On Generalized Fiducial Inference

Jan Hannig

jan.hannig@colostate.edu

University of North Carolina at Chapel Hill

Parts of this talk are based on joint work with:

Hari Iyer, Thomas C.M. Lee,

Paul Patterson, Lidong E, Damian Wandler and Derek Sonderegger

Colorado State University

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- Oxford English Dictionary
  - **adjective** TECHNICAL (of a point or line) used as a fixed basis of comparison.
  - ORIGIN from Latin fiducia 'trust, confidence'
- Merriam-Webster dictionary
  1. taken as standard of reference *a fiducial mark*
  2. founded on faith or trust
  3. having the nature of a trust : FIDUCIARY

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- related work: Fraser (1960), Dawid and Stone (1982), Dempster (1968, 2008).
- it is fair to say that fiducial inference failed to occupy an important place in mainstream statistics

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- Hannig, Iyer & Patterson (2006) noted that every published generalized confidence interval was obtainable using the fiducial arguments
- and they proved the asymptotic frequentist correctness of such intervals
- Hannig (2008) have developed/modified these ideas further — termed the resulting work generalized fiducial inference

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- the switching of the roles of $\boldsymbol{X}$ and θ

Simplistic Example 1

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- We can simulate this distribution using $\mathcal{R}_\mu = 10 - Z^\star$, where $Z^\star \sim N(0, 1)$ independent of Z.

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- After simplification the fiducial distribution is $N(\bar{x}, 1/n)$.
- We have non-uniqueness due to Borel paradox.

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- Let $\mathbf{X} = \mu + \sigma \mathbf{Z}$.
  - If we observe $a_i < X_i < b_i$ we need to generate $\mathbf{Z}^\star$ keeping only those values that agree with $a_i < \mu + \sigma \mathbf{Z}_i^\star < b_i$ for all i.

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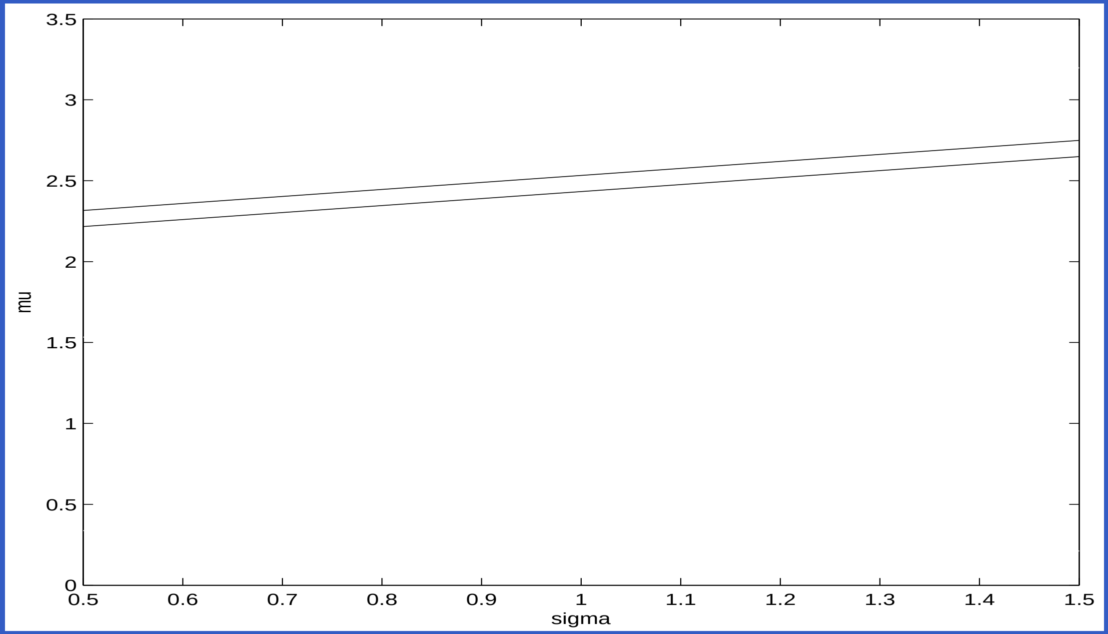

