NONLINEAR DIFFUSION FROM A DELOCALIZED SOURCE: AFFINE SELF-SIMILARITY, TIME REVERSAL, & NONRADIAL FOCUSING GEOMETRIES*

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ABSTRACT. A family of explicit solutions is described, to the porous medium equation in its full range of nonlinearities (plus some analogous fourth-order diffusions), in which the pressure is given by a quadratic function of space at each instant in time. These include spreading solutions whose source is concentrated on any conic region of dimension lower than the ambient space, and solutions which focus at conic regions. The singular limiting distributions are affine projections of Barenblatt type solutions (with arbitrary signature) onto lower dimensional subspaces. All affine images of Barenblatt solutions form an invariant space on which the dynamics can be integrated explicitly. A time-reversal symmetry is revealed for the pressure equation which transforms spreading solutions to focusing solutions, and vice-versa. This yields new information about the long and short time asymptotics of finite-mass solutions, about the instability of focusing, and about singularity geometry.

RÉSUMÉ. On décrit une famille de solutions explicites pour l'équation des milieux poreux pour toute l'échelle des nonlinéarités (ainsi que pour d'autres équations de diffusion analogues de quatrième degré). Pour ces solutions, la pression à chaque instant est une fonction quadratique en $\boldsymbol{x}$. Certaines de ces solutions sont diffusées à partir d'une source concentrée sur un domaine conique arbitraire de dimension inférieure à celle de l'espace ambient; d'autres se focalisent sur des régions coniques. Pour les solutions focalisantes, on obtient à la limite une distribution singulière qui est une projection affine d'une solution de type Barenblatt (à signature arbitraire) sur un espace de dimension inférieure. L'ensemble des images affines des solutions de type Barenblatt est un espace invariant, sur lequel la dynamique peut s'intégrer explicitement. Par un changement de variables qui inverse le sens du temps, on arrive à transformer une solution diffusive en une solution focalisante. Ceci donne de nouvelles informations sur l'asymptotique en temps court et long pour des solutions de masse finie, sur l'instabilité des solutions focalisantes, ainsi que sur la géométrie des singularités.

1. INTRODUCTION

Since the 1950s, the nonlinear diffusion equation

$$\frac{\partial \rho}{\partial t} = \frac{1}{m}\Delta(\rho^m) \tag{1}$$

has been a central model for strongly viscous spreading phenomena. With various boundary conditions and different interpretations for the evolving density $\rho(\boldsymbol{x},t) \geq 0$ on $\mathbf{R}^n \times [0,\infty[$, this *porous medium* equation has been used to model population spreading, fluid seeping through soil ($m = 2$), creeping thin films ($m = 4$), heat propagation in plasmas ($m = 6$), avalanches in sandpiles ($m = 1 - \frac{n}{2}$), and geometrical curvature flows ($m = \frac{n-2}{n+2}$). An understanding of this equation is now well-developed; with $m > 1$, its phenomenology includes finite speed propagation of fluid into dry regions (or heat into cold regions), contrasting sharply with the linear diffusion $m = 1$. The radial self-similar solution to these equations

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discovered long ago [7] [48] enjoys a prominent role in this theory: it models the eventual behavior of finite-mass spreading solutions, whether or not they enjoy spherical symmetry initially [28]. See Barenblatt [8] and Vázquez [43] [44] for reviews.

More recently, (1) has emerged as a model for focusing phenomena, like the filling in of a dry spot by thin lava-like films in the viscous gravity current experiments of Diez et al [27], or the eventual disappearance of groundwater from soil into fissures in the filtration-absorption model of Barenblatt et al [9]. For viscous gravity currents, careful experiments supported by detailed numerics [27] [5] leave little doubt that typical holes become more and more elongated as they shrink, eventually disappearing non-selfsimilarly. However, no explicit solutions have been known with this form, so one has only a formal asymptotic construction called the *closing-eye solution* by Angenent et al [5] as a model. (Self-similar solutions with k-fold axial symmetry were shown to exist for $k \geq 3$ and $m > m_k > 1$ sufficiently large by Angenent and Aronson [4]; they are not explicit, and they capture only the bifurcation from roundness [6] and not the generic disappearances observed, which correspond rather to a $k = 2$ instability for all $m > 1$).

The present manuscript describes closed-form solutions to (1) which are not radially symmetric, but enjoy a property we call *affine* self-similarity, meaning the density at each instant is a sheared image of its initial profile. These solutions were discovered independently some twenty years ago by Luc Tartar, who described them in an unpublished manuscript [39], which we learned of only after this work had been largely completed; his solutions are also documented and developed in the forthcoming book of Vázquez [45]. In this respect, as in others, Tartar was ahead of his time; many of the applications we discuss, which enhance the relevance of these solutions, had not yet been conceived. Like Tartar's, our solutions are obtained by quadrature; in suitable coordinates they remain reflection symmetric with respect to each spatial variable, and form an $\frac{n(n+1)}{2}$ parameter family whose evolution is completely encoded in the finite-dimensional dynamics of the shearing maps. Their level sets remain conic sections (ellipsoids or hyperboloids) for as long as the solutions exist. The spatial dimensions of these conics evolve at different rates, given by a coupled nonlinear system of first order differential equations (8), until one or more of these dimensions degenerates to zero. The number and signature of the dimensions which collapse, and their relative rates, determine the geometry of the focusing singularity if it occurs. These rates are not exact power laws, except in the classical case of spherical focusing or spreading solutions, but they are asymptotic to power laws at the time of collapse, with scaling exponents that we calculate. In suitable variables the affine dynamics can actually be integrated independently of m, whose effect is therefore reduced to prescribing how the affine orbits are parameterized.

Unfortunately, these affinely self-symmetric solutions have no nontrivial continuation beyond the time of the singularity, so do not replace the solutions of Graveleau [30] and Angenent et al [5] as models for hole-filling in the experiments of Diez et al [27]. However, they provide appropriate solutions to the filtration-absorption problem of Barenblatt et al [9], since our solutions remain zero after the last of the groundwater drains into the fissures. Moreover, our solutions predict that any perturbation from initial roundness will be amplified as the fluid drains, leading to a last drop of fluid which becomes more and more elongated as it is absorbed, eventually disappearing in the form of a flattened ellipsoid, with one axis collapsed and the others having generically different positive lengths. This reflects the fact that the smallest dimension collapses faster than the others, since surface tension is absent from the model. This is in marked contrast to the *linearized* stability result for circular drainage asserted by Chertock [21].

These affinely self-similar solutions also provide new information concerning the long-time asymptotics of the spreading solution, a subject of considerable interest; see for example the works [2] [18] [15] [17] [24] [38] [32] [35] [36] [40] of Angenent, Carrillo, Del Pino, Di Francesco, Dolbeault, Jüngel, Kim, Markowich, Otto, Slepčev, Toscani, Unterreiter, and Vázquez. In the porous medium regime $m > 1$, they suggest that normalized initial profiles with compact support and the same center of mass evolve toward each other at rate $t^{-2/(2+(m-1)n)}$ in $L^1(\mathbf{R}^n)$ — the square of the rate for uncentered solutions [16] [38] [36] — and certainly no faster than that in dimensions $n > 1$. This rate is consistent with the extrapolation of the linearized dynamics we computed in [25] for the fast diffusion equation $1 - 2/n < m < 1$, but due to an eigenvalue crossing at $m = 1$ it is slower than the $1/t$ rate established at both extremes [33] [35] of that regime. It also explains Angenent & Aronson's observation, that for nonradial solutions with $m \geq 1$ [3], dimensions $n \geq 2$ are distinguished from one-dimension [47] [42] [2], in that no choice of delay t_0 improves the rate of decay beyond that obtained by centering the data.

A final section discusses analogous solutions to certain fourth-order diffusion equations (22) and (27) which parallel the porous medium family. These equations arise as gradient flows of the generalized Fisher information (28) with respect to the 2-Wasserstein distance. In particular, the special case $m = 1$ is a quantum drift diffusion model studied by Jüngel & Pinnau in one-dimension (see [31] for references); it is also the generalization to higher dimensions of a model proposed by Derrida, Lebowitz, Speer & Spohn for the statistical mechanical fluctuations around a pinned interface [26].

2. Solution concept

The evolution (1) can also be written as a conservation law

$$\frac{\partial \rho}{\partial t} - \nabla \cdot (\rho \nabla \pi) = 0, \tag{2}$$

where $\pi = \gamma \rho^{1/\gamma}$ is the *pressure* and $\gamma = 1/(m-1)$ is the adiabatic exponent of the gas (also called the polytropic index, but not consistently through the literature). The pressure satisfies

$$\frac{\partial \pi}{\partial t} = \frac{1}{\gamma} \pi \Delta \pi + |\nabla \pi|^2 \tag{3}$$

whenever the density $\rho(\boldsymbol{x}, t)$ satisfies (1). The unusual normalization factor γ in $\pi = \gamma \rho^{1/\gamma}$ is convenient both from the following mathematical and physical points of view: It allows us to avoid case distinctions between the porous medium ($m > 1 \Leftrightarrow \gamma > 0$) and fast diffusion ($m < 1 \Leftrightarrow \gamma < 0$) regimes, while guaranteeing that matter moves from higher to lower pressures independently of the sign of the pressure.

We argue that the truncation $\pi_* = \gamma(\pi/\gamma)_+ := \gamma \max\{\pi/\gamma, 0\}$ of any smooth solution $\pi(\boldsymbol{x}, t)$ also represents a physically meaningful notion of solution for (3), provided $\nabla \pi$ does not vanish on the zero set of π, and hence leads to a non-negative solution ρ for (1): Indeed, if π solves (3), then the zero set $\mathcal{Z}_t := \{\boldsymbol{x} \in \mathbf{R}^n \mid \pi(\boldsymbol{x}, t) = 0\}$, which is a manifold if $\nabla \pi \neq 0$, moves in normal direction with velocity $-\pi_t \nabla \pi / |\nabla \pi|^2 = -\nabla \pi$, because $\pi \Delta \pi = 0$ on the interface. Moreover, it is exactly consistent with Darcy's law as expressed in (2), so the support changes exactly as it should under Darcy's law.

We are not purporting any general theory for (3), which is parabolic in forward time only where $\pi/\gamma > 0$, and parabolic in backwards direction only where $\pi/\gamma < 0$ and therefore ill-posed in the absence of a sign condition on the data. Rather, we construct non-local source solutions for (1) in the physically meaningful range $\pi/\gamma > 0$ from explicit (quadratic) solutions to (3) by means of truncation of a sporadic classical solution to (3). Or equivalently, by time reversal $-\pi(\boldsymbol{x}, -t) =: \bar{\pi}(\boldsymbol{x}, t)$, solutions with $\pi/\gamma < 0$ which evolve backwards in time.

Example 1 (Focusing by time-reversed Barenblatt)**.** *The $\gamma > 0$ self-similar solution of Barenblatt* [7]*, Pattle* [37]*, Zel'dovich & Kompaneets* [48] *corresponds to (the positive truncation of) a 'downwardly oriented' quadratic pressure*

$$\pi_b(\boldsymbol{x}, t) = \frac{b\gamma}{|t|^{n/(n+2\gamma)}} - \frac{\gamma}{(n+2\gamma)} \frac{|\boldsymbol{x}|^2}{|2t|}, \qquad t > 0. \tag{4}$$

The mass of $\rho = (\pi_b/\gamma)_+^\gamma$ is determined by $b > 0$; its zero set spreads at rate $|\boldsymbol{x}| \sim t^{\gamma/(n+2\gamma)}$. Its negated time-reversal

$$\bar{\pi}_b(\boldsymbol{x}, t) = -\frac{b\gamma}{|t|^{n/(n+2\gamma)}} + \frac{\gamma}{(n+2\gamma)} \frac{|\boldsymbol{x}|^2}{|2t|}, \qquad t < 0$$

is an 'upward' quadratic pressure, whose positive truncation vanishes in a ball shrinking to zero like $|t|^{\gamma/(n+2\gamma)}$ at the moment $t \to 0$ of the focusing singularity.

Note that the time-reversed solution described above has no non-trivial continuation beyond the singularity, so is a poor model for hole filling. Nevertheless, it is important to note that the focusing geometry is unstable precisely because it is the backwards in time flow of the spreading solution, which was shown by Friedman & Kamin [28] and Vázquez [44] to be a global attractor. In particular this implies that perturbations from sphericity will increase.

The partial differential equation $\partial u/\partial t = u\Delta u + \gamma|\nabla u|^2$, which is simply (3) with $u = \pi/\gamma$, has been studied in the fast diffusion range $\gamma \le 0$ by Bertsch, Dal Passo & Ughi [10] [12] [11], using methods from Dal Passo & Luckhaus [23], and earlier in one dimension in [41]. (Their γ is our $-\gamma$). They define a nonnegative function u as a weak solution for initial value $0 \le u_0 \in L^\infty_{\rm loc} \cap C(\mathbf{R}^n)$ if

$$(5)\qquad \begin{aligned} &u \in L^\infty_{\rm loc}([0,\infty[\times \mathbf{R}^n) \cap L^2_{\rm loc}\Big([0,\infty[\to W^{1,2}_{\rm loc}(\mathbf{R}^n)\Big) \quad \text{and} \\ &\int u_0\psi(0,\cdot)\,d\boldsymbol{x} + \iint \Big(u\frac{\partial\psi}{\partial t} - u\nabla u\cdot\nabla\psi + (\gamma-1)|\nabla u|^2\psi\Big)\,d\boldsymbol{x}dt = 0 \end{aligned}$$

for all $\psi \in C^{1,1}([0,\infty[\times\mathbf{R}^n)$ with compact support in $[0,\infty[\times\mathbf{R}^n$. This definition makes sense for either sign of γ. We prove in an appendix that the solutions we find are weak solutions in the sense of (5). Bertsch, Dal Passo & Ughi actually require $u \in L^\infty([0,\infty[\times\mathbf{R}^n)$, but the distinction between L^∞ and $L^\infty_{\rm loc}$ is not essential. Since only some, but not all, of our solutions are globally bounded, we prefer the $L^\infty_{\rm loc}$ formulation (5).

Even for bounded pressures, the Cauchy problem features vast nonuniqueness of weak solutions [11] for fast diffusion ($\gamma \le 0$): Roughly speaking, any smoothly shrinking support for u can be prescribed. The unique viscosity solution constructed in [12] has constant support in time. This will not be true for the fast diffusions we construct which have bounded pressure, so they are not viscosity solutions. We also construct solutions with non-vanishing unbounded pressures which, analogously to (4), lead to classical solutions of the fast diffusion equation in the whole space. Finally, for $\gamma < 0$, we construct signed solutions $\pi(\boldsymbol{x},t)$ which are neither bounded above nor below; these correspond to unbounded densities $\rho = (\pi/\gamma)^\gamma_+ \ge 0$ which are infinite (singular) on the set $\pi > 0$. Solutions of this kind have singular sets which expand, and were studied by Chasseigne & Vázquez in [20, §8].

