obviously, our proof goes through for any perturbation functions for which $\mathrm{A}_p = p^j\int_0^\infty G(t)\,dt$ with a function $G(t) = st^{-\alpha} + O(t^\beta)$ and some $-1 < -\alpha < \beta$. In particular, *all* rational functions A_p are included in the theorem, not only the ones that come from 4π–periodic trigonometric polynomials. As a sample, this includes $\mathrm{A}_p = 4/(p^2-1)$, the sequence corresponding to

$$\Delta(u) = 3\sin\frac{u}{2} - 2\sin u \ln\cos\frac{u}{4} \ .$$

Using lemma 10 amounts to rescuing the asymptotic information of A_p as $p \to \infty$, through the inverse Laplace transform. This was possible here by means of explicit calculation of the inverse Laplace transform.

6.3. **Concluding Remarks and Open Problems.** The results of this paper concern a borderline situation from which different paths can be pursued. If one is willing to make the assumption that the (processed) Taylor coefficients A_p of the perturbation function are in the range of the Laplace transform, theorem 1 describes a case where this Taylor series has radius of convergence 1. For perturbations chosen "at random", this is the "most likely" situation because of the factor $1/(1+z^2)$ introduced in (2.1). It is however a degenerate choice from the point of view of the asymptotic behaviour of R_n, due to the monkey saddle determining this behaviour. It should therefore be even easier to do an asymptotic analysis for bigger (or smaller) radii of convergence (with a nondegenerate saddle), but the author has not carried this out, believing that perturbation functions like (1.2) are a more natural choice.

On the other hand, one might wish to use the full information $R_n \equiv 0$ instead of the asymptotic behaviour only. A conjecture would be that for functions $G \in C^0[0,\infty[$ (growing at most exponentially), the transformation (2.6) is injective. This would easily make theorem 1 cover much more perturbations. An approach along the lines of [13] might be promising, but the author has not succeeded by now. Note that such an injectivity result would not contradict theorem 5, because for the functions S^μ, C^μ given there, G is a distribution supported in $\{0\}$.

In principle, one could try to check the second order conditions for persistence of a breather for the functions of theorem 5 (equations (4.4) and (3.17) of [8] can immediately be used). However, the exact Taylor coefficients of the higher orders cannot be dispensed with in such an attempt, unlike in the case where persistence of a whole family of breathers is studied.

As a consequence of theorem 5, any function of the form $\sum_0^\infty(a_\mu S^\mu(u)+b_\mu C^\mu(u))$ for which the series converges uniformly for real u will admit a solution to (1.1) for the value of m selected. This is immediate from the integral form $\int \Delta(u^*)\chi_n\,dxdt = 0$ of the solvability conditions, as explained in [2] or [7]. Fine tuning of the coefficients will produce non-analytic examples. Up to now, analyticity has been crucial for the arguments.

As soon as one gives up analyticity of $\Delta(\cdot)$, one may assume 4π–periodicity instead for free, because the persistence conditions (in integral form, for any order of ε) depend only on how Δ behaves on the range of the breather. This is the main reason, why the author believes 4π–periodic trigonometric polynomials to be a natural object of study in this context; they are the easiest case of theorem 1.

The author is not aware how crucial the assumption that (A_p) can be inverse Laplace transformed will be in the end. By a different fine tuning of coefficients, one may find analytic $\sum(a_\mu S^\mu + b_\mu C^\mu)$ for which A_p grows slower than any exponential, but faster than any polynomial. It is dubious whether such a sequence can be incorporated in (2.5), whereas the summands can. Is the condition introduced by the inverse Laplace transform effectively immaterial, because the forward transform

(2.6), whose kernel has a similar qualitative behaviour as the Laplace kernel, follows it immediately?

Transforms similar to (2.6) with ${}_2F_1$ in the kernel have been considered in the literature. In [12], the transformed variable appears in 2 parameters (as opposed to 3, plus an extra factor, here). See also [18],[19]. Apparently more commonly considered are transforms where the transformed variable appears in the *argument* of the hypergeometric function rather than in the parameters. See e.g. [16]. The author is not aware of a reference treating the case needed here.

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For the readers' convenience: the above are referred to in the following sections: [1]*: 5.1.2, 5.1.3, 5.2.2, 5.2.4 —* [2]*: 1, 2, 3.1, 6.3 —* [3]*: 5.1.3 —* [4]*: 5.2.1, 5.2.2 —* [5]*: 5.1.3 —* [6]*: 3.3 —* [7]*: 1, 2, 3.1, 6.3 —* [8]*: 3.1, 6.3 —* [10]*: 5.1.3 —* [11]*: 6.1 —* [12],[13]*:*

6.3 — [14],[15]*: 5.1.3* — [16]*: 6.3* — [18],[19]*: 6.3* — [9],[17]*: general ref. for Laplace transform*

As the items [12] *and* [19] *may be difficult to obtain in many libraries, some readers may find the e-mail addresses of the respective authors useful, namely:* `bgonzalez@ull.es, enegrin@ull.es, nhayek@ull.es` *and* `semen@mmf.bsu.minsk.by` *respectively. The e-mail address of the Revista Técnica is* `retecin@europa.ica.luz.ve`.

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