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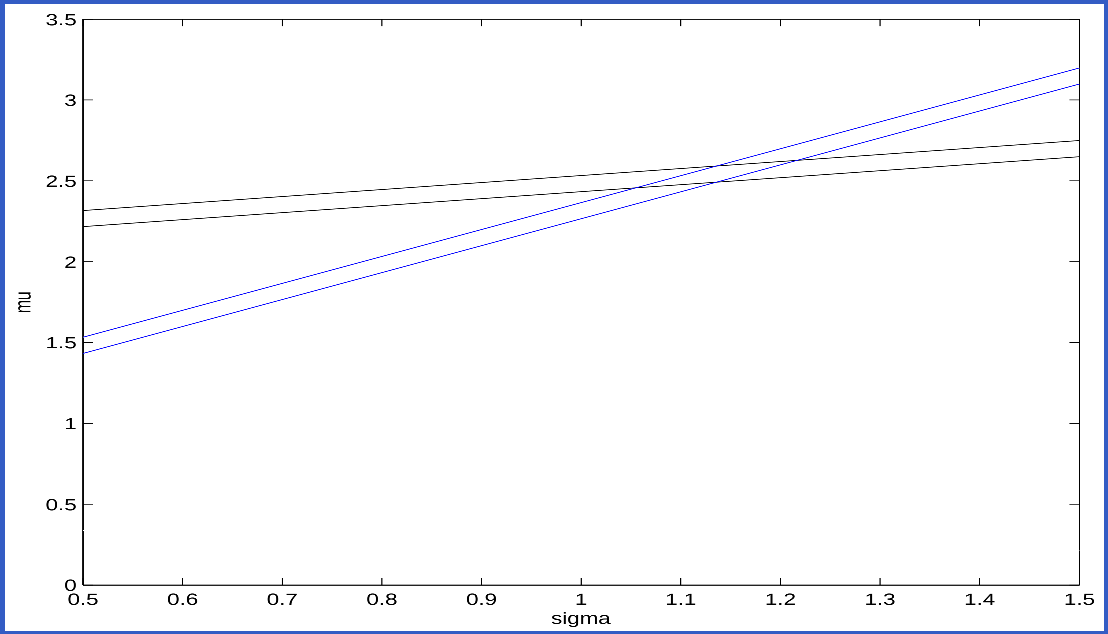

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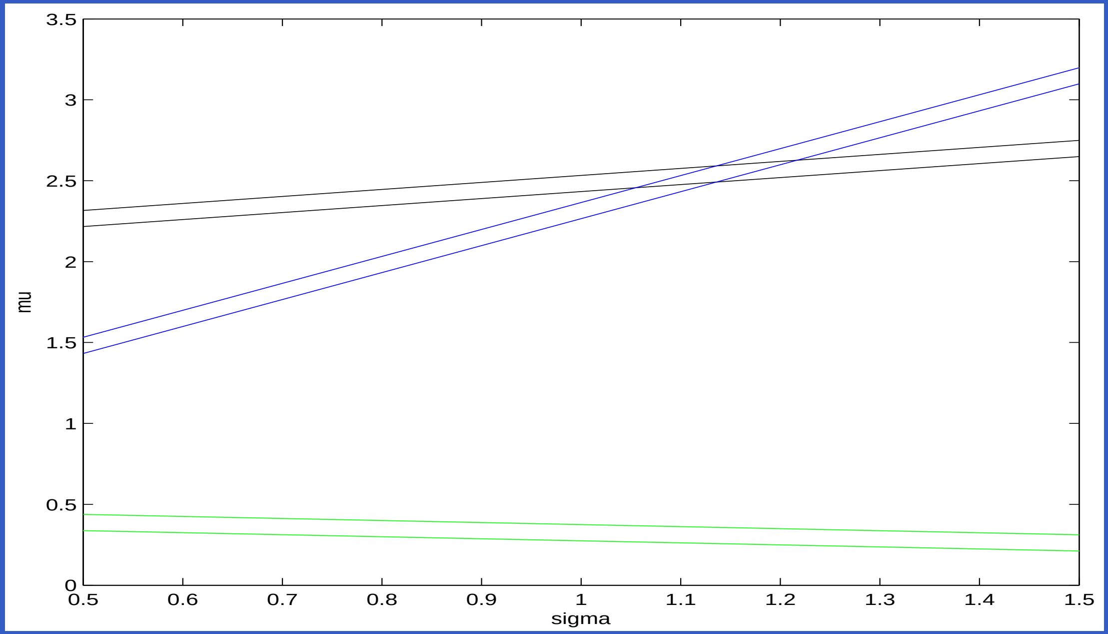

