CONVEXITY ESTIMATES FOR HYPERSURFACES MOVING BY CONVEX CURVATURE FUNCTIONS

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Abstract. We consider the evolution of compact hypersurfaces by fully non-linear, parabolic curvature flows for which the normal speed is given by a smooth, convex, degree one homogeneous function of the principal curvatures. We prove that solution hypersurfaces on which the speed is initially positive become weakly convex at a singularity of the flow. The result extends the convexity estimate [HS99b] of Huisken and Sinestrari for the mean curvature flow to a large class of speeds, and leads to an analogous description of 'type-II' singularities. We remark that many of the speeds considered are positive on larger cones than the positive mean half-space, so that the result in those cases also applies to non-mean-convex initial data.

1. Introduction

Given a smooth, compact immersion $X_0 : M^n \to \mathbb{R}^{n+1}$, $n > 1$, we consider smooth families $X : M \times [0, T) \to \mathbb{R}^{n+1}$ of smooth immersions $X(\cdot, t)$ solving the curvature flow

$$\begin{aligned} \frac{\partial X}{\partial t}(x,t) &= -\, s(x,t)\nu(x,t)\,, \\ X(\cdot,0) &= X_0\,, \end{aligned} \tag{1.1}$$

where ν is the outer unit normal field of the solution, and the speed s is determined by a function of the principal curvatures κ_i (with respect to ν). That is,

$$s(x,t) = f(\kappa_1(x,t), \ldots, \kappa_n(x,t))\,. \tag{1.2}$$

We require that the speed function f satisfies the following conditions:

Conditions.

(i) that $f \in C^\infty(\Gamma)$ for some connected, open, symmetric cone $\Gamma \subset \mathbb{R}^n$;
(ii) that f is monotone increasing in each argument;
(iii) that f is homogeneous of degree one;
(iv) that $f > 0$; and
(v) that Γ is preserved by the flow (1.1).

Condition (v) is intended as follows: Let X be a solution of (1.1)–(1.2) such that the initial hypersurface satisfies $(\kappa_1(x,0), \ldots, \kappa_n(x,0)) \in \Gamma$ for all $x \in M$. Then there is a connected, open, symmetric subcone Γ_0 of Γ satisfying $\bar{\Gamma}_0 \setminus \{0\} \subset \Gamma$ such that the principle curvatures of the solution satisfy $(\kappa_1(x,t), \ldots, \kappa_n(x,t)) \in \Gamma_0$ for all $(x,t) \in M \times [0,T)$. We refer to Γ_0 as a *preserved cone* of the flow. This is discussed further below.

Observe that, since the normal points outwards and f is homogeneous, we lose no generality in assuming further that $(1, \ldots, 1) \in \Gamma$, and that f is normalised such that $f(1, \ldots, 1) = 1$. Furthermore, since f is symmetric, we may at each point reorder the principal curvatures such that $\kappa_n \geq \cdots \geq \kappa_1$.

For most of the paper, we will also require that f satisfies the following two conditions, which are somewhat distinct from Conditions (i)–(v):

Conditions.

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(vi) *that f is locally convex; and*
(vii) *that* $\left(\frac{\partial f}{\partial z_p} - \frac{\partial f}{\partial z_q}\right)\Big|_z \geq 0$ *whenever $z \in \Gamma$ is such that $z_p \geq z_q$.*

We will say that s is an *admissible speed* for the flow (1.1) if s is given by (1.2) such that f satisfies Conditions (i)–(vii).

Some discussion of Conditions (i)–(vii) is in order: The symmetry of f is a geometric condition– it allows us to write s as a smooth function of the Weingarten map of the solution, which ensures geometric invariance of the flow. The monotonicity of f then ensures that the flow is parabolic, which guarantees short time existence of a solution if the principal curvatures of the initial immersion lie in Γ. Condition (v) is then a requirement that the principle curvatures do not 'move out of' Γ during the flow. In general, some such condition is necessary (c.f. [AMZ13, Theorem 3]), although, in particular, it automatically holds in each of the following situations (c.f. Lemma 2.4):

Ancillary Conditions.