The results [11] of Bertsch, Dal Passo & Ughi (namely solutions with shrinking support) are only intended for the fast diffusion case $\gamma \le 0$, and in the density formulation they translate to unbounded focusing solutions. We construct such solutions (unbounded, but this time locally bounded) for the porous medium case ($\gamma > 0$). But we also construct bounded pressure solutions with spreading support, analogous (and forward asymptotic) to the Barenblatt solutions (4). Translated back into density variables, they give weak solutions to the porous medium equation with spreading support. Recall a weak solution [44] to the porous medium equation (1) with $m > 1$ is a function $\rho \ge 0$ such that $\rho^m, \nabla(\rho^m) \in L^1_{\rm loc}(\mathbf{R}^n\times[0,\infty[)$ and

$$(6)\qquad \int \varphi(\cdot,0)\rho_0\,d\boldsymbol{x} + \iint (\rho\frac{\partial\varphi}{\partial t} - \frac{1}{m}\nabla\rho^m\cdot\nabla\varphi)\,d\boldsymbol{x}dt = 0$$

for all $\varphi \in C^1([0,\infty[\times\mathbf{R}^n)$ with compact support in $[0,\infty[\times\mathbf{R}^n$. Lemma 10 of the appendix below asserts that weak solutions $u(\boldsymbol{x},t) \ge 0$ of the pressure evolution (5) give rise to weak solutions $\rho = u^\gamma$ of the density evolution (6). Thus our solutions $\pi(\boldsymbol{x},t)$, after truncation, give weak porous medium flows $\rho = (\pi/\gamma)^\gamma_+$.

3. Results

We now show that any quadratic initial pressure $\pi(\boldsymbol{x},0)$ leads to a family of classical solutions

$$(7)\qquad \pi(\boldsymbol{x},t) = \tfrac{1}{2}\langle\boldsymbol{x},\mathtt{P}(t)\boldsymbol{x}\rangle - \tfrac{1}{2}p_0(t)$$

which are quadratic in space at each instant in time; here $\mathtt{P}(t)$ is a symmetric $n\times n$ matrix. It costs no generality to exclude a linear term in $\boldsymbol{x}$ from (7), which corresponds to centering the solution, nor to choose coordinates on $\mathbf{R}^n$ which diagonalize the initial matrix $\mathtt{P}(0)$. The truncation π_* of (7) with the same sign as γ yields the pressure corresponding to the affinely self-similar solutions we are interested in.

The quadratic pressure ansatz (7) reduces the partial differential equation (3) to an ordinary differential equation

$$(8)\qquad \frac{d\mathtt{P}}{dt} = \frac{\operatorname{tr}\mathtt{P}}{\gamma}\mathtt{P} + 2\mathtt{P}^2$$

for the matrix $\mathtt{P}(t)$, plus a condition determining the constant

$$\frac{dp_0}{dt} = p_0 \frac{\operatorname{tr} \mathtt{P}}{\gamma}. \tag{9}$$

For functional calculus purposes below, note from (8) that $d\mathtt{P}/dt$ commutes with $\mathtt{P}$. The standard theory of differential equations guarantees that this system admits a smooth solution for short times; the only question is under what conditions the quadratic right hand side causes some entries in the matrix to blow-up in a finite time. Remarkably, as Tartar also discovered [39], a nonlinear change of variables reduces this system to a single quadrature.

The symmetric matrix $t \mapsto \mathtt{P}(t)$ can be diagonalized in a flow-invariant manner, and we assume therefore, with no loss of generality, that $\mathtt{P}(t)$ is a diagonal matrix. Let its diagonal elements be called p_1, p_2,..., p_n. The hyperplanes $p_i = p_j$ are clearly invariant, and so are the hyperplanes $p_i = 0$. Therefore any ordering of the p_i and the constant 0 in the initial values will be preserved under the flow within the maximal interval of existence. If any p_j vanishes, then it vanishes identically and can be dropped from the ODE, reducing the dimension and not affecting the other coordinates. We will therefore assume, with no loss of generality, that none of the p_j vanishes.

Under this hypothesis, we first deduce invariance of the quantity (10). This expresses the height $p_0(t)$ of the pressure plateau in terms of the evolution of the matrix $\mathtt{P}(t)$. Note the *moment index* $\mu := n+2\gamma$ used in our previous works [24] [25] [32] [33] [35] (where it was denoted by $-p = n + 2\gamma$) enters naturally as a scaling exponent. As Lemma 2 indicates, the conserved ratio (10) indexes the total mass $\|\rho_0\|_{L^1(\mathbf{R}^n)} \leq \infty$ of the density $\rho = (\pi/\gamma)_+^\gamma$ in (1).

Lemma 1 (Evolution of pressure plateau). *Fix $\gamma \neq 0$ and let $\pi(\boldsymbol{x}, t) = \frac{1}{2}(\langle \boldsymbol{x}\mathtt{P}(t)\boldsymbol{x}\rangle - p_0(t))$ satisfy (8)–(9) on an interval $]t_\alpha, t_\omega[\subset \mathbf{R}$. If $\det \mathtt{P}(t) \neq 0$ then*

$$z^2 = |p_0(t)|/|\det \mathtt{P}(t)|^{1/(n+2\gamma)} = \mathit{const}. \tag{10}$$

Proof. There is no problem solving (9) on any interval of time where (8) has a smooth solution. Reformulating these dynamics as

$$\frac{d}{dt} \log |\mathtt{P}| = \frac{\operatorname{tr} \mathtt{P}}{\gamma} \mathtt{I} + 2\mathtt{P}, \tag{11}$$

the invariance of (10) follows by comparing

$$\frac{d}{dt} \log |p_0| = \frac{\operatorname{tr} \mathtt{P}}{\gamma},$$

with the trace of the first equation:

$$\frac{d}{dt} \log \det |\mathtt{P}| = (n + 2\gamma) \frac{\operatorname{tr} \mathtt{P}}{\gamma}.$$

■

An equally elementary theorem holds the key to describing the dynamics. Before proving this theorem, observe that if $\mathtt{P}(t) \geq 0$ and $p_0(t) > 0$, the zero set $\mathcal{Z}_t$ of the pressure (7) is given by $|\mathtt{P}(t)^{1/2}\boldsymbol{x}| = p_0(t)^{1/2}$. So $|\mathtt{P}(t)/p_0(t)|^{-1/2}$ scales as a length. More generally, the invariance of (10), which corresponds to conservation of mass, implies $\mathtt{S}(t) := -|\det \mathtt{P}(t)|^{1/(n+2\gamma)}\mathtt{P}(t)^{-1}$ scales as a length squared. The matrix $\mathtt{S}$ will turn out to trivialize the dynamics. To relate this matrix directly to the evolving profile $\rho = (\pi/\gamma)_+^\gamma$, order the eigenvalues of $\mathtt{P}(t)$ and $\mathtt{S}(t)$ so that $p_1(t) \geq p_2(t) \geq \cdots \geq p_n(t)$ and hence

$$0 > s_1(t) \geq s_2(t) \geq \cdots \geq s_j(t) > -\infty; \quad +\infty > s_{j+1}(t) \geq \cdots \geq s_n(t) > 0, \tag{12}$$

where j counts the number of positive eigenvalues of $\mathtt{P}(t)$. Define the function $f_{(n,j)}$ (encoding the signature of $\mathtt{P}$) by

$$2\gamma f_{(n,j)}(\boldsymbol{x}) := -\frac{p_0(t)}{|p_0(t)|} + \sum_{i=1}^{j} x_i^2 - \sum_{i=j+1}^{n} x_i^2. \tag{13}$$

Think of $2\gamma(f_{(n,j)})_+^\gamma$ as the density of a Borel measure. For any affine matrix $\mathtt{A}$, let $\rho_0 = \mathtt{A}_\#(f_{(n,j)})_+^\gamma$ denote the unique Borel measure satisfying

$$\int_{\mathbf{R}^n} \xi d\rho_0 = \int \xi(\mathtt{A}\boldsymbol{x})(f_{(n,j)}(\boldsymbol{x}))_+^\gamma \, d\boldsymbol{x} \tag{14}$$

for all continuous test functions $0 \le \xi \in C_c(\mathbf{R}^n)$ of compact support. A straightforward application of the change of variable formula shows that ρ_0 has the density $(f_{(n,j)})_+^\gamma(\mathtt{A}^{-1}\boldsymbol{y})/|\det \mathtt{A}|$.

Setting $\mathtt{A}(t) = |\mathtt{S}(t)|^{1/2}$ as above, the evolving density $\rho(t) = \mathtt{A}(t)_\#(f_{(n,j)})_+^\gamma$ gives the pressure profile corresponding to (7), where we choose the constant in (10) to be $z = 1$ without loss of generality hereafter. More explicitly:

Lemma 2 (Density evolution). *For $0 \neq \gamma \neq -n/2$ let $\pi(\boldsymbol{x}, t) = \frac{1}{2}(\langle \boldsymbol{x}, \mathtt{P}(t)\boldsymbol{x}\rangle - p_0(t))$ solve (8)–(9) on $]t_\alpha, t_\omega[\subset \mathbf{R}$. Let j be the number of positive eigenvalues of $\mathtt{P}(t)$ and define $z > 0$ and $f_{(n,j)}(\boldsymbol{x})$ via (10) and (13). If $p_0(t)\det \mathtt{P}(t) \neq 0$ then $\rho(t) = (\pi/\gamma)_+^\gamma = z^{n+2\gamma}(z\mathtt{A}(t))_\#(f_{(n,j)})_+^\gamma$ where $\mathtt{A}(t)^{-2} = |\mathtt{P}(t)|/|\det \mathtt{P}(t)|^{1/(n+2\gamma)}$.*

Proof. The definition of z gives $|p_0(t)| = z^2|\det \mathtt{P}(t)|^{1/(n+2\gamma)} = z^2|\det \mathtt{A}(t)|^{-1/\gamma}$, so

$$\begin{aligned}
\pi(\boldsymbol{y}, t) &= \frac{|p_0(t)|}{2}\left(\left\langle \boldsymbol{y}, \frac{\mathtt{P}(t)}{z^2|\det \mathtt{P}(t)|^{1/(n+2\gamma)}}\boldsymbol{y}\right\rangle - \frac{p_0(t)}{|p_0(t)|}\right) \\
&= \gamma z^2|\det \mathtt{A}(t)|^{-1/\gamma} f_{(n,j)}(\mathtt{A}(t)^{-1}\boldsymbol{y}/z).
\end{aligned}$$

Integrating $\xi(\boldsymbol{y})$ against $\rho = (\pi/\gamma)_+^\gamma$ yields

$$\begin{aligned}
\int_{\mathbf{R}^n} \xi(\boldsymbol{y})\rho(\boldsymbol{y}, t) d\boldsymbol{y} &= z^{2\gamma+n}\int_{\mathbf{R}^n} \xi(\boldsymbol{y})(f_{(n,j)}(\mathtt{A}(t)^{-1}\boldsymbol{y}/z))_+^\gamma \det(z\mathtt{A}(t))^{-1}\, d\boldsymbol{y} \\
&= z^{2\gamma+n}\int_{\mathbf{R}^n} \xi(z\mathtt{A}(t)\boldsymbol{x})(f_{(n,j)}(\boldsymbol{x}))_+^\gamma \, d\boldsymbol{x}.
\end{aligned}$$

The change of variables $\boldsymbol{y} = z\mathtt{A}(t)\boldsymbol{x}$ is invertible since $p_0(t)\det \mathtt{P}(t) \neq 0$. ■

Theorem 3 (Integrability). *Fix $0 \neq \gamma \neq -n/2$, and let $\mathtt{P}(t)$ be a diagonal invertible matrix solving (8) on a maximal interval $]t_\alpha, t_\omega[\subset \mathbf{R}$. Then the eigenvalues s_i of $\mathtt{S}(t) = -\mathtt{P}^{-1}(t)|\det \mathtt{P}(t)|^{1/(n+2\gamma)}$ satisfy*

$$\frac{ds_i}{dt} = 2\,|\det \mathtt{S}(t)|^{-\frac{1}{2\gamma}} \qquad \text{for} \qquad 1 \le i \le n. \tag{15}$$

Proof. Differentiating $\log|s_i(t)|$ yields

$$\begin{aligned}
\frac{1}{s_i}\frac{ds_i}{dt} &= -\frac{d\log|p_i|}{dt} + \frac{1}{n+2\gamma}\sum_{k=1}^n \frac{d\log|p_k|}{dt} \\
&= -\frac{\operatorname{tr}\mathtt{P}}{\gamma} - 2p_i + \frac{1}{n+2\gamma}\left(\frac{n}{\gamma}\operatorname{tr}\mathtt{P} + 2\operatorname{tr}\mathtt{P}\right) \\
&= -2p_i \\
&= +\frac{2}{s_i}|\det \mathtt{P}|^{1/(n+2\gamma)} \\
&= \frac{2}{s_i}|\det \mathtt{S}|^{-1/(2\gamma)}
\end{aligned}$$

where the second identity utilized (11). ■

Corollary 4 (Fate of quadratic pressures). *The orbits of the dynamical system (15) consist of all lines $\mathtt{S}(t) = \mathtt{S}(t_0) + (\tau(t) - \tau(t_0))\mathtt{I}$ which terminate at the coordinate hyperplanes $\det \mathtt{S}(t) = 0$; see Figure 1a). The traceless part $\mathtt{B}$ of $\mathtt{S}(t)$ is a constant of motion. Away from the coordinate hyperplanes, $d\tau/dt = 2\,|\det \mathtt{S}|^{-1/(2\gamma)} > 0$ is smooth and positive, and $\tau(t) = \frac{1}{n}\operatorname{tr}\mathtt{S}(t)$.*

Proof. The corollary follows immediately from (15). ∎

It may seem surprising that the orbits of $\mathtt{S}(t)$ are completely independent of the nonlinearity γ; the precise value γ affects only the rate at which these orbits are parameterized. As a partial explanation, observe that the evolution (1) formally preserves the ρ-expectation of every harmonic function. Since the solutions $\rho = (\pi/\gamma)_+^\gamma$ that we study can be determined, within their family, by their second moments, i.e., by testing them with quadratic polynomials, and since the harmonic polynomials are codimension-1 in this space of test functions, it follows that all but one degree of freedom of our special solutions should be fixed, independent of γ.