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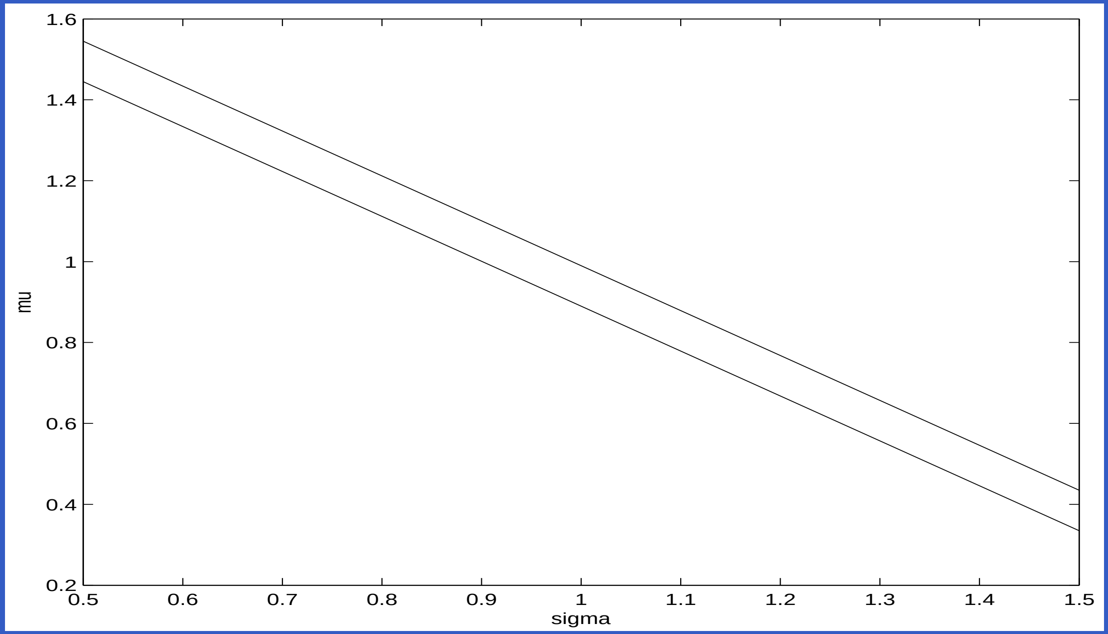

$$Z_1^\star = 1.102$$

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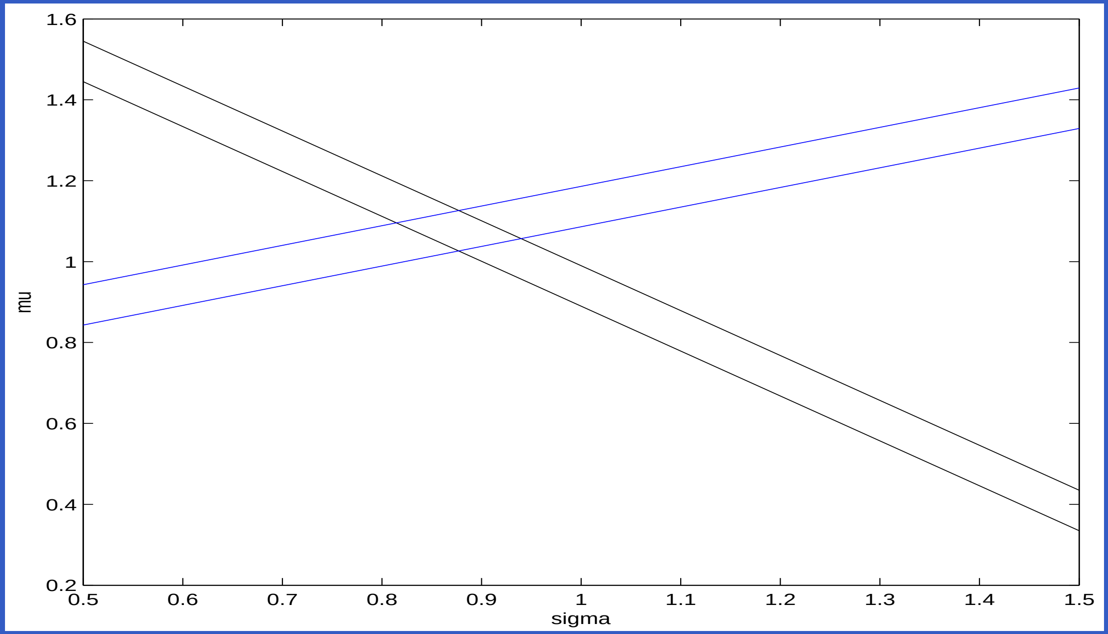