(viii) *that Conditions (i)–(iv) and (vi) hold, and Γ is convex; or*
(ix) *that Conditions (i)–(iv) and (vi) hold, and $f\big|_{\partial\Gamma} = 0$; or*
(x) *that Conditions (i)–(iv) hold, and $n = 2$.*

For the purposes of Theorem 1.1, however, we need only assume that the weaker condition (v) holds. We remark that Ancillary Condition (ix) makes sense because any function satisfying Conditions (i)–(iv) has a continuous extension to $\partial\Gamma$. This is proved for $\Gamma = \Gamma_+$ in [AMZ13], but the proof is easily modified for the present situation.

In the presence of Condition (i), Conditions (vi)–(vii) are equivalent to requiring that that the speed is a smooth, convex function of the Weingarten map (c.f. Lemma 2.1). We note that Condition (vii) is automatically true in each of the following situations:

Ancillary Conditions.

(xi) *that Conditions (i)–(iii) and (vi) hold, and Γ is convex; or*
(xii) *that Conditions (i)–(iii), and (vi) hold, and f extends as a convex function to $\mathbb{R}^n$ (for example, if $f\big|_{\partial\Gamma} = 0$); or*
(xiii) *that Conditions (i)–(iv), and (vi) hold, and $n = 2$.*

The above assertions are discussed in greater detail in section 2.

We now list some examples of admissible speeds.

Examples 1.1. *The following functions define admissible speeds for the flow* (1.1)*:*

(1) The arithmetic mean: $f(z_1, \ldots, z_n) = z_1 + \cdots + z_n$ *on the half-space* $\Gamma = \{z \in \mathbb{R}^n : z_1 + \cdots + z_n > 0\}$. *The corresponding flow is the (mean convex) mean curvature flow.*
(2) The power means: $f_p(z_1, \ldots, z_n) = (z_1^p + \cdots + z_n^p)^{\frac{1}{p}}$, $p \geq 1$, *on the positive cone* $\Gamma_+^n = \{z \in \mathbb{R}^n : z_i > 0 \text{ for all } i\}$. *The case $p = 2$ corresponds to the flow by the norm of the Weingarten map.*
(3) Positive linear combinations: *If $f_1, \ldots, f_k$ are admissible on Γ, then, for all $(s_1, \ldots, s_k) \in \Gamma_+^k$, the function $f = s_1 f_1 + \cdots + s_k f_k$ is admissible on Γ. For example, the function* $f(z_1, \ldots, z_n) = z_1 + \cdots + z_n + \sqrt{z_1^2 + \cdots + z_n^2}$ *on the cone Γ_+ defines an admissible speed. In fact, the functions* $f_\alpha(z_1, \ldots, z_n) = z_1 + \cdots + z_n + \alpha\sqrt{z_1^2 + \cdots + z_n^2}$ *for* $\alpha \in [0, 1]$ *on the larger cones* $\Gamma_\alpha = \{z \in \mathbb{R}^n : z_1 + \cdots + z_n + \alpha\sqrt{z_1^2 + \cdots + z_n^2} > 0\}$ *define admissible speeds. We remark that the cones Γ_α contain the half-space* $\{z \in \mathbb{R}^n : z_1 + \cdots + z_n > 0\}$.
(4) Concave functions: *If $g \in C^\infty(\Gamma)$ is symmetric, homogeneous degree one and concave, then an admissible speed is defined by the function $f = H - \varepsilon g$ on the subcone of Γ for which $H > \varepsilon g$ and $\dot{g}^i < \frac{1}{\varepsilon}$ for all i. The class of concave functions discussed in* [An07] *then provide an interesting class of admissible speeds.*
(5) Convex homogeneous combinations: *Let ϕ satisfy Conditions (i)–(iv) and (vi)–(vii) on a cone $\widetilde{\Gamma} \subset \mathbb{R}^k$, and suppose that the functions $f_1, \ldots, f_k$ define admissible speeds on a cone $\Gamma_k \subset \mathbb{R}^n$. Then the function* $f(z_1, \ldots, z_n) := \phi\left(f_1(z_1, \ldots, z_n), \ldots, f_k(z_1, \ldots, z_n)\right)$ *on*