Let us record the asymptotics of the solutions $\mathtt{S}(t)$ as $t \nearrow t_\omega$, which are easily read off of (15). Since $-\mathtt{S}(-t)$ solves the same equation, the asymptotics as $t \searrow t_\alpha$ follow immediately. In the porous medium case $\gamma > 0$, the vector field (15) diverges on each of the coordinate hyperplanes $s_i = 0$, so the dynamics accelerate towards and away from this singular set, blowing up at a finite time $t_\omega < \infty$. When $\gamma < 0$, the vector field $|\det \mathtt{S}(t)|^{-1/2\gamma}(1, \ldots, 1)$ vanishes continuously along these hyperplanes, so the dynamics decelerates towards (and away from) this singular set. Whether or not the singularity is reached in finite time is then determined by how quickly the vector field vanishes; i.e., by the modulus α of Hölder continuity along its integral curves (relative to 1). Where the orbit simultaneously intersects n_1 of the coordinate hyperplanes, this Hölder exponent is given by $\alpha = -n_1/2\gamma$. Borrowing the classification philosophy for solutions of Ricci flow equation, e.g. [22], this gives a convenient criterion for deciding when a quadratic pressure solution will be *eternal* (i.e. both *ancient* ($t_\alpha = -\infty$) and *immortal* ($t_\omega = \infty$)). To be eternal requires the multiplicities n_1 and n_n of the largest negative and smallest positive eigenvalues (12) of $\mathtt{S}(t)$ to satisfy $n_1 + 2\gamma > 0$ and $n_n + 2\gamma > 0$ with $\gamma < 0$. Thus for $\gamma \in [-1/2, 0[$, each $\mathtt{P}(t)$ of indefinite sign leads to an eternal solution, but for $\gamma \notin [-n/4, 0]$ no quadratic pressure is eternal save the constant $\mathtt{P}(t) = 0$. Note however, that for $\gamma < -n/2$ the orbits $\mathtt{S}(t)$ of the *finite-dimensional* dynamics can be extended continuously (but not smoothly) to the entire time domain $t \in \mathbf{R}$, simply by electing to spend (say) as little time on the singular set as possible. This choice specifies the extension uniquely, but it does not seem to have a meaningful interpretation at the level of the partial differential equation.

Corollary 5 (Asymptotics)**.** *Fix $0 \neq \gamma \neq -n/2$, and let $\mathtt{P}(t)$ be a diagonal invertible matrix solving (8) on a maximal interval $]t_\alpha, t_\omega[\subset \mathbf{R}^n$. Set $\tau(t) = \operatorname{tr} \mathtt{S}(t)/n$ and denote the eigenvalues of $\mathtt{S}(t) = -\mathtt{P}^{-1}(t)|\det \mathtt{P}(t)|^{1/(n+2\gamma)}$ by $s_i(t)$ and those of $\mathtt{B} = \mathtt{S}(t) - \tau(t)\mathtt{I}$ by b_i. Then there exist constants $c = |\frac{\gamma}{n_1+2\gamma}| \prod_{i>n_1}^n |s_i(t_\omega)|^{1/(2\gamma)}$ specified below and t_1 such that*

(a) **Spreading:** *if $\mathtt{S}(t) > 0$ then $\tau(t) \to \infty$ as $t \to t_\omega \leq +\infty$ and*

$$\tau(t) = \begin{cases} \left[\frac{n+2\gamma}{\gamma}(t - t_1)\right]^{2\gamma/(n+2\gamma)} + O_*((t-t_1)^{-2\gamma/(n+2\gamma)}) & \text{or} \\ \left[\frac{n+2\gamma}{|\gamma|}(t_\omega - t)\right]^{2\gamma/(n+2\gamma)} + O((t_\omega - t)^{-2\gamma/(n+2\gamma)}) \end{cases} \tag{16}$$

where the first formula holds for $(n + 2\gamma)/\gamma > 0$, hence $t_\omega = \infty$, and the second formula holds for $(n + 2\gamma)/\gamma < 0$, hence t_ω finite. The symbol O_* is the same as O, except when $\gamma = n/2$: In that case an extra factor $\log t$ enters in the O term.

(b) **Focusing:** *Otherwise, suppose $s_1(t)$ has the smallest absolute value among negative eigenvalues (12) of $\mathtt{S}(t)$, and let n_1 denote its multiplicity. Then $\mathtt{S}(t)$ converges to a rank $(n - n_1)$ finite limit matrix $\mathtt{S}(t_\omega) = \mathtt{B} + \tau_\omega \mathtt{I}$ as $t \to t_\omega$, with $s_1(t_\omega) = 0$ and $\tau_\omega := \tau(t_\omega)$, while*

$$\tau_\omega - \tau(t) = \begin{cases} [(t_\omega - t)/c]^{2\gamma/(n_1+2\gamma)} + O\left((t_\omega - t)^{4\gamma/(n_1+2\gamma)}\right) & \text{or} \\ [(t - t_1)/c]^{2\gamma/(n_1+2\gamma)} + O_*\left((t - t_1)^{4\gamma/(n_1+2\gamma)}\right) \end{cases} \tag{17}$$

where the first formula holds for $(n_1 + 2\gamma)/\gamma > 0$, hence $t_\omega < \infty$. The second formula holds for $(n_1 + 2\gamma)/\gamma < 0$, hence $t_\omega = \infty$. The symbol O_* denotes an extra logarithmic factor in the case $\gamma = -n_1/4$.

Proof. Recall from Corollary 4 that

$$\frac{d\tau}{dt} = 2\prod_{i=1}^{n} |b_i + \tau(t)|^{-1/2\gamma}, \tag{18}$$

If $\mathtt{S} > 0$, then clearly $\tau(t) \to \infty$ as $t \to t_\omega$ by the previous corollary and the continuation theorem. The absolute values in (18) are redundant in this case; for the moment assume $\gamma > 0$ or $\gamma < -n/2$, hence $1 + \frac{n}{2\gamma} > 0$. Separation of variables and integration (using $\operatorname{tr}\mathtt{B} = 0$) yields

$$\frac{\gamma}{n+2\gamma}\tau^{1+\frac{n}{2\gamma}}\left(1 + O_*(\tau^{-2})\right) = t - t_1 \tag{19}$$

with a yet undetermined constant of integration t_1. The $*$ in $O_*(\tau^{-2})$ is only relevant for $\gamma = n/2$ and in this case refers to an extra factor $\ln\tau$ in the $O(\cdot)$ term. From (19) and $(n+2\gamma)/\gamma > 0$, one gets $t_\omega = \infty$, and also

$$\tau\left(1 + O_*(\tau^{-2})\right) = \left[\frac{n+2\gamma}{\gamma}(t - t_1)\right]^{2\gamma/(n+2\gamma)}.$$

Hence the claim (16). Essentially the same calculation can also be used for $-\frac{n}{2} < \gamma < 0$. However in this case, $t - t_1 \to 0$ as $\tau \to \infty$; so we have t_ω finite, equal to t_1. We then get the corresponding second formula in (16).

Now in the case where $s_1 < 0$, the hyperplane $s_1 = 0$ (actually, the space $s_1 = \ldots = s_{n_1} = 0$) is reached for a finite value of $\tau_\omega := \tau(t_\omega)$. The remaining factors $\prod_{i=n_1+1}^{n}\cdots$ are analytic near τ_ω. Separation of variables in (18) gives, for $1 + n_1/2\gamma > 0$,

$$c(\tau_\omega - \tau)^{1+n_1/2\gamma}(1 + O(\tau_\omega - \tau)) = t_\omega - t$$

with $t_\omega < \infty$ and $c = \frac{\gamma}{n_1+2\gamma}\prod_{i=n_1+1}^{n}|b_i + \tau_\omega|^{1/2\gamma}$. In contrast, for $1 + n_1/2\gamma < 0$ we get

$$c(\tau_\omega - \tau)^{1+n_1/2\gamma}(1 + O_*(\tau_\omega - \tau)) = t - t_1$$

with some constant of integration t_1, with $c = \frac{|\gamma|}{n_1+2\gamma}\prod_{i=n_1+1}^{n}|b_i + \tau_\omega|^{1/2\gamma}$. and $t_\omega = \infty$; the O_* refers to a logarithmic term instead of $(\tau_\omega - \tau)^0$, i.e., if $\gamma = -n_1/4$. The claim follows immediately. ■

Note that in the immortal case (16a), for $\gamma > 0$ a time shift may or may not be absorbed into the error term, depending on the size of $\gamma > 0$ relative to $n/2$. This suggests phase transitions in the long-time porous medium asymptotics, similar to those observed [17] [24] in the fast diffusion regime $\gamma < 0$. Higher asymptotics would follow from a closer study of the ordinary differential equation (18) with $\sum b_i = \operatorname{tr}\mathtt{B} = 0$. Let us rather examine the solutions described. Table 1 gives an overview.

Since eigenvalues do not change sign, the corollary asserts that a quadratic pressure $\pi(\boldsymbol{x}, t)$ represents a spreading solution if and only if its quadratic part $\mathtt{P} = -\mathtt{S}^{-1}|\det\mathtt{S}|^{-1/2\gamma}$ is negative definite. In this case the density $\rho = (\pi/\gamma)_+^\gamma$ is compactly supported if $\gamma > 0$, and otherwise has tails that decay like $|\boldsymbol{x}|^{2\gamma}$. Thus $\rho \in L^1(\mathbf{R}^n)$ if and only if $\gamma \notin [-n/2, 0]$. We call this range the *mass-preserving* regime; it coincides with the range of γ for which spreading solutions are immortal ($t_\omega = +\infty$), and all other solutions focus in finite time. As the solution $\rho(t) = \mathtt{A}(t)_\#(f_{(n,0)})_+^\gamma$ spreads out, its isobars (= level sets of the pressure) are ellipsoids which become rounder and rounder as $t \to t_\omega$, since $\mathtt{A}(t) = |\tau(t)\mathtt{I} + \mathtt{B}|^{1/2} \sim \tau(t)^{1/2}(\mathtt{I} + \frac{1}{2\tau(t)}\mathtt{B})$ as τ grows. This is in accordance with the attractivity of the Barenblatt solution proved in the mass-preserving regime by Friedman & Kamin [28]; see also Vázquez [44]. Comparing different eigenvalues $a_i(t)$ of $\mathtt{A}(t)$, for $\gamma > 0$ the departure from sphericity $2(a_i - a_k)/\tau(t)^{1/2} \sim (b_i - b_k)/|\frac{n+2\gamma}{\gamma}t|^{2\gamma/(n+2\gamma)}$ disappears more slowly than the discrepancy $(a_i(t + t_0) - a_i(t))/\tau(t)^{1/2} \sim |\frac{\gamma}{n+2\gamma}|\frac{t_0}{t}$ between a solution and its time delay. These precise decay exponents are also predicted by the linearization we carried out in the mass-preserving fast diffusion $\gamma < -n/2$ range [25]. Note the constants $b_i \neq b_k$ are distinct unless the eigenvalues $a_i(t) = a_k(t)$ coincide for all $t \in]t_\alpha, t_\omega[$.

Turning now to the focusing case, we observe that when $\mathtt{S}(t)$ is not positive definite, its orbit terminates on the i-th coordinate hyperplane, where $s_i(t)$ is the smallest in absolute value among the negative eigenvalues (12) of $\mathtt{S}(t)$; $i = 1$ in our numeration (12). This eigenvalue decays to zero at a rate $|t_\omega -$

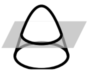
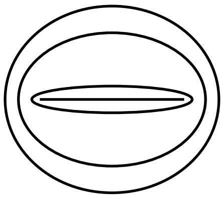
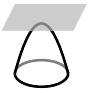
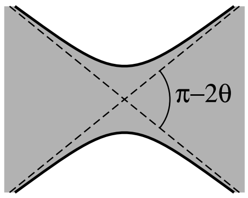
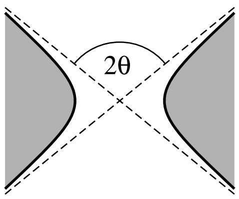
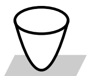
π figure	$\gamma > 0$ Porous Medium	$\gamma < -n/2$ Conservative Fast Diffusion
$\mathtt{P} < 0,\ -p_0 > 0$ 		$\mathtt{P} \sim \frac{-\gamma/(2\gamma+n)}{t}\mathtt{I} + O(t^{-1-2\gamma/(\gamma+2n)})$; $p_0 \sim \frac{const}{t^{n/(2\gamma+n)}}$.
	spreading and getting rounder; increasing elliptical support. Affine image of Barenblatt	elliptical singular support (infinite density), spreads out and gets rounder as $t \nearrow \infty$.
$\mathtt{P} < 0,\ -p_0 < 0$ 	N/A	Affine image of a Barenblatt solution
$\mathtt{P}$ indef., $-p_0 > 0$		support of $\mathtt{P}$ spreading as $t \nearrow t_\omega$: $\theta \sim (t_\omega - t)^{\gamma/(2\gamma+n_1)}$; 'waistline' $\to$ const. As $t \searrow t_\alpha$, support shrinks to horizontal axis.
	density supported in shaded area	density infinity in shaded area, finite in unshaded area
$\mathtt{P}$ indef., $-p_0 < 0$		support of $\mathtt{P}$ spreading as $t \nearrow t_\omega$: $\theta \sim (t_\omega - t)^{\gamma/(2\gamma+n_1)}$; 'gap' $\to 0$. As $t \searrow t_\alpha$, support shrinks to two rays with finite gap
	density supported in shaded area	density infinity in shaded area, finite in unshaded area
$\mathtt{P} > 0,\ -p_0 > 0$ 	unbounded, locally finite solution supported in whole space	N/A
$\mathtt{P} > 0,\ -p_0 < 0$ 	unbounded, locally finite solution supported in the complement of an elliptical hole. As $t \nearrow t_\omega$, this hole shrinks to a slit or elliptical disc depending on dimension.	singular support (infinite density) outside ellipse/ellipsoid; finite density inside shrinks to a to slit or elliptical disc.