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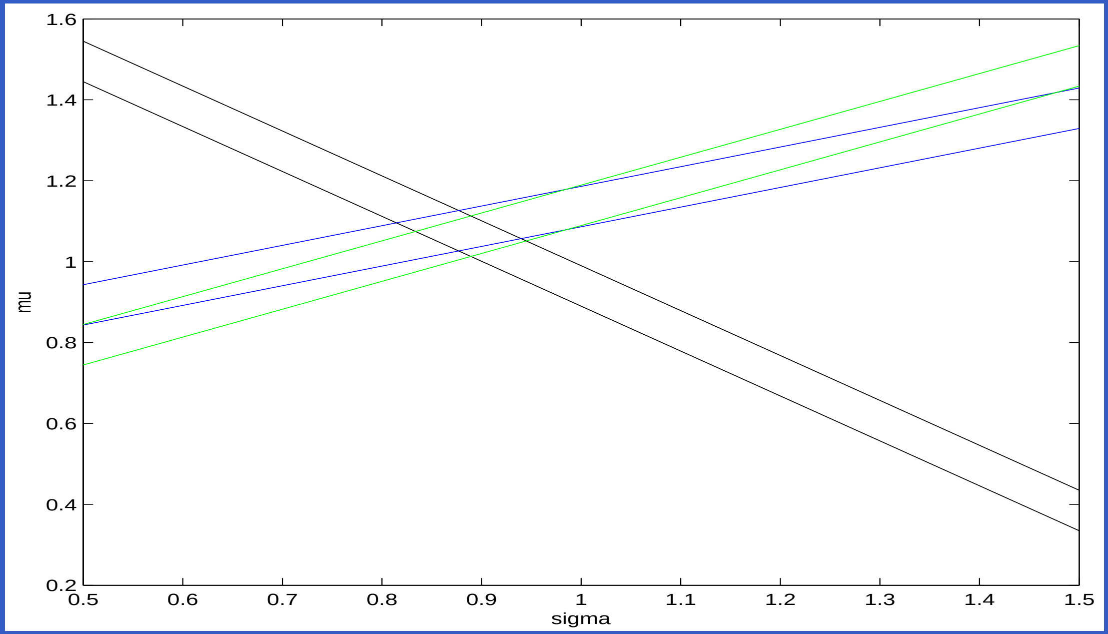

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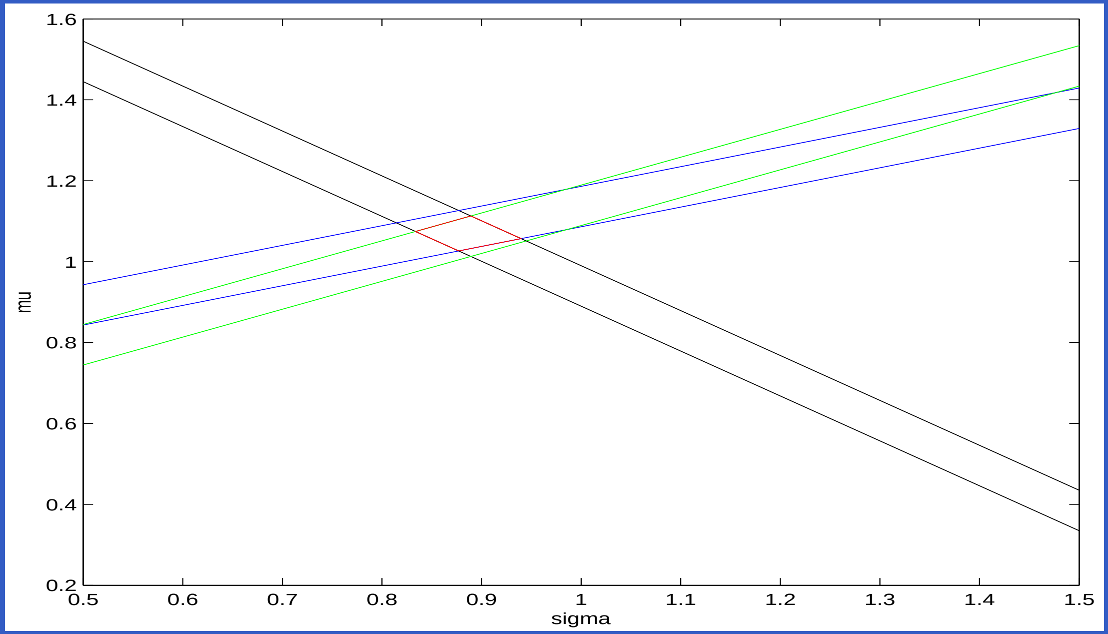

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- Denote the intersection by Q.