the cone $\{z \in \Gamma : (f_1(z), \ldots, f_k(z)) \in \widetilde{\Gamma}\}$ *defines an admissible speed. For example, the function* $f_\varepsilon(z_1, \ldots, z_n) = H_p(z_1 + \varepsilon H, \ldots, z_n + \varepsilon H)$ *on the cone* $\Gamma_\varepsilon := \{z \in \mathbb{R}^n : z_i + \varepsilon H > 0$ *for all* $i\}$ *defines an admissible speed.*

Curvature problems of the form (1.1)–(1.2) have been studied extensively, although mostly under the assumption that the initial hypersurface is locally convex, that is, having Weingarten map everywhere positive definite. The most well-known result in this case is Huisken's Theorem [Hu84], which states that, when the speed is given by the mean curvature, uniformly locally convex initial hypersurfaces remain uniformly locally convex and shrink to round points, 'round' meaning that the solution approaches total umbilicity at the final point. Chow showed that this behaviour is true also for the flows by the n-th root of the Gauss curvature [Ch85], and, if an initial curvature pinching condition is assumed, the square root of the scalar curvature [Ch87]. Each of these flows satisfy Conditions (i)–(iv) on the positive cone $\Gamma = \Gamma_+ := \{x \in \mathbb{R}^n : x_i > 0,\ i = 1, \ldots, n\}$. More general degree one homogeneous speeds were treated by the first author in [An94a, An07, An10], where it was shown that uniformly convex hypersurfaces will contract to round points under the flow 1.1–(1.2), so long as the speed satisfies Conditions (i)-(iv) and, in addition, either:

(a) $n = 2$; or
(b) f is convex; or
(c) f is concave, and *inverse concave*, that is, the function

$$f_*(z_1, \ldots, z_n) = f\left(z_1^{-1}, \ldots, z_n^{-1}\right)^{-1}$$

is concave.

These conditions were weakened in [AMZ13], and their necessity demonstrated by the construction, in dimensions $n > 2$, of concave speed functions satisfying Conditions (i)-(iv) for which convex initial hypersurfaces do not remain convex under the corresponding flow [AMZ13, Theorem 3].

In the case of non-convex initial hypersurfaces, much less is known about the behaviour of solutions of (1.1), although in many cases the analogy with the mean curvature flow continues. For example, a simple calculation shows that spheres shrink to points in finite time under flows (1.1)–(1.2) satisfying Conditions (i)-(iv). The avoidance principle (c.f.[1] [ALM12a, Theorem 5]) then implies that any compact solution of (1.1) must become singular in finite time. If, in addition, the flow admits second derivative Hölder estimates (for example, if the speed function is a concave or convex function of the principal curvatures [Ev82, Kr82], or if $n = 2$ [An04]), one can deduce, by standard methods, that a singularity is characterised by a curvature blow-up (c.f. [ALM12b]).

For the mean curvature flow, a crucial part of the current understanding of singularities is the asymptotic convexity estimate of Huisken and Sinestrari, which states that any mean convex initial hypersurface flowing by mean curvature becomes weakly convex at a singularity [HS99b]. This, together with the monotonicity formula of Huisken [Hu90] and the Harnack inequality of Hamilton [Ham95a] allows a rather complete description of singularities in the positive mean curvature case. We note that asymptotic convexity is necessary for the application of the Harnack inequality to deduce that "fast-forming" or "type-II" singularities are asymptotic to convex translation solutions of the flow.