TABLE 1. Overview of solutions

$t|^{2\gamma/(n_1+2\gamma)}$ determined by its multiplicity n_1; (we tacitly replace $t_\omega - t$ by $t - t_1$ in this and similar formulae if $-n_1/2 < \gamma < 0$, since the solution in that case is immortal, meaning the focusing takes infinitely long). Since the other distinct eigenvalues of $\mathtt{S}(t)$ remain at fixed distances to s_1, they converge to non-zero values at the same rate. Thus n_1 spatial dimensions $a_1 = \cdots = a_{n_1} = |s_i|^{1/2}$ of the solution $\rho(t) = \mathtt{A}(t)_\#(f_{(n,j)})_+^\gamma$ collapse as $t \to t_\omega$, the others tending to as many distinct non-zero values as their multiplicities permit. In the filtration-absorption model of Barenblatt, Bertsch, Chertock & Prostokishin [9] ($j = n$), this suggests that the last drop of fluid drains not as a round drop, but rather as a flattening ellipsoid with generically different axes, the shortest disappearing at a rate $x_1 \sim (t_\omega - t)^{\gamma/(2\gamma+n_1)}$, and the others tending to distinct non-zero limits as $t \to t_\omega$. Since the physical relevance of quadratic focusing solutions may be debatable, let us observe via time-reversal that the future asymptotics $t \to t_\omega$ for the $\mathtt{S}(t) < 0$ solutions above (which are ancient in the mass-preserving regime), also describe the past asymptotics $t \to \bar{t}_\alpha = -t_\omega$ of spreading solutions at their moment of origin. Thus our spreading solutions with $\mathtt{S}(t) > 0$ originate not at a point source (unless they are perfectly round), but rather as a characteristic density on a finite size rod, disc, or flattened ellipsoid. The next lemma shows this characteristic density is the sheared image of a Barenblatt profile in dimension $k = n - n_n$, but associated to a different nonlinearity $m' \neq m$ so that its moment index $n + 2\gamma =: \mu = \mu' := k + 2/(m' - 1)$ is the same.

The shape of this profile can also be realized by projecting the mass of $\rho_0 = (f_{(n,0)})_+^\gamma$ from $\mathbf{R}^n$ to $\mathbf{R}^k$ using a rank k matrix $\mathtt{A}(t_\alpha)$. Corollary 4 makes it clear that each projection $\mathtt{A}(t_\alpha)_\#\rho_0$ with $\det \mathtt{A}(t_\alpha) = 0$ is actually realized as the $t \searrow t_\alpha$ limit of a (unique) orbit composed of densities diffusing with quadratic pressure, and that affine images of the Barenblatt solution form an invariant subspace for the porous medium / fast diffusion dynamics (1). We remind the reader that j continues to denote the number of positive eigenvalues in $\mathtt{P}$. However, for the backwards-in-time limit $t \to t_\alpha$, it is not $p_1 > 0$ but $p_n < 0$ that blows up, and we will assume its multiplicity to be $n - k$, so that $p_k \neq p_{k+1} = \ldots = p_n < 0$.

Lemma 6 (Singular initial profiles preserve moment index). *Suppose $\mathtt{A}(t) > 0$ is a positive definite diagonal $n \times n$ matrix for each $t \in]t_\alpha, t_\omega[$, and converges to a rank k (with $k < n$) limit matrix $\mathtt{A}(t_\alpha) = \operatorname{diag}(a_1, \ldots a_k, 0, \ldots, 0)$ as $t \to t_\alpha$. Given $j \leq k$ and $\gamma \notin [(k-n)/2, 0]$ define $\rho(t) = \mathtt{A}(t)_\#(f_{n,j})_+^\gamma$ via (13)–(14). If $\gamma < 0$, assume $j = 0$ and $p_0(t) > 0$ to ensure $(f_{(n,j)})_+^\gamma$ is a Radon measure. Up to a normalization constant $z' > 0$,*

$$\lim_{t \to t_\alpha} \int_{\mathbf{R}^n} \xi(\boldsymbol{x})\rho(\boldsymbol{x}, t)\, d\boldsymbol{x} = \int_{\mathbf{R}^n} \xi(\mathtt{A}(t_\alpha)\boldsymbol{x})\, (f_{(n,j)}(\boldsymbol{x}))_+^\gamma d\boldsymbol{x} = z' \int_{\mathbf{R}^n} \xi d\sigma, \tag{20}$$

for all continuous test functions $\xi \in C_c(\mathbf{R}^n)$ of compact support. Here $\sigma = \mathtt{A}(t_\alpha)_\#(f_{(n,j)})_+^\gamma$ is a measure supported on the subspace $x_{k+1} = \cdots = x_n = 0$; its density with respect to Hausdorff k-dimensional measure $\mathcal{H}^k$ is given by

$$\frac{d\sigma}{d\mathcal{H}^k}(x_1, \ldots, x_k, 0, \ldots, 0) = \left(f_{(k,j)}(\frac{x_1}{a_1}, \ldots, \frac{x_k}{a_k}) \right)_+^{\gamma + (n-k)/2}. \tag{21}$$

Proof. The definition $\rho(t) := \mathtt{A}(t)_\#(f_{(n,j)})_+^\gamma$ gives

$$\int_{\mathbf{R}^n} \xi(\boldsymbol{y})\rho(\boldsymbol{y}, t) d\boldsymbol{y} = \int_{\mathbf{R}^n} \xi(\mathtt{A}(t)\boldsymbol{x})(f_{(n,j)}(\boldsymbol{x}))_+^\gamma\, d\boldsymbol{x}.$$

The $t \to t_\alpha$ limit (20) will be derived using the dominated convergence theorem. For $\gamma > 0$ a constant function on a compact set supporting all integrands is a majorant: $(f_{(n,j)})_+^\gamma$ is continuous, hence $L^1_{\mathrm{loc}}(\mathbf{R}^n)$, and the support of $\xi(\mathtt{A}(t)\boldsymbol{x}) \in C_c(\mathbf{R}^n)$ is eventually contained in a cylinder $K \times \mathbf{R}^{n-k}$ with compact cross-section $K \subset \mathbf{R}^k$. The set $\{\boldsymbol{x} | f_{(n,j)}(\boldsymbol{x}) > 0\}$ is asymptotic to the cone $\sum_{i=j+1}^n x_i^2 \leq \sum_{i=1}^j x_i^2$, which has compact intersection with $K \times \mathbf{R}^{n-k}$ because $j \leq k$. For $\gamma < (k-n)/2 < 0$, our hypotheses $j = 0$ and $p_0(t) > 0$ ensure that $(f_{(n,j)}(\boldsymbol{x}))_+^\gamma$ is globally positive and integrable on $K \times \mathbf{R}^{n-k}$, so $\|\xi\|_{L^\infty(\mathbf{R}^n)}$ times this function is a majorant.

Finally, $\xi(\mathtt{A}(t_\alpha)\boldsymbol{x}) = \xi(a_1x_1, \ldots, a_kx_k, 0, \ldots, 0)$ implies σ is supported on the subspace $x_{k+1} = \cdots = x_n = 0$. We can obtain its Lebesgue density there by integrating out x_i inductively for $n \geq i \geq k+1$: Since $f_{(i,j)} := f_{(i,j)}(x_1, \ldots, x_i) = f_{(i-1,j)} - x_i^2/\gamma$, we know for $\gamma > 0$ that $f_{(i-1,j)} > 0$ holds wherever

$f_{(i,j)} > 0$, while for $\gamma < 0$ then $f_{(i,j)} > 0$ for each $i \le n$ by our choices $j = 0$ and $p_0(t) > 0$. We integrate

$$\begin{aligned}\int_{-\infty}^{\infty} (f_{(i,j)})_+^\beta dx_i &= (f_{(i-1,j)})_+^{\beta+1/2}\sqrt{|\gamma|}\int_{-\infty}^{\infty}\Big(1-\frac{x_i^2}{\gamma f_{(i-1,j)}}\Big)_+^\beta \frac{dx_i}{\sqrt{|\gamma| f_{(i-1,j)+}}} \\ &= (f_{(i-1,j)})_+^{\beta+1/2}\sqrt{|\gamma|}\int_{-\infty}^{\infty}\Big(1-|\gamma|r^2/\gamma\Big)_+^\beta dr\end{aligned}$$

for $i = n, n-1, \dots, k+1$, and $\beta = \gamma, \gamma + \frac{1}{2}, \dots, \gamma + \frac{n-k-1}{2}$. The dr integral is finite (if $\gamma > 0$, then $\beta > 0$; if $\gamma < (k-n)/2$, then $\beta < -\frac{1}{2}$) and could be evaluated in terms of the Gamma function, but we simply absorb its value into the normalization constant z'. Absorbing the Jacobian $\prod_{i=1}^k a_i$ into z' also, the invertible change of variables $\mathtt{A}(t_\alpha) : \mathbf{R}^k \longrightarrow \mathbf{R}^k$ yields (21). ■

This lemma appears natural, and is divorced from the dynamics. It is therefore interesting to see the hypothesis $\gamma \notin [(k-n)/2, 0]$ required to guarantee local finiteness of the limiting measure σ is the same which permits nonlinear diffusion from this measure to occur in a finite time $t_\alpha > -\infty$. The forward focusing limit $t \to t_\omega < \infty$ would be more awkward to address since, as in the case $\gamma < 0$ and $\mathtt{P}(t)$ of indefinite sign, infinities make (20) diverge for all non-vanishing $0 \le \xi \in C(\mathbf{R}^n)$.

Finally, the solutions with $\mathtt{P}(t)$ of indefinite sign have more complicated focusing geometries (and a finite lifetime $]t_\alpha, t_\omega[$ in the mass-preserving regime). Some of these are included in the lemma above, and illustrated in Table 1. Assuming the eigenvalues $p_1(t) > \cdots > p_n(t)$ are distinct and non-zero, the isobars will be hyperbolas / hyperboloids (in two and three-dimensions) asymptotic to a double-cone emitted by the plane $x_n = 0$ at $t = t_\alpha$, and collapsing anisotropically onto the plane $x_1 = 0$ as $t \to t_\omega$. In two-dimensions, the initial region of positive pressure can be (i) two opposite rays along $x_2 = 0$ or (ii) the entire line $x_2 = 0$. At the focusing time $t = t_\omega$ it will finally occupy all but (i) the line $x_1 = 0$ or (ii) two opposite rays along it. In three-dimensions the asymptotic cone encircles either (a) the x_3-axis or (b) the x_1-axis, depending on the sign of p_2: the positive pressure region then begins as (a1) the entire plane, (a2) the complement of an ellipse, or the region (b1) between or (b2) outside the two sheets of a hyperbola in the plane $x_3 = 0$; it evolves to fill (and diverge on) all but a region (b2), (b1), (a2) or (a1) in the plane $x_1 = 0$ by time $t = t_\omega$. When $n = 3$, various degenerate limits are possible if two eigenvalues are equal. Higher dimensions allow for more exotic geometries and singularities.

4. Fourth order diffusions

4.1. **A naive logarithmic diffusion in several dimensions.** In this section, we observe that the same strategy can be used to expose and to analyze affinely self-similar solutions for certain fourth-order equations of recent interest governing diffusion of a non-negative scalar $\rho(\boldsymbol{x}, t) \ge 0$ on $\mathbf{R}^n \times]0, \infty[$. The first of these is the naive generalization

$$\frac{\partial \rho}{\partial t} = -\Delta(\rho \Delta \log \rho) \tag{22}$$

to higher dimensions of an evolution proposed by Derrida, Lebowitz, Speer & Spohn [26] as a model governing the size x of interfacial fluctuations in a statistical mechanical spin system, as a function of the distance t to a point where the interface is pinned by the boundary conditions. The exponential change of variables $\pi(\boldsymbol{x}, t) = \log \rho(\boldsymbol{x}, t)$ converts this model to a form [31]

$$-\frac{\partial \pi}{\partial t} = \Delta^2 \pi + 2\nabla\pi \cdot \nabla\Delta\pi + (\Delta\pi)^2 + \Delta\pi|\nabla\pi|^2$$

which makes clear that quadratic solutions $2\pi(\boldsymbol{x}, t) = \langle \boldsymbol{x}\mathtt{P}(t)\boldsymbol{x}\rangle - p_0(t)$ exist for any symmetric initial matrix $\mathtt{P}(t_0)$ and constant $p_0(t_0)$. The resulting differential equations

$$\frac{d\mathtt{P}(t)}{dt} = -2\mathtt{P}(t)^2 \operatorname{tr} \mathtt{P}(t) \tag{23}$$

$$\frac{dp_0(t)}{dt} = 2(\operatorname{tr} \mathtt{P}(t))^2 \tag{24}$$

are smooth, fix the harmonic polynomials $\operatorname{tr}\mathtt{P} = 0$, and can be integrated on some maximal time interval $]t_\alpha, t_\omega[\subset \mathbf{R}$ to give invariance of $e^{p_0(t)} \det \mathtt{P}(t) = e^{p_0(t_0)} \det \mathtt{P}(t_0)$ and orbits $\mathtt{P}(t)^{-1} = \mathtt{P}(t_0)^{-1} - (\tau(t) - \tau(t_0))\mathtt{I}$ parallel to the principal diagonal, as in Lemma 1 and Corollary 4; see Figure 1b), which we now justify. The parametrization $\tau(t)$ is governed by

$$\frac{d\tau}{dt} = 2\sum_{i=1}^{n} \frac{1}{b_i + \tau(t)},$$

where b_i denote the eigenvalues of the traceless part $\mathtt{B}$ of $\mathtt{S}(t) := -\mathtt{P}(t)^{-1}$. The dynamics governing $\mathtt{S}(t) = \mathtt{B} + \tau(t)\mathtt{I}$ is smooth away from the set $\det \mathtt{S} = 0$. It is clear from (23) that all orbits are immortal: $\operatorname{tr}\mathtt{P}$ cannot change sign because $\{\operatorname{tr}\mathtt{P} = 0\}$ is invariant; so if $\operatorname{tr}\mathtt{P} > 0$, all p_i decrease, but none can go to $-\infty$ because the trace is bounded below; an analogous argument applies if $\operatorname{tr}\mathtt{P} < 0$. The bound then gives global existence. All orbits apart from the equilibria $\operatorname{tr}\mathtt{P} = 0$ fail to be ancient: If $\operatorname{tr}\mathtt{P} > 0$, then this trace increases in backwards time, and also some $p_i > 0$. So $d/d(-t)\, p_i \geq c p_i^2$ implies finite blow up. Similarly for $\operatorname{tr}\mathtt{P} < 0$. So the orbits originate at $\det \mathtt{S} = 0$ as $t \searrow t_\alpha$, departing from the intersection of n_k coordinate hyperplanes in a finite time $\tau(t) \sim \tau(t_\alpha) + 2\sqrt{n_k(t - t_\alpha)}$. If $\mathtt{S}(t_0) > 0$ or $\mathtt{S}(t_0) < 0$ has a definite sign, then the orbit $\mathtt{S}(t) = \mathtt{B} + \tau(t)\mathtt{I}$ grows like $\tau(t) \sim 2\sqrt{nt}$ as $t \to \infty$; otherwise $\mathtt{S}(t) = \mathtt{S}_\omega + O(e^{-\kappa t})$ converges exponentially to the surface $\operatorname{tr}\mathtt{S}^{-1} = 0$ as $t \to \infty$, with exponent $\kappa = 2\operatorname{tr}(\mathtt{S}_\omega^{-2})$. The corresponding densities $\rho = e^\pi > 0$ are classical solutions to equation (22), given as the image $\rho(t) = \mathtt{A}(t)_\# \exp(\gamma f_{(n,j)})$ under $\mathtt{A}(t) = |\mathtt{S}(t)|^{1/2}$ of a Gaussian (13), j counting the number of negative eigenvalues of $\mathtt{S}(t)$. When $j = 0$, this evolves from a Gaussian measure concentrated on the subspace $\mathtt{A}(\tau(t_\alpha))(\mathbf{R}^n)$, toward the self-similar solution $(4t)^{1/4}{}_\# \exp(\gamma f_{(n,0)})$ found by Derrida et al [26] in the one-dimensional case. Since the evolution (22) preserves center of mass $\boldsymbol{x}_0 \in \mathbf{R}^n$, from $\mathtt{A}(t) \sim \tau(t)^{1/2}(\mathtt{I} + \frac{1}{2\tau(t)}\mathtt{B})$ as $t \to \infty$ we deduce the relative rates of convergence of translations in space $\boldsymbol{x}_0/\tau(t)^{1/2} \sim \boldsymbol{x}_0/(4t)^{1/4}$, and in time $(a_i(t + t_0) - a_i(t))/\tau(t)^{1/2} \sim t_0/(4t)$, with the deviation from sphericity $2(a_i - a_k)/\tau(t)^{1/2} \sim (b_i - b_k)/(4t)^{1/2}$. For $j = n$ the solution instead grows exponentially as $\boldsymbol{x} \to \infty$ and $t \to \infty$, while for $1 < j < n$ it is a logarithmically quadratic function which becomes log-harmonic exponentially fast in the long time limit.