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- $P(Q \neq \emptyset) \geq P(\mathbf{X} \in (\mathbf{a},\mathbf{b})) > 0$, so there is no Borel paradox in the definition of fiducial distribution (1).
- Q typically contains more than one element. We can either use Dempster-Shafer calculus to interpret its meaning, or additionally choose (randomly) an element from Q.

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- Let $\mathbf{X} = \mu + \sigma\mathbf{Z}$. We observe (2.0,2.1), (0.6,0.7), (0.4, 0.5), (1.4,1.5), (0.7,0.8), (0.8,0.9), (1.2,1.3), (1.2,1.3), (1.1,1.2), (1.5,1.6), (1.4,1.5), (0.4,0.5), (1.2,1.3), (0.7,0.8), (0.5,0.6).

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- The exact distribution of $Q((\mathbf{a},\mathbf{b}),\mathbf{Z}^\star) \mid \{Q((\mathbf{a},\mathbf{b}),\mathbf{Z}^\star) \neq \emptyset\}$ is complicated. We use Gibbs sampler to get a sample from this distribution.

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- Each Q is a polygon. When sampling an element of Q we take a random vertex.
- Notice that this approach does not assume that the true value of $\mathbf{X}$ is uniform in the observed interval!

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A sample from $Q((\mathbf{a},\mathbf{b}),\mathbf{Z}^\star) \mid \{Q((\mathbf{a},\mathbf{b}),\mathbf{Z}^\star) \neq \emptyset\}$.

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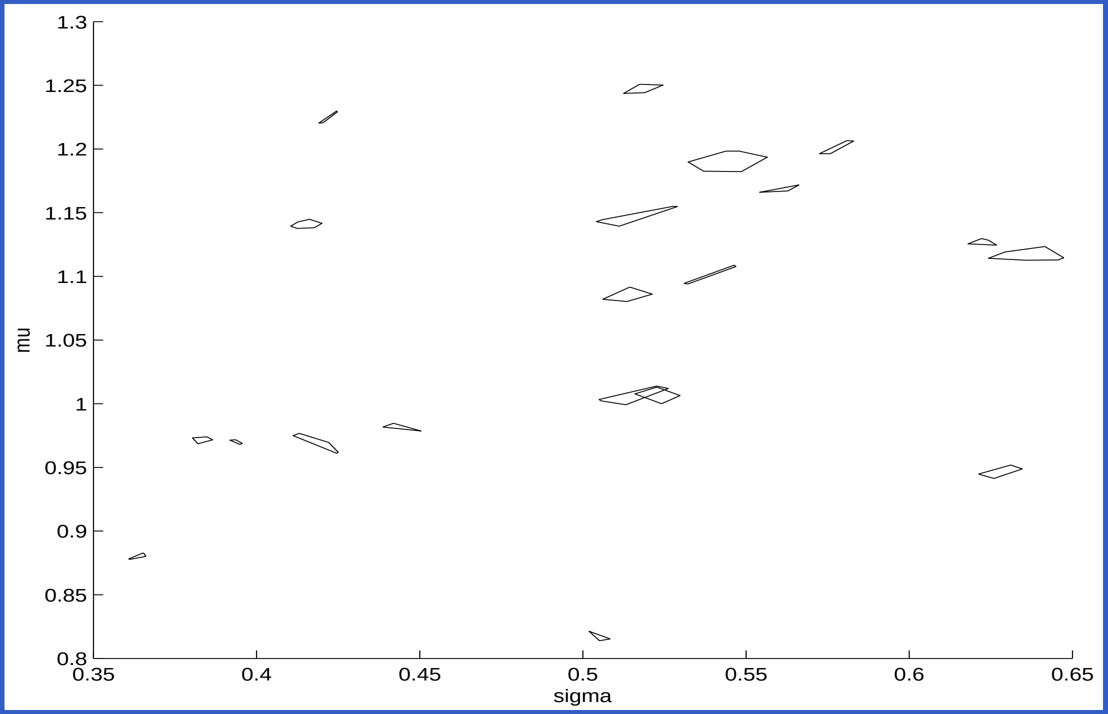

20 observations.

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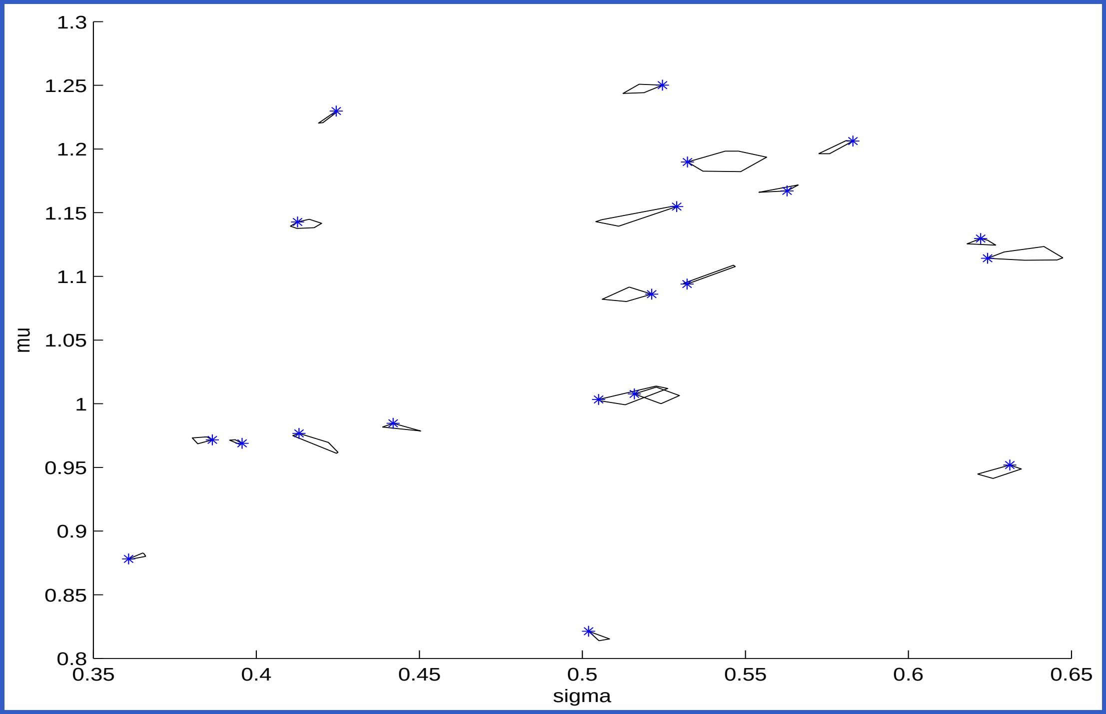

20 observations, final sampled value shown.

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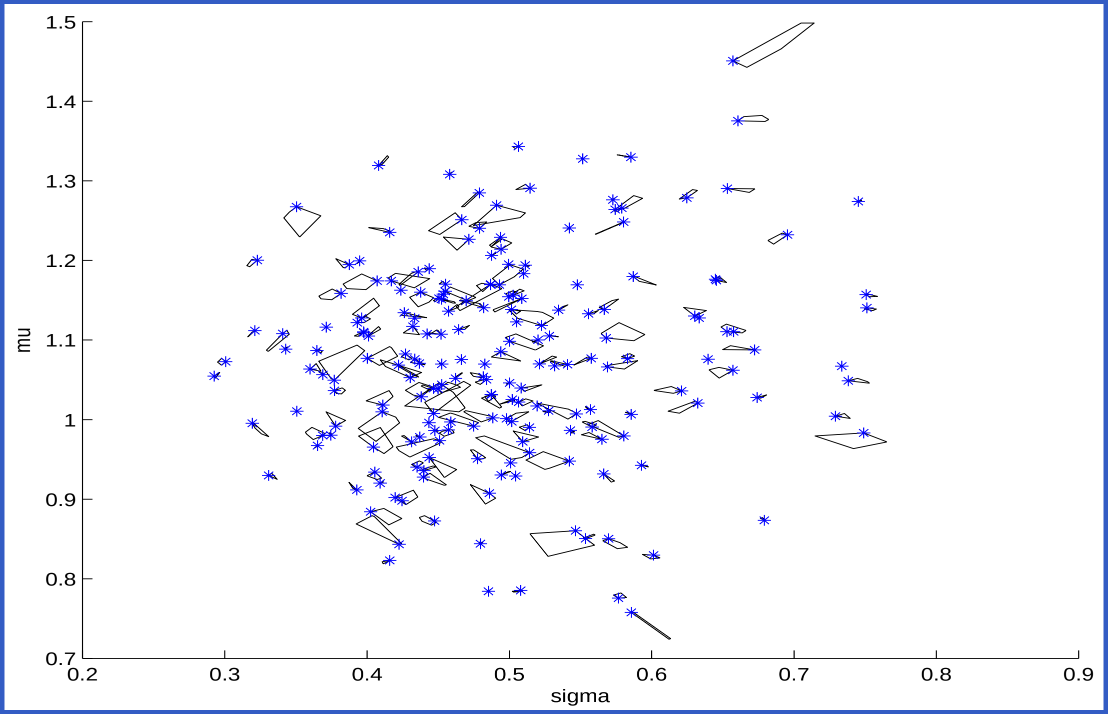