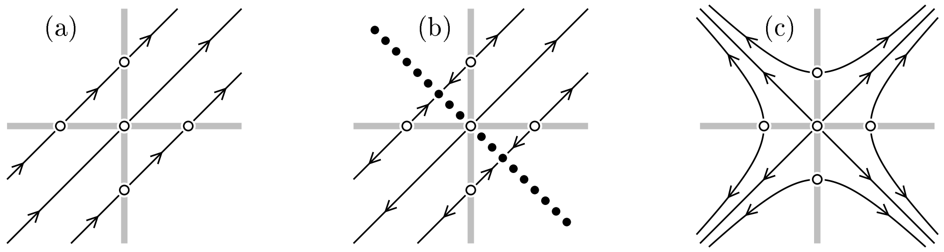
4.2. **A diffusion that is the 2-Wasserstein gradient flow of the Fisher information.** Another diffusion coinciding with (22) in one-dimension is the evolution

$$\frac{\partial \rho}{\partial t} = -\sum_{i,j=1}^{n} \partial^2_{x_i x_j}(\rho \partial^2_{x_i x_j} \log \rho) \tag{25}$$

explored by Jüngel and Pinnau as a model for transport phenomena in quantum systems [31] (damped by a magnetic field, for example). The same exponential change of variables converts this equation to the form

$$-\frac{\partial \pi}{\partial t} = \Delta^2 \pi + 2\langle \nabla \pi, \nabla \Delta \pi \rangle + \operatorname{tr}(D^2 \pi D^2 \pi) + \langle \nabla \pi, D^2 \pi \nabla \pi \rangle \tag{26}$$

where $D^2\pi$ denotes the Hessian matrix of π. The quadratic ansatz $2\pi(\boldsymbol{x}, t) = \langle \boldsymbol{x}, \mathtt{P}(t)\boldsymbol{x} \rangle - p_0(t)$ proves self-consistent, reducing the dynamics to a system of ordinary differential equations $d\mathtt{P}/dt = -2\mathtt{P}(t)^3$ and $dp_0/dt = 2\operatorname{tr}(\mathtt{P}^2)$ which can be integrated directly to yield $\mathtt{P}(t)^{-2} = \mathtt{P}(t_0)^{-2} + 4(t - t_0)\mathtt{I} = 4\mathtt{B} + 4t\mathtt{I}$ and $e^{p_0(t)} \det \mathtt{P}(t) = e^{p_0(t_0)} \det \mathtt{P}(t_0)$ on a maximal time interval $]t_\alpha, \infty[$ with $t_\alpha > -\infty$ (unless $P \equiv 0$). The ordering among the eigenvalues $a_i(t)$ of $\mathtt{A}(t) = |\mathtt{P}(t)|^{-1/2}$ is preserved; all are increasing functions of time, whose asymptotics can be deduced from $\mathtt{A}(t) = (4t)^{1/4}(\mathtt{I} + \mathtt{B}/t)^{1/4}$. In this case, focusing is not consistent with our quadratic ansatz; all solutions spread forwards in time (and focus backwards in time). The corresponding densities $\rho = \mathtt{A}(t)_\# \exp(\gamma f_{(n,j)})$ are classical solutions to the evolution (25) which represent Gaussians (if $\mathtt{P}(t) < 0$ so $j = 0$); they begin evolving from a lower dimensional Gaussian at $t = t_\alpha$, and spread at rate $a_i(t) \sim (4t)^{1/4}$ so that $a_i/a_{i+1} \to 1$ as $t \to \infty$. The finer asymptotics $(a_i(t + t_0) - a_i(t))/(4t)^{1/4} \sim t_0/(4t)$ and $(a_i - a_k)/(4t)^{1/4} \sim (b_i - b_k)/(4t)$ parallel the heat equation, though translations converge at the slower rate $1/|4t|^{1/4}$. Figure 1 gives a quick comparison of the dynamics of quadratic solutions for the different types of diffusion considered here.



Porous medium family: $\mathtt{S} = -\mathtt{P}^{-1} |\det \mathtt{P}|^{-1/(n+2\gamma)}$ see Corollary 4.

The naive 4th order model (22): $\mathtt{S} = -\mathtt{P}^{-1}$; let $\gamma = \infty$ for comparison purposes. Note $\operatorname{tr}\mathtt{P} = 0 \Leftrightarrow \operatorname{tr}\mathtt{S} = 0$ in 2D.

Equation (25), the $\gamma \to \infty$ limit of (27). Again $\mathtt{S} := -\mathtt{P}^{-1}$.

FIGURE 1. Comparison of quadratic dynamics for three diffusion equations in 2D: In each case, the axes denote the eigenvalues of $\mathtt{S}$. Gray zones denote singularities, as do white dots, to stress that the orbits run into singularities. Black dots denote equilibria.

4.3. **A fourth order analog for the porous medium family.** Recently, the global well-posedness of (25) was announced by Gianazza, Savaré & Toscani, who exploited the gradient flow structure of the dynamics with respect to 2-Wasserstein distance [29]. This structure makes it clear that (25) represents the case $m = 1$ in a family of fourth-order diffusion equations analogous to the porous medium family discussed above. In conservation law form, these equations

$$\frac{\partial \rho}{\partial t} = -\frac{1}{m - 1/2} \nabla \cdot [\rho \nabla (\rho^{m-3/2} \Delta(\rho^{m-1/2}))], \tag{27}$$

govern the evolution of a non-negative scalar $\rho(\boldsymbol{x}, t) \geq 0$ on $\mathbf{R}^n \times]0, \infty[$, and depend on a parameter $m \in \mathbf{R}$. Formally, this dynamics arises as the gradient flow of the *generalized Fisher information*

$$I_m(\rho) := \frac{1}{2} \frac{1}{(m - 1/2)^2} \int_{\mathbf{R}^n} |\nabla (\rho^{m-1/2}(\boldsymbol{x}, t))|^2 \, d\boldsymbol{x}, \tag{28}$$

with respect to 2-Wasserstein distance, in the same way that the porous medium flow (1) arises as the 2-Wasserstein gradient flow of the *entropy*

$$E_m(\rho) := \frac{1}{m(m-1)} \int_{\mathbf{R}^n} (\rho(\boldsymbol{x}, t)^m - \rho(\boldsymbol{x}, t)) \, d\boldsymbol{x} \tag{29}$$

in the work of Otto [36]; see Ambrosio, Gigli & Savaré [1] and Carrillo, McCann & Villani [19] for related developments.

Remarkably, the dynamics (27) can also be formulated in terms of a pressure equation (30) which possesses quadratic solutions. The limit $m \to 1$ motivates nomenclature, while the fact that $dE_m(\rho)/dt = -2I_m(\rho) < 0$ under the porous medium dynamics (1) explains some of the parallels between these two families. To see directly the equivalence of (25) with the $m = 1$ instance of (27), i.e., to show

$$2\partial_i \left(\rho \partial_i \left(\rho^{-1/2} \partial_j \partial_j \rho^{1/2} \right) \right) = \partial_i \partial_j (\rho \partial_i \partial_j \log \rho)$$

evaluate the rightmost derivative on either side; after canceling the common 'factor' ∂_i, it the suffices to show the identity $\rho \partial_i \rho^{-1/2} \partial_j \rho^{-1/2} \rho_j = \partial_j \rho \partial_i \rho^{-1} \rho_j$. The left side equals $\rho \partial_i \partial_j \rho^{-1} \rho_j - \rho \partial_i (-\frac{1}{2} \rho^{-2} \rho_j \rho_j)$. The right hand side equals $\rho \partial_j \partial_i \rho^{-1} \rho_j + \rho_j \partial_i \rho^{-1} \rho_j$. Equality can now be checked easily. For $m \leq 1$, some quadratic solutions discussed below will be classical; but for $m > 1$, the discussion is purely formal, since we do not attempt to identify a notion of weak solution.

As in the porous medium case, applying the identity

$$\frac{\pi^{1-\gamma/2}}{1 + \gamma/2} \Delta(\pi^{1+\gamma/2}) = \pi \Delta \pi + \gamma |\nabla \pi|^2 / 2$$

to $\gamma = 1/(m-1)$ and $\pi = \gamma\rho^{1/\gamma}$ reexpresses the evolution (27) in the simpler form

$$-\frac{\partial \pi}{\partial t} = \frac{\pi}{\gamma}\Delta\Big(\frac{\pi}{\gamma}\Delta\pi + \frac{1}{2}|\nabla\pi|^2\Big) + \nabla\pi\cdot\nabla\Big(\frac{\pi}{\gamma}\Delta\pi + \frac{1}{2}|\nabla\pi|^2\Big). \tag{30}$$

Unlike the pressure version (3) of the porous medium equation, this equation is invariant under the negation $\pi(\boldsymbol{x},t) \leftrightarrow -\pi(\boldsymbol{x},t)$. It degenerates only where $\pi = 0$ or $\pi = \pm\infty$. The quadratic ansatz $\pi(\boldsymbol{x},t) = \frac{1}{2}\left(\langle \boldsymbol{x}, \mathtt{P}(t)\boldsymbol{x}\rangle - p_0(t)\right)$ proves self-consistent, and yields a system of ordinary differential equations

$$-\frac{d\mathtt{P}(t)}{dt} = 2\mathtt{P}^3 + 2\mathtt{P}^2\frac{\operatorname{tr}\mathtt{P}}{\gamma} + \mathtt{P}\frac{\operatorname{tr}(\mathtt{P}^2)}{\gamma} + \mathtt{P}\frac{(\operatorname{tr}\mathtt{P})^2}{\gamma^2} \tag{31}$$

$$-\frac{1}{p_0}\frac{dp_0}{dt} = \frac{\operatorname{tr}(\mathtt{P}^2)}{\gamma} + \frac{(\operatorname{tr}\mathtt{P})^2}{\gamma^2} \tag{32}$$

which clearly admit local solutions. Any initially diagonal matrix $\mathtt{P}(0)$ remains diagonal under the evolution, and any eigenvalue $p_i(0) = 0$ which vanishes initially remains zero at all subsequent times. We can thus discard any zero eigenvalues from the system (31), instead studying the analogous evolution for a smaller, non-degenerate matrix. We therefore assume $\det\mathtt{P}(t) \neq 0$ hereafter without loss of generality.

Since distinct eigenvalues remain distinct under the flow (31), and since zero eigenvalues can be ignored, we can define the effective dimension $\check{n}$ to be the number of distinct nonzero eigenvalues of $\mathtt{P}$.

As in (10), it is straightforward to deduce the invariance of

$$z^2 = |p_0(t)|/|\det\mathtt{P}(t)|^{1/(n+2\gamma)} = \mathit{const} \tag{33}$$

from (31)–(32). Reexpressed in terms of

$$\mathtt{S}(t) := -\mathtt{P}^{-1}|\det\mathtt{P}(t)|^{1/(n+2\gamma)}, \quad |\det\mathtt{S}| = |\det\mathtt{P}|^{-2\gamma/(n+2\gamma)}, \tag{34}$$

a calculation similar to the proof of Theorem 3 reduces the dynamics (31) to the form

$$\frac{d\mathtt{S}}{dt} = 2|\det\mathtt{S}(t)|^{-1/\gamma}\Big(\mathtt{S}^{-1} + \frac{\operatorname{tr}\mathtt{S}^{-1}}{\gamma}\mathtt{I}\Big). \tag{35}$$

Notice that Lemma 2 now asserts the density $\rho = z^{n+2\gamma}(z\mathtt{A}(t))_\#(f_{(n,j)})^\gamma_+$ is a formal solution to our fourth order evolution (27), where $\mathtt{A}(t) = |\mathtt{S}(t)|^{1/2}$, $f_{(n,j)}$ is defined by (13) and j counts the number of negative eigenvalues of $\mathtt{S}(t)$. It becomes a classical solution if $j = 0 > \gamma$.

As it turns out, (31) and (35) can be reduced to quadratures in 2 dimensions (or if the effective dimension $\check{n}$ is 2), but apparently not otherwise. When the effective dimension is larger, a qualitative analysis leads to Theorem 7. But first we study the case $\check{n} = 2$, both for the $\mathtt{S}$ equation (35) in rescaled time $d\mathtt{S}/d\tau = \mathtt{S}^{-1} + (\operatorname{tr}\mathtt{S}^{-1})\mathtt{I}/\gamma$, and for the $\mathtt{P}$ equation (31). We omit the calculations for $\mathtt{S}$, because they are easier than the ones for $\mathtt{P}$, which we do explain subsequently. The translation between (31) and (35) involves case distinctions from the value of the exponent in $|\det\mathtt{S}| = |\det\mathtt{P}|^{-2\gamma/(n+2\gamma)}$, and both equations are useful for extracting different dynamical information. For example, the $t \searrow t_\alpha$ asymptotics are more easily accessed via $\mathtt{S}$, whereas $\mathtt{P}$ is more convenient for the $t \nearrow t_\omega$ limit.

Assume $\mathtt{S}$ has eigenvalues s_1 and s_2 with respective multiplicity n_1 and n_2, so $n = n_1 + n_2$. The axes $s_1 s_2 = 0$ are excluded from the phase space, which is therefore disconnected. The orbits satisfy the homogeneous ODE

$$\frac{ds_2}{ds_1} = \frac{(\gamma+n_2)s_1 + n_1 s_2}{(\gamma+n_1)s_2 + n_2 s_1},$$

which has the invariant directions $s_2/s_1 = -(n_2+\gamma)/(n_1+\gamma)$ and $s_2/s_1 = 1$ unless $\gamma = -n_1 = -n_2 = -n/2$; its solutions satisfy

$$|s_2 - s_1|^{\gamma+n}|(\gamma+n_1)s_2 + (\gamma+n_2)s_1|^\gamma = \mathit{const}.$$

For $\gamma = -n_1 = -n_2 = -n/2$, the solutions are $s_2/s_1 = \mathit{const}$, but for $\gamma = -n/2$, the $\mathtt{P}$ and $\mathtt{S}$ equations are not related anyway, because the transformation connecting them becomes singular. Phase portraits for both $\mathtt{S}$ and $\mathtt{P}$ in the case $n_1 = n_2 = n/2$ are collected in Figure 2. Portraits for the case $n_1 = 3$, $n_2 = 1$, are collected in Figures 3 and 4 below.

Returning to the P equation, let us assume that P has entry $p_1 =: x$ with multiplicity n_1, and $p_2 =: y$ with multiplicity n_2, with $n = n_1 + n_2$. Then orbits are described by the homogeneous ODE, which follows immediately from (31):

$$\frac{dy}{dx} = \frac{2y^3 + 2(n_1x + n_2y)y^2/\gamma + (n_1x^2 + n_2y^2)y/\gamma + (n_1x + n_2y)^2y/\gamma^2}{2x^3 + 2(n_1x + n_2y)x^2/\gamma + (n_1x^2 + n_2y^2)x/\gamma + (n_1x + n_2y)^2x/\gamma^2}$$

(or, of course, the reciprocal of this equation, to capture orbits where $dx/dy = 0$). It is easy to check for radial lines that are flow invariant and determine the stability or instability of the origin for the flow restricted to these lines: Equating the huge fraction with y/x gives a fourth degree polynomial equation of which three solutions $x = 0$, $y = 0$ and $y = x$ are obvious and allow a fourth solution to be found. The stability of the origin under the reduced dynamics on the radial direction is determined by going back to (31). The results of these routine calculations are listed in the following table, in which the origin is stable if the given sign is negative, and unstable if it is positive. The ray consists of fixed points if the sign is 0:

(36)

invariant direction	stability of origin given by sign of
$y = 0$	$-(n_1 + \gamma)(n_1 + 2\gamma)$
$x = 0$	$-(n_2 + \gamma)(n_2 + 2\gamma)$
$y = x$	$-(n + \gamma)(n + 2\gamma)$
$\frac{y}{x} = -\frac{n_1+\gamma}{n_2+\gamma}$	$-(n_1 + \gamma)(n_2 + \gamma)(n + 2\gamma)/\gamma$

An explicit solution by quadratures can be found when $\check{n} = 2$, using the usual substitution $v = y/x$:

$$\frac{dx}{x} = \left(-\frac{n_1 + 2\gamma}{2\gamma}\frac{1}{v} + \frac{n + \gamma}{2\gamma}\frac{1}{v - 1} + \frac{1}{2}\frac{1}{v + \frac{n_1+\gamma}{n_2+\gamma}} \right) dv$$

which implies, after integration, the equation for the orbits:

$$|x|^{n_2+2\gamma}|y|^{n_1+2\gamma}|y - x|^{-n-\gamma}|(n_2 + \gamma)y + (n_1 + \gamma)x|^{-\gamma} = C. \tag{37}$$

As can be seen from table (36), a wide variety of dynamics occurs over the range of nonlinearities already in two dimensions. We give a graphical sketch for the case where $n_1 = n_2 = n/2$ in Figure 2. For general dimension, we prove the following theorem.

Theorem 7 (Asymptotics of fourth order diffusion family). *(a) For any $\gamma \neq 0$, the distance between distinct eigenvalues s_i of the same sign decreases as a function of time under the evolution (35).*
(b) If $\gamma > 0$ or $\gamma < -n$, then all maximal orbits $\mathtt{P}(t)$ of (31) converge to 0 as $t \to t_\omega = +\infty$, whereas $\|\mathtt{P}\| \to \infty$ as $t \to t_\alpha > -\infty$ (unless $\mathtt{P} \equiv 0$). If P has a definite sign, then a more precise asymptotic is

$$\mathtt{P}(t) = \pm\frac{\gamma\mathtt{I}}{(2(n + \gamma)(n + 2\gamma)t)^{1/2}}(1 + o(1)) + O(t^{-(n+3\gamma)/2(n+\gamma)}) \qquad \textit{as } t \to \infty, \tag{38}$$

where the $O(t^{-(n+3\gamma)/2(n+\gamma)})$ term is traceless. As $t \to t_\alpha$, the eigenvalue of P with largest absolute value blows up (the corresponding eigenvalue of S with smallest absolute value goes to 0). All the other distinct eigenvalues of S go to distinct finite nonzero limits. The leading order of the smallest (in absolute value) eigenvalue s_1 is $s_1 \sim c(t - t_\alpha)^{\gamma/(n_1+2\gamma)}$ as $t \to t_\alpha$.
(c) If $\gamma = -n$, then still $t_\omega = \infty$, and each orbit converges to some multiple of the identity matrix: nonzero, if P has a definite sign, 0 otherwise. Unless $\mathtt{P} = c\mathtt{I}$, the same claims about t_α and its asymptotics hold as in (b).
(d) If P has a specific sign and $-n < \gamma < -n/2$, then $\det \mathtt{P} \to \infty$ as $t \to t_\omega$.

PROOF: Claim (a) is immediate from $d(s_i - s_j)/d\tau = s_i^{-1} - s_j^{-1}$.
Write (31) as

$$-\frac{d}{dt}p_i = p_i \left[p_i^2 + \frac{1}{\gamma}\operatorname{tr}\mathtt{P}^2 + \left(p_i + \frac{1}{\gamma}\operatorname{tr}\mathtt{P} \right)^2 \right] \tag{39}$$

Let $\|\mathtt{P}\|_\infty := \max_i |p_i| =: |p_j|$. For $\gamma > 0$, it obviously follows that $\frac{d}{dt}\|\mathtt{P}\|_\infty \leq -\|\mathtt{P}\|_\infty^3$. A similar conclusion for $\gamma \leq -n$ follows from $\operatorname{tr}\mathtt{P}^2 \leq np_j^2$, namely $\frac{d}{dt}\|\mathtt{P}\|_\infty \leq -(1 - \frac{n}{|\gamma|})\|\mathtt{P}\|_\infty^3$. These estimates give $t_\omega = \infty$

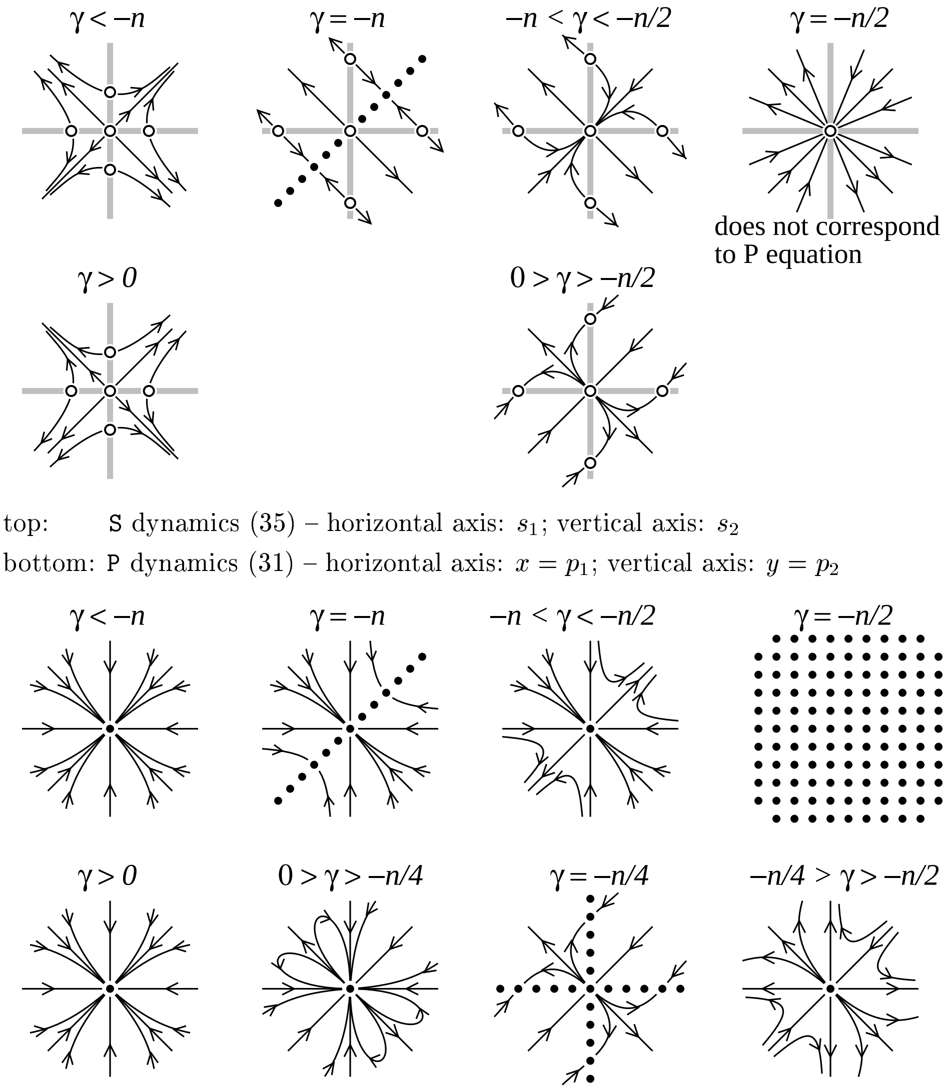


FIGURE 2. Dynamics of equations (31) and (35) in the case $n_1 = n_2$ ($\check{n} = 2$). The graphs are qualitative and represent only topology and tangency, but not symmetries. Note that $\gamma < -n$ and $\gamma > 0$ have homeomorphic phase portraits, hence the cyclic arrangement of graphs.

for $\gamma \leq -n$, as well as $t_\alpha > -\infty$ for $\gamma \notin [-n, 0]$, and hence the qualitative part of claim (b), and the first claim of (c).

If $\mathtt{P}$ has a definite sign, we may assume $\mathtt{P} > 0$. (If $\mathtt{P} < 0$, consider the same argument for $-\mathtt{P}$.) The function $1 - \frac{(\operatorname{tr}\mathtt{P})^2}{n \operatorname{tr}\mathtt{P}^2}$ describes the sine square of the angular distance of a point from the diagonal $\mathbf{R}\mathtt{I}$, and it turns out to be a Lyapunov function in the positive cone (for the dynamics projected on the unit sphere): Indeed, a straightforward calculation gives

$$\frac{d}{dt}\left(1 - \frac{(\operatorname{tr}\mathtt{P})^2}{n \operatorname{tr}\mathtt{P}^2}\right) = \frac{4 \operatorname{tr}\mathtt{P}}{n(\operatorname{tr}\mathtt{P}^2)^2}\left(\operatorname{tr}\mathtt{P}^3 \operatorname{tr}\mathtt{P}^2 - \operatorname{tr}\mathtt{P}^4 \operatorname{tr}\mathtt{P} + \frac{1}{\gamma}(\operatorname{tr}\mathtt{P}^2)^2 \operatorname{tr}\mathtt{P} - \frac{1}{\gamma}\operatorname{tr}\mathtt{P}^3(\operatorname{tr}\mathtt{P})^2\right) . \tag{40}$$

Now the usual Hölder style interpolation of ℓ^p norms in $\mathbf{R}^n$ gives

$$\operatorname{tr} \mathtt{P}^3 \le (\operatorname{tr} \mathtt{P}^4)^{1/2} (\operatorname{tr} \mathtt{P}^2)^{1/2} \qquad \text{hence} \qquad \operatorname{tr} \mathtt{P}^4 \ \operatorname{tr} \mathtt{P} \ge \frac{\operatorname{tr} \mathtt{P}^3}{\operatorname{tr} \mathtt{P}^2} \times (\operatorname{tr} \mathtt{P}^3 \ \operatorname{tr} \mathtt{P}).$$

With this estimate used on the second term in the parenthesis of (40), we conclude

$$\frac{d}{dt}\left(1 - \frac{(\operatorname{tr} \mathtt{P})^2}{n \operatorname{tr} \mathtt{P}^2}\right) \le \frac{-4 \operatorname{tr} \mathtt{P}}{n (\operatorname{tr} \mathtt{P}^2)^2} \left(\frac{\operatorname{tr} \mathtt{P}^3}{\operatorname{tr} \mathtt{P}^2} + \frac{1}{\gamma} \operatorname{tr} \mathtt{P}\right) \times \left(\operatorname{tr} \mathtt{P}^3 \ \operatorname{tr} \mathtt{P} - (\operatorname{tr} \mathtt{P}^2)^2\right) \tag{41}$$

The first parenthesis on the right is trivially positive if $\gamma > 0$, and positive by Hölder's inequality if $\gamma < -n$. If $\gamma = -n$, it is still positive, unless $\mathtt{P} \in \mathbf{R}\mathtt{I}$. The second factor on the right is nonnegative by Hölder style interpolation again.

The orbits of (31), projected on the unit sphere, are orbits of a dynamical system there. By homogeneity of (31) different orbits whose initial points project onto the same point on the sphere, will, as a whole, have the same projection onto the sphere for all time, even though the time parametrization will depend on the radial coordinate of the initial point. We have shown that on the orthant of the unit sphere that comes from the intersection with the positive cone, $1 - (\operatorname{tr} \mathtt{P})^2 / n \operatorname{tr} \mathtt{P}^2$ is a strict Lyapunov function minimized at the diagonal point $\mathtt{I}/\|\mathtt{I}\|_2$. So this point is the $t \to t_\omega$ limit of the projected dynamics.