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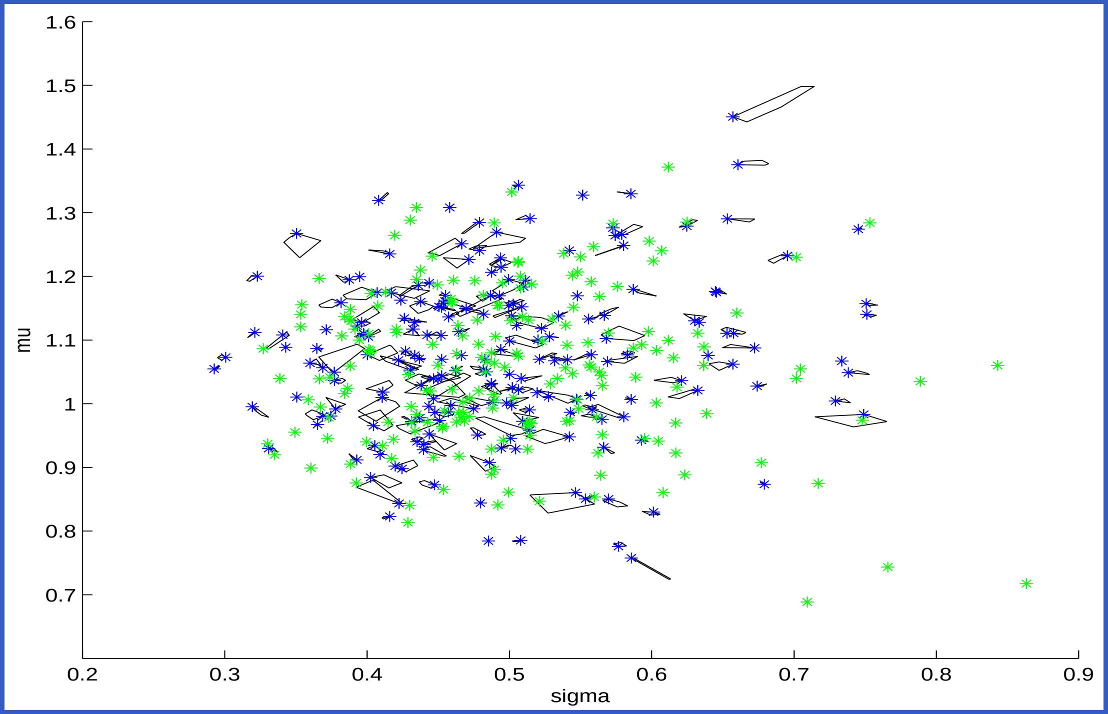

green sample from "usual fiducial" computed with fully known observation.

Questions

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  - Random environment: The observations X are random. But only partially observed.
  - The set we condition on has an extremely small probability.
  - Geometry is complicated. It is an intersection of large number of random, dependent parallelograms. However it has typically surprisingly low number of vertexes.

What can we do?

- Let $d = (d_1, d_2) \in \mathbb{S}^2$ and define $Q_d((\mathbf{a}, \mathbf{b}), \mathbf{Z}^\star)$ the most extreme point along the direction d. (It is one of the vertexes a.s.)

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$$\sum_{i<j} \frac{|c_i^{ij} - c_j^{ij}|}{\sigma^{-3}} \phi\left(\frac{c_i^{ij} - \mu}{\sigma}\right) \phi\left(\frac{c_j^{ij} - \mu}{\sigma}\right) \prod_{k \neq i,j} \left(\Phi\left(\frac{b_i - \mu}{\sigma}\right) - \Phi\left(\frac{a_i - \mu}{\sigma}\right)\right)$$

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- This allows us to compute
  - Find a limit as $\mathbf{b} - \mathbf{a} \to 0$. (More on this later. 2)

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- The distribution of $Q_d((\mathbf{a}, \mathbf{b}), \mathbf{Z}^\star) \mid \{Q((\mathbf{a}, \mathbf{b}), \mathbf{Z}^\star) \neq \emptyset\}$ is proportional to

$$\sum_{i<j} \frac{|c_i^{ij} - c_j^{ij}|}{\sigma^{-3}} \phi\left(\frac{c_i^{ij} - \mu}{\sigma}\right) \phi\left(\frac{c_j^{ij} - \mu}{\sigma}\right) \prod_{k \neq i,j} \left(\Phi\left(\frac{b_i - \mu}{\sigma}\right) - \Phi\left(\frac{a_i - \mu}{\sigma}\right)\right)$$

  c_i^{ij} is either a_i or b_i depending on d.
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  - Find a limit as $\mathbf{b} - \mathbf{a} \to 0$. (More on this later. 2)
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  - Would love to know $n(Q - Q_d) \mid \{Q \neq \emptyset\}$

Generalized Fiducial Recipe

- Let $\mathbb{X}$ be a random vector with a distribution indexed by a parameter $\xi \in \mathbb{R}^p$. Assume that

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- Assume that for any measurable S there is a random variable $V(S)$ with support S .

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Based on $\mathbb{X} = G(U, \xi)$ define a generalized fiducial distribution as the conditional distribution of

$$V(Q(\mathcal{A}, U^\star)) \mid \{ Q(\mathcal{A}, U^\star) \neq \emptyset\}. \qquad (1)$$

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- Let $\mathcal{R}_\xi(\mathcal{A})$ be a random variable with distribution (1). If $\theta = \pi(\xi)$ is of interest use $\mathcal{R}_\theta = \pi(\mathcal{R}_\xi)$. We will call these GFQs.

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    - This is caused by Borel paradox.
- Under suitable conditions the fiducial distribution leads to procedures with asymptotically correct frequentist properties.

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- The generalized fiducial distribution is then calculated to be

$$r(\xi|\mathbf{x}) = \frac{f_{\mathbf{X}}(\mathbf{x}|\xi) J(\mathbf{x}, \xi)}{\int_{\Xi} f_{\mathbf{X}}(\mathbf{x}|\xi') J(\mathbf{x}, \xi')\, d\xi'}, \tag{2}$$

where $J(\mathbf{x}, \xi) = \binom{n}{p}^{-1} \sum_{\mathbf{i}} \left| \frac{\det\left(\frac{d}{d\xi} \mathbf{G}_{\mathbf{i}}^{-1}(\mathbf{x}_{\mathbf{i}}, \xi)\right)}{\det\left(\frac{d}{d\mathbf{x}_{\mathbf{i}}} \mathbf{G}_{\mathbf{i}}^{-1}(\mathbf{x}_{\mathbf{i}}, \xi)\right)} \right|$.

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- Roughly speaking, there is a centering T such that conditionally on the data $\mathbf{X} = \mathbf{x}$ the generalized fiducial quantity $\mathcal{R}_\theta \approx N(T(\mathbf{x}), \sigma_n^2)$. Moreover unconditionally $T(\mathbb{X}) \approx N(\theta, \sigma_n^2)$.
- The lower CI is approximately $(-\infty, T(\mathbf{x}) + z_\alpha \sigma_n)$. The coverage of this CI is approximately $P(\theta < T(\mathbb{X}) + z_\alpha \sigma_n) = P(-z_\alpha \sigma_n < T(\mathbb{X}) - \theta) = \alpha$.

Why does it work asymptotically?

Theorem (Hannig, 2007)**.** *Assume that* $J(x, \bullet)$ *is continuous in* θ*,* $\pi(\theta) = E_{\theta_0} J(X, \theta)$ *is finite,* $\pi(\theta_0) > 0$*, and on some neighborhood of* θ_0 $E_{\theta_0}\left(\sup_{\theta \in (\theta_0 - \delta_0, \theta_0 + \delta_0)} J(X, \theta)\right) < \infty$*. Then under regularity conditions*

$$\int_{\mathbb{R}} \left| r(\theta, \mathbf{x}) - \frac{e^{-\frac{s^2}{2/I(\theta_0)}}}{\sqrt{2\pi/I(\theta_0)}} \right| d\theta \overset{P_{\theta_0}}{\longrightarrow} 0.$$

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- The rough idea of the proof is to show that $J(x, \theta) \to \pi(\theta)$ uniformly and use Bernstein-von Mises theorem for Bayesian posterior. There is a technical problem caused by the fact that $\pi(\theta)$ is typically improper.

Concluding Remarks

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- Many simulation studies show that generalized fiducial solutions have very good small sample properties.
- Current popularity of generalized inference in some applied circles suggests that if computers were available 70 years ago, fiducial inference might not have been rejected.

Quotes

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