We have therefore shown that $\mathtt{P}(t) = \sigma(t)\mathtt{I} + \mathtt{T}(t)$ as $t \to t_\omega = \infty$, where the error term $\mathtt{T}(t)$ is $o(\sigma(t))$; with no loss of generality, by a small modification of σ we may assume $\mathtt{T}(t)$ to be traceless. The function σ is yet unspecified, except that we know $\sigma(t) \to 0$ as $t \to \infty$. Using this decomposition in (31), we conclude

$$-\frac{d\sigma}{dt} = \left(2 + \frac{n}{\gamma}\right)\left(1 + \frac{n}{\gamma}\right)(1 + o(1))\sigma^3 , \quad -\frac{d\mathtt{T}}{dt} = \left(2 + \frac{n}{\gamma}\right)\left(3 + \frac{n}{\gamma}\right)(1 + o(1))\sigma^2 \mathtt{T} ,$$

which can be integrated to obtain (38): first the asymptotic for σ, then for $\mathtt{T}$. The $o(1)$-term in σ is actually $O(\mathtt{T}^2/\sigma^2)$ and could subsequently be improved from the $\mathtt{T}$ asymptotics if so desired. This proves the t_ω asymptotics of (b).

Still in (b), from $d \operatorname{tr} \mathtt{S}/d(-\tau) = -(1 + n/\gamma) \operatorname{tr} \mathtt{S}^{-1}$ it is clear that all $s_i > 0$ remain bounded above as $t \to t_\alpha$. Denoting the smallest s_i as s_1, we find, in view of (a), that $s_1 = \frac{1}{n}[\operatorname{tr} \mathtt{S} - \sum(s_i - s_1)]$ decreases as $t \searrow t_\alpha$. So $\lim s_1$ exists. Now from (a) again, all $\lim s_i$ exist by monotonicity. If s_1 were positive, then all s_i would be bounded away from both ∞ and 0, and then the same would follow for the p_i, in contradiction to $\|\mathtt{P}\|_\infty \to \infty$ as $t \searrow t_\alpha$. We therefore conclude $s_1 \to 0$. All eigenvalues distinct from

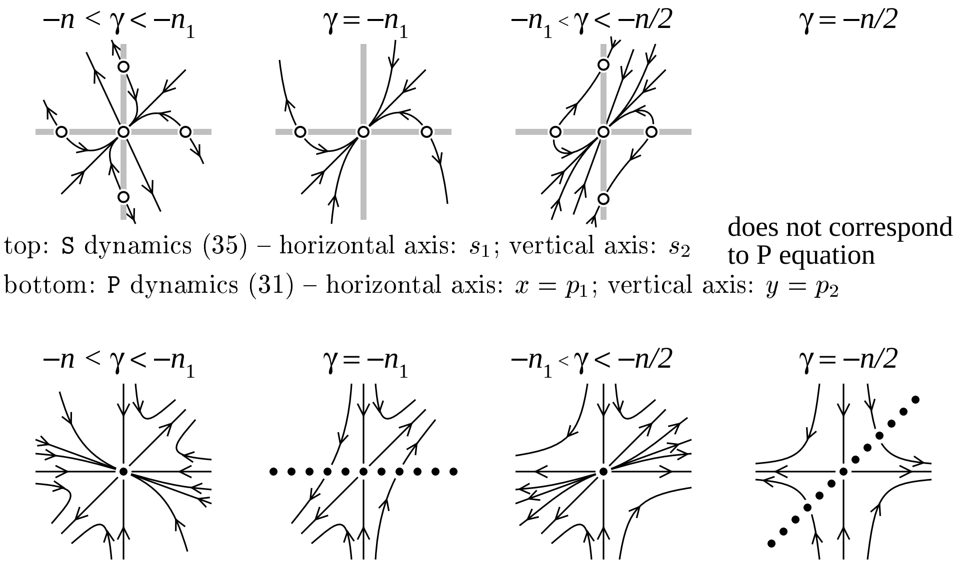


FIGURE 3. Some cases of the dynamics of (35) and (31) for $\check{n} = 2$, $n_1 = 3$, $n_2 = 1$, $n = 4$. A variety of behavior in the range $-n < \gamma \le -n/2$ already.

s_1 must have a positive limit, due to (a). Absorbing all $\lim s_i$ other than 0 in a constant, equation (35) becomes $\frac{d}{dt}s_1 = \tilde{c}s_1^{-1-n_1/\gamma}(1+O(s_1))$ and hence $s_1 \sim c(t-t_\alpha)^{\gamma/(n_1+2\gamma)}$ as $t \to t_\alpha$.

Concerning (c), we have already obtained the boundedness of the orbits and $t_\omega = \infty$ from the previous calculation $\frac{d}{dt}\|\mathtt{P}\|_\infty \leq 0$. By LaSalle's invariance principle (see, e.g., [46, VII§30]), the ω-limit set consisting of all forward accumulation points of a given orbit must be a connected invariant subset of the set on which $\frac{d}{dt}\|\mathtt{P}\|_\infty$ vanishes. Vanishing of this derivative implies that $\mathtt{P}$ is a multiple of $\mathtt{I}$, so the subset in question could only be an interval on the diagonal $\mathbf{R}\mathtt{I}$. Each $\mathtt{P}$ on the diagonal is an equilibrium when $\gamma = -n$. If $\mathtt{P}$ is not (positive or negative) definite, this (together with the fact that the evolution preserves the signs of the p_i) only leaves the possibility $\mathtt{P} \to 0$.

Now assume $\mathtt{P}$ (and hence $\mathtt{S}$, too) has a definite sign; we still want to show that the compact interval on the diagonal must be a singleton. Use (35) with a rescaled time τ, namely $\frac{d}{d\tau}\mathtt{S} = \mathtt{S}^{-1} + \frac{1}{\gamma}\operatorname{tr}\mathtt{S}^{-1}$, hence $\frac{1}{2}\frac{d}{d\tau}\operatorname{tr}\mathtt{S}^2 = n + \frac{1}{\gamma}\operatorname{tr}\mathtt{S}\operatorname{tr}\mathtt{S}^{-1} \leq n - n^2/|\gamma|$, which equals 0 for $\gamma = -n$. By monotonicity, $\lim_{\tau\to\tau_\omega}\operatorname{tr}\mathtt{S}^2$ exists. This, together with the previous conclusion about the ω-limit set, forces the ω-limit set to be a singleton, i.e., $\lim_{t\to\infty}\mathtt{P}(t) = c\mathtt{I}$ for some c. Now $c \neq 0$ because else $\operatorname{tr}\mathtt{S}^2 \to \infty$. This proves the t_ω claims of (c).

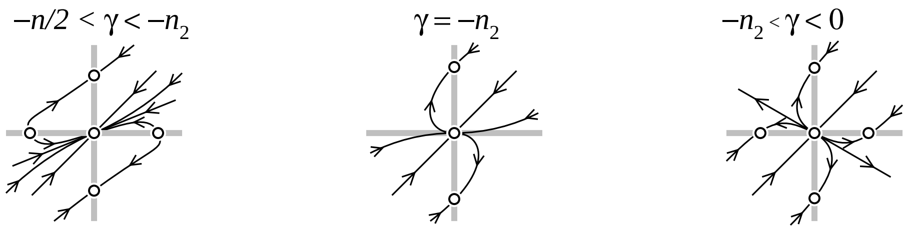
For the t_α limit, assume $\mathtt{P} \neq c\mathtt{I}$. Then we still get from (41) that $\frac{d}{dt}(1 - (\operatorname{tr}\mathtt{P})^2/(n\operatorname{tr}\mathtt{P}^2)) < 0$. So there exists an $\varepsilon > 0$ such that $(\operatorname{tr}\mathtt{P})^2/n\operatorname{tr}\mathtt{P}^2) < (1-\varepsilon)$ as $t \to t_\alpha$. Then by an immediate indirect proof, there exists also some $\delta > 0$ such that $\operatorname{tr}\mathtt{P} < n(1-\delta)p_j$. This, with (39), gives $-\frac{d}{dt}\|\mathtt{P}\|_\infty \geq \delta^2\|\mathtt{P}\|^3$, hence $\|\mathtt{P}\|_\infty \to \infty$ as $t \to t_\alpha > -\infty$. The arguments from (b) for the quantitative asymptotics carry over literally now.

For $-n < \gamma$, we even have $d\operatorname{tr}\mathtt{S}^2/d\tau \leq -c < 0$ (and hence $\tau_\omega < \infty$). We cannot conclude $\mathtt{S} \to 0$ by Lyapunov's method, due to the singularity on the hyperplanes $\det\mathtt{S} = 0$. But as either Lyapunov's conclusion applies, or else a singularity is hit, we can conclude $\det\mathtt{S} \to 0$ as $\tau \to \tau_\omega$ either way. For $\gamma < -n/2$, this implies $\det\mathtt{P} \to \infty$, thus proving (d). ■

A study of asymmetric multiplicities $n_1 \neq n_2$ in effective dimension $\check{n} = 2$ reveals that the exact rate of blowup of one or several eigenvalues may depend further on the initial data. Depending on γ and initial data, only some or all s_i may go to 0; see Figure 3. In the few cases studied, the relative asymptotics observed implies $t_\omega < \infty$, but deciding validity of this conjecture in general remains a task for some further research. Moreover, for $\gamma = -n/2$, there arise affine images of Barenblatt with $t_\alpha = -\infty$ whose initial profile as $t \searrow t_\alpha$ is not a singular measure, but a regular density. It is remarkable that in the cases $\gamma = -n$ (or equivalently $m = 1 - 1/n$) and $\gamma = -n/2$ ($m = 1 - 2/n$), the Barenblatt profile is a stationary point of the dynamics (27). According to Figure 3, we see that this profile is unstable for $\gamma = -n/2$, but is stable at least within the invariant subspace of quadratic pressure solutions for $\gamma = -n$, due to Theorem 7(c). Whether it remains globally attractive for fourth order diffusions with $\gamma \notin [-n, 0]$ and a wide class of initial data, as in the second order case, remains an open question. For the special case $(n,m) = (1, 3/2)$ it was addressed by Carrillo & Toscani [14]. The analogous question in a periodic domain — attractivity of the constant solution — was addressed in their subsequent work with Cáceres when $(n,m) = (1,1)$ [13].

Let us conclude our study of fourth order nonlinear diffusions by showing that every degenerate matrix $\det\mathtt{S}_0 = 0$ is either the initial or final point of at least one orbit of the dynamical system (35), and complement Theorem 7(b) by giving the asymptotics of the orbit at that limit. To do so, it is convenient to replace $\mathtt{S}(t)$ by an effective matrix $\mathtt{Q}(t)$ whose eigenvalues $q_i(t)$ are distinct, but coincide with those of $\mathtt{S}(t)$ otherwise; they come with multiplicities $n_1, \ldots, n_{\check{n}}$ in $\mathtt{S}(t)$. Obviously $\mathtt{Q}(t)$ and the integer matrix $\mathtt{N}$ of multiplicities are $\check{n} \times \check{n}$ diagonal matrices satisfy the evolution

$$\begin{aligned} \frac{dq_i}{d\tau} &:= \frac{1}{2|\det\mathtt{S}(t)|^{-1/\gamma}}\frac{dq_i}{dt} \\ &= q_i^{-1} + \frac{1}{\gamma}\operatorname{tr}\mathtt{N}\mathtt{Q}^{-1}. \end{aligned} \tag{42}$$



top: $\mathtt{S}$ dynamics (35) – horizontal axis: s_1; vertical axis: s_2

bottom: $\mathtt{P}$ dynamics (31) – horizontal axis: $x = p_1$; vertical axis: $y = p_2$

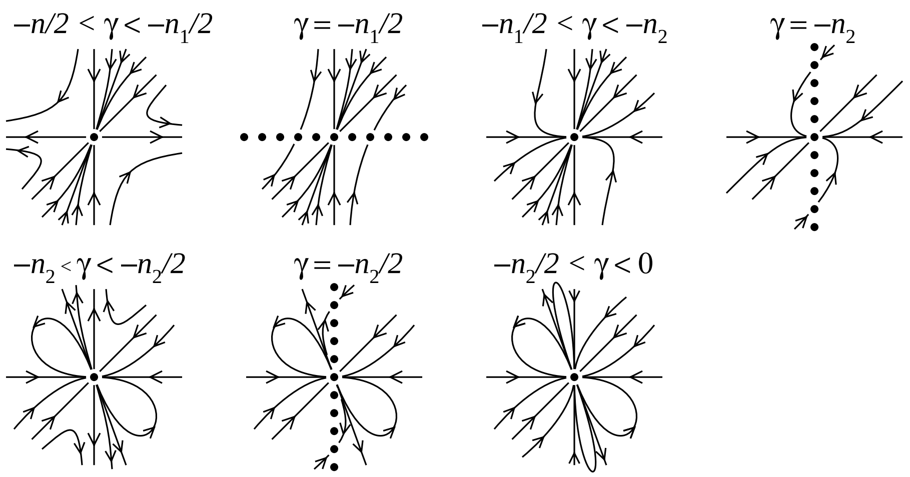


FIGURE 4. More cases of the dynamics of (35) and (31) for $\check{n} = 2$, $n_1 = 3$, $n_2 = 1$, $n = 4$.

Theorem 8 (Affinely self-similar fourth-order dynamics from sources concentrated on subspaces)**.** *Let $\mathtt{Q}_0$ and $\mathtt{N}$ be diagonal $\check{n} \times \check{n}$ matrices, with eigenvalues $\bar{q}_1, \ldots, \bar{q}_{\check{n}} \in \mathbf{R}$ and $n_1, \ldots, n_{\check{n}} \in \mathbf{N}$. Suppose $\bar{q}_1 = 0$ and $0 \neq \gamma \neq -n_1$. If the eigenvalues of $\mathtt{Q}_0$ are distinct, then precisely two maximal smooth orbits $\det \mathtt{Q}(t) \neq 0$ of (42) include $\mathtt{Q}_0$ in their closures. These two orbits $\mathtt{Q}(t)$ have unit tangent vectors which parallel $\pm(1 + \frac{\gamma}{n_1}, 1, \ldots, 1)$ at $\mathtt{Q}_0$. If $\gamma \notin [-n_1, 0]$ then $\mathtt{Q}_0 = \lim_{t \to t_\alpha} \mathtt{Q}(t)$ and $t_\alpha \in \mathbf{R}$; otherwise $\mathtt{Q}_0 = \lim_{t \to t_\omega} \mathtt{Q}(t)$ with $t_\omega \in \mathbf{R}$ if $\gamma \in]-n_1, -n_1/2[$, and $t_\omega = +\infty$ if $\gamma \in [-n_1/2, 0[$, as in (46)-(47).*

PROOF: The orbits $\mathtt{Q}(t) = (q_1(t), \ldots, q_{\check{n}}(t))$ of the dynamical system (42) are tangent to the field of unit vectors whose directions are given by

$$\frac{dq_k}{dq_1} = \frac{q_1(q_k^{-1} + \operatorname{tr} \mathtt{N}\mathtt{Q}^{-1}/\gamma)}{1 + q_1 \operatorname{tr} \mathtt{N}\mathtt{Q}^{-1}/\gamma} \tag{43}$$

for $\det \mathtt{Q} \neq 0$ and $k > 1$. Since the eigenvalues of $\mathtt{Q}_0$ are distinct, we see $q_1 \operatorname{tr} \mathtt{N}\mathtt{Q}^{-1}$ is a smooth function of $\mathtt{Q} = (q_1, \ldots, q_{\check{n}})$ in a neighbourhood of $\mathtt{Q}_0 = \operatorname{diag}(0, \bar{q}_2, \ldots \bar{q}_3)$, with limit $q_1 \operatorname{tr} \mathtt{N}\mathtt{Q}^{-1} \to n_1$ as $q_1 \to 0$. Thus the direction field (43) of unit tangent vectors extends smoothly to a neighbourhood of $\mathtt{Q}_0$, with

$$\left.\frac{dq_k}{dq_1}\right|_{\mathtt{Q}_0} = \frac{n_1/\gamma}{1 + n_1/\gamma}. \tag{44}$$

The theory of ordinary differential equations asserts this smooth direction field has a (unique) integral curve passing through $\mathtt{Q}_0$; it intersects the plane $q_1 = 0$ transversally since (44) is non-vanishing. Since orbits of the dynamical system (42) lie on integral curves of (43), at most two such orbits $\det \mathtt{Q}(t) \neq 0$ approach $\mathtt{Q}_0$. On the other hand, for $\mathtt{Q}(t)$ near $\mathtt{Q}_0$, the sign of

$$q_1 \frac{dq_1}{d\tau} = 1 + \frac{n_1}{\gamma} + O(q_1) \qquad \text{as } q_1 \to 0, \tag{45}$$

shows $\mathtt{Q}_0$ is the initial point $\tau \searrow \tau_\alpha > -\infty$ of both orbits if $\gamma \notin [-n_1, 0]$, and the final limit $\tau \nearrow \tau_\omega < +\infty$ of both orbits $\mathtt{Q}(t)$ otherwise. This shows $\mathtt{Q}_0$ is either a fixed point of the dynamics, as for $\gamma = -n_1/2$, or a singular point. Either way, the lemma is proved. Integrating (45) yields

$$\frac{q_1(\tau)^2}{2} = \begin{cases} -(1+\frac{n_1}{\gamma})(\tau_\omega - \tau) + O(|\tau_\omega - \tau|^{3/2}) & \text{if } \gamma \in]-n_1, 0[\\ (1+\frac{n_1}{\gamma})(\tau - \tau_\alpha) + O(|\tau - \tau_\alpha|^{3/2}) & \text{if } \gamma \notin [-n_1, 0]. \end{cases} \tag{46}$$

Thus $q_i(\tau) - \bar{q}_i = O(|\tau - \tau_0|^{1/2})$ as $\tau \to \tau_0$ for $i = 1, \ldots, \check{n}$, where $\tau_0 = \tau_\alpha$ or τ_ω and the limit is taken from above or below according to the sign of $1 + n_1/\gamma$. Using this to integrate

$$\begin{aligned} \frac{dt}{d\tau} &= \frac{1}{2}\prod_{k=1}^{\check{n}} |q_k(\tau)|^{n_k/\gamma} \\ &= \left|2\Big(1+\frac{n_1}{\gamma}\Big)(\tau - \tau_0)\right|^{n_1/2\gamma}\Big(\frac{1}{2} + O(|\tau-\tau_0|^{1/2}\Big)\prod_{k=2}^{\check{n}} |\bar{q}_k|^{n_k/\gamma} \end{aligned}$$

yields

$$\begin{array}{llll} t - t_0 \sim \frac{\gamma c}{n_1+2\gamma}|\tau - \tau_0|^{(n_1+2\gamma)/2\gamma} & \text{with } t_0 \in \mathbf{R} & \text{if } \gamma \notin [-n_1/2, 0], \\ t \sim -\frac{c}{2}\log|\tau_\omega - \tau| & \text{with } t_\omega = +\infty & \text{if } \gamma = -n_1/2, \\ t \sim -\frac{\gamma c}{n_1+2\gamma}|\tau_\omega - \tau|^{(n_1+2\gamma)/2\gamma} & \text{with } t_\omega = +\infty & \text{if } \gamma \in]-n_1/2, 0], \end{array} \tag{47}$$

as $\tau \to \tau_0$, where $c = |2(1+\frac{n_1}{\gamma})|^{n_1/2\gamma}\prod_{k=2}^{\check{n}}|\bar{q}_k|^{n_k/\gamma} > 0$. ■

Notice the limiting tangent direction $\pm(1+\frac{\gamma}{n_1}, 1, \ldots, 1)$ of orbits in this result always lies in a plane, but rotates 180 degrees over the full range of $\gamma \in \mathbf{R}$, which helps to explain the diversity of dynamics represented in Figures 2, 3, and 4.

5. Appendix: Weak pressure and density evolutions

This appendix is devoted to showing the positive truncation $\rho = (\pi/\gamma)_+^\gamma$ of our quadratic pressure (7) yields a weak solution to the porous medium equation in the pressure form (5) with $u = \rho^{1/\gamma}$ and in the density form (6).

Lemma 9 (Non-negative solutions from signed solutions). *If $u \in C^2(\mathbf{R}^n \times [0,\infty[)$ is a classical solution to $\partial u/\partial t = u\Delta u + \gamma|\nabla u|^2$, then $u_+ := \max\{u, 0\}$ is a weak solution to this same equation in the sense of (5).*

Proof: Since $u_+ \in W^{1,2}_{\text{loc}}$, the term $\int u_+\psi_t\, dt$ in (5) can be integrated by parts with $(u_+)_t = u_t\,\chi_{\{u>0\}}$, as in e.g. Lieb & Loss [34, §6]. Likewise $\nabla u_+ = (\nabla u)\chi_{\{u>0\}}$. To show that for each t,

$$\int_{\{u(t,\cdot)>0\}}\Big((-u\Delta u - \gamma|\nabla u|^2)\psi - u\nabla u\cdot\nabla\psi + (\gamma-1)|\nabla u|^2\psi\Big)\, d\boldsymbol{x} = 0$$

we note that the left hand side reduces to $-\int_{\{u>0\}}\operatorname{div}(u\psi\nabla u)\, d\boldsymbol{x}$, which vanishes without further geometric hypotheses on the boundary of $\Omega := \{\boldsymbol{x} \in \mathbf{R}^n \mid u(\boldsymbol{x}, t) > 0\}$, because the C^1 vector field $v := u\psi\nabla u$ vanishes on the boundary of the open set $\Omega \subset \mathbf{R}^n$. To prove this, fix a compact set $\Omega_\varepsilon \subset \Omega$ whose boundary is in an ε-neighbourhood of $\partial\Omega$, and a smooth cutoff function $\eta_\varepsilon \in C_c^\infty(\Omega; [0,1])$, compactly supported in Ω and with $\eta_\varepsilon = 1$ on Ω_ε. Specifically, we can mollify $\hat{\eta}_\varepsilon := \min\{1, \frac{3}{\varepsilon}(\operatorname{dist}(x, \partial\Omega) - \frac{\varepsilon}{3})_+)\}$. Then we have $\operatorname{div} v = O(1)$ on $\Omega \setminus \Omega_\varepsilon$, and the measure of $\Omega\setminus\Omega_\varepsilon$ goes to 0 as $\varepsilon \to 0$. Therefore

$$\begin{aligned} \int_\Omega \operatorname{div} v &= \int_{\Omega_\varepsilon}\operatorname{div}(v\eta_\varepsilon) + \int_{\Omega\setminus\Omega_\varepsilon}\operatorname{div} v \\ &= \int_\Omega \operatorname{div}(v\eta_\varepsilon) - \int_{\Omega\setminus\Omega_\varepsilon}((\operatorname{div} v)\eta_\varepsilon + v\cdot\nabla\eta_\varepsilon) + o(1) \qquad \text{as } \varepsilon \to 0. \end{aligned}$$

The first integral vanishes, since Ω can be deformed inwards to a smooth domain on which the divergence theorem applies, still avoiding the compact support of η_ε. The remaining terms are small, since

$\|v\|_\infty = O(\varepsilon)$ balances $\|\nabla\eta_\varepsilon\|_\infty = O(1/\varepsilon)$ as $\varepsilon \to 0$, while $\operatorname{div} v$ and $\|\eta_\varepsilon\|_\infty$ are bounded and the domain of integration has measure going to 0. ∎

A final lemma asserts that the weak solutions (5) we described to the pressure evolution (3) also give rise to weak solutions of the density evolution (1).

Lemma 10 (Weak pressure evolutions yield weak density evolutions)**.** *Fix $\gamma > 0$. Assume a weak solution u to $u_t = u\Delta u + \gamma|\nabla u|^2$, in the sense of (5) is Lipschitz (uniformly on compact subsets of $[0,\infty[\, \times \mathbf{R}^n$). Moreover, if $\gamma < 1$, assume $\nabla u^\gamma \in L^1_{\mathrm{loc}}([0,\infty[\, \times \mathbf{R}^n)$, or $\nabla u^{(1+\gamma)/2} \in L^2_{\mathrm{loc}}([0,\infty[\, \times \mathbf{R}^n)$. Then $\rho := u^\gamma$ is a weak solution of the porous medium equation in the sense of (6).*

Proof: For a Lipschitz function u to be a weak solution, the expression

$$V := \int u_0\psi_0\, d\boldsymbol{x} + \iint \left(\psi_t u + (\gamma-1)|\nabla u|^2\psi - u\nabla u \cdot \nabla\psi\right)\, d\boldsymbol{x}\, dt$$

vanishes for all ψ Lipschitz with compact support. Guided by the calculation with classical solutions, we'd like to choose $\psi := \gamma u^{\gamma-1}\varphi$. We can choose $\psi_\varepsilon := \gamma(u+\varepsilon)^{\gamma-1}\varphi$ for $\varphi \in C^1$ with compact support. Then

$$\begin{aligned} V/\gamma = &\int u_0(u_0+\varepsilon)^{\gamma-1}\varphi_0\, d\boldsymbol{x} + \iint (\gamma-1)(u+\varepsilon)^{\gamma-2} u u_t \varphi\, d\boldsymbol{x}\, dt \\ &+ \iint (u+\varepsilon)^{\gamma-1} u\varphi_t\, d\boldsymbol{x}\, dt + (\gamma-1)\iint |\nabla u|^2 (u+\varepsilon)^{\gamma-1}\varphi\, d\boldsymbol{x}\, dt \\ &- \iint u\nabla u(\gamma-1)(u+\varepsilon)^{\gamma-2}\nabla u\varphi\, d\boldsymbol{x}\, dt - \iint u\nabla u(u+\varepsilon)^{\gamma-1}\nabla\varphi\, d\boldsymbol{x}\, dt. \end{aligned}$$

The first, third and sixth terms converge to their formal limits by the dominated convergence theorem. Terms 4 and 5 can be merged; term 2 gets split into an easy part and one with an explicit factor ε:

$$\begin{aligned} V/\gamma = &\int u_0^\gamma \varphi_0\, d\boldsymbol{x} + \iint u^\gamma \varphi_t\, d\boldsymbol{x}\, dt - \iint u^\gamma \nabla u \nabla\varphi\, d\boldsymbol{x}\, dt \\ &+ (\gamma-1)\iint (u+\varepsilon)^{\gamma-1} u_t\varphi\, d\boldsymbol{x}\, dt \\ &- (\gamma-1)\,\varepsilon \iint (u+\varepsilon)^{\gamma-2}(u_t - |\nabla u|^2)\varphi\, d\boldsymbol{x}\, dt + o(1) \quad \text{as } \varepsilon \to 0. \end{aligned}$$

Writing $(u+\varepsilon)^{\gamma-1}u_t = \gamma^{-1}\partial_t(u+\varepsilon)^\gamma$ and $(\gamma-1)(u+\varepsilon)^{\gamma-2}u_t = \partial_t(u+\varepsilon)^{\gamma-1}$, we integrate the u_t terms by parts, set $u = \rho^{m-1} = \rho^{1/\gamma}$, and use dominated convergence again to get

$$\begin{aligned} V/\gamma = &\gamma^{-1}\int \rho_0\varphi_0\, d\boldsymbol{x} + \gamma^{-1}\iint \rho\varphi_t\, d\boldsymbol{x}\, dt - \frac{1}{\gamma+1}\iint \nabla\rho^{(\gamma+1)/\gamma}\nabla\varphi\, d\boldsymbol{x}\, dt \\ &+ (\gamma-1)\,\varepsilon\iint (u+\varepsilon)^{\gamma-2}|\nabla u|^2\varphi\, d\boldsymbol{x}\, dt + o(1) \end{aligned}$$

as $\varepsilon \to 0$. (This uses the estimate $\varepsilon(u+\varepsilon)^{\gamma-1} = O(\varepsilon^{\min(1,\gamma)})$ on the support of φ.) We have to show that the last integral disappears as $\varepsilon \to 0$. For $\gamma \geq 2$ the dominated convergence theorem completes the proof. If $1 < \gamma < 2$, then $\varepsilon(u+\varepsilon)^{\gamma-2} \leq \varepsilon^{\gamma-1}$ implies this term is $O(\varepsilon^{\gamma-1})$. If $\gamma = 1$, its coefficient is zero anyways. Finally, for $0 < \gamma < 1$, we have $\varepsilon(u+\varepsilon)^{\gamma-2} \leq (u+\varepsilon)^{\gamma-1} \leq u^{\gamma-1}$. Since ∇u and φ are bounded on the support of φ, dominated convergence again applies in view of our hypotheses on ∇u^γ or $\nabla u^{(1+\gamma)/2}$. In the limit we retrieve (6). ∎

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