For other flows, the understanding of singularities is far less developed. There are several reasons for this: First, there is no analogue available for the monotonicity formula, which is used to show that "slowly forming" or "type-I" singularities of the mean curvature flow are asymptotically self-similar. Second, there is in general no Harnack inequality available sufficient to classify type-II singularities, although the latter is known for quite a wide sub-class of flows [An94b]. And finally, there is so far no analogue of the Huisken-Sinestrari asymptotic convexity estimate for most other flows, with the notable exception of the recent result of Alessandroni and Sinestrari, which applies to a class of flows by functions of the mean curvature having a certain asymptotic behaviour [AS10]. In a companion paper [ALM12b], we wera able to exploit the simplified structure of the evolution

[1]We remark that the avoidance principle proved in [ALM12a, Theorem 5] is not in general true when the cone of definition of the speed is non-convex. However, a slight modification reveals that it is still possible to compare compact solutions with spheres.

equation for the second fundamental form in two dimensions (see also [Sc06, An07, Mc11]) to prove that an asymptotic convexity estimate holds in surprising generality for flows of surfaces, namely for any surface flow (1.1)–(1.2) satisfying Conditions (i)–(iv). On the other hand, one would expect this result should fail in higher dimensions in such generality, due to the aforementioned examples of 'nice' speeds which fail to preserve local convexity of initial data. In this paper, we show that an asymptotic convexity estimate is possible in higher dimensions in the presence of the additional convexity Conditions (vi)-(vii).

Theorem 1.1. *Let $X : M \times [0,T) \to \mathbb{R}^{n+1}$ be a solution of* (1.1) *with s an admissible speed. Then for all $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that*

$$-\kappa_1(x,t) \le \varepsilon s(x,t) + C_\varepsilon \,.$$

for all $(x,t) \in M \times [0,T)$.

The proof of Theorem 1.1 utilises a Stampacchia-De Giorgi iteration procedure analogous to those of [Hu84, HS99a, HS99b, Ch85, Ch87] (see also [ALM12b]), in contrast to the result of [AS10] (see also [Sc06]), which is proved using the maximum principle. We remark that, by carefully constructing our curvature pinching function, we are able to avoid the rather technical induction on the elementary symmetric functions of curvature that is necessary in [HS99b].

Combining Theorem 1.1 with the Harnack estimate of [An94b] (c.f. [Ham95a]) as in [HS99a, HS99b], we are led to the following classification of type-II blow-up limits about type-II singularities.

Corollary 1.2. *If s is an admissible speed, then any type-II blow-up limit of a solution of the corresponding flow* (1.1) *about a type-II singularity decomposes as a product $X : (\Sigma^k \times \mathbb{R}^{n-k}) \times \mathbb{R} \to \mathbb{R}^{n+1}$, such that $X\big|_{\Sigma^k} : \Sigma^k \times \mathbb{R} \to \mathbb{R}^{k+1} \subset \mathbb{R}^{n+1}$ is a strictly convex (k-dimensional) translation solution of the flow* (1.1).

Corollary 1.2 is proved in section 6.

2. Notation and Preliminary Results

We now describe some important background results necessary for the subsequent sections. We begin with flow independent results to do with symmetric functions, and prove, in Lemma 2.2, that each of the Ancillary Conditions (xi)–(xiii) implies Condition (vii). We then discuss flow dependent results, and prove, in Lemma 2.4, that each of the Ancillary Conditions (viii)–(x) implies Condition (v). We follow the conventions of [AMZ13, An07, An10, Mc05], where proofs or references for much of this section may be found. Many of the results can also be found in the book [Ge06] by Gerhardt.

The curvature function f is a smooth, symmetric function defined on an open, convex, symmetric cone Γ. Denote by $\mathcal{S}_\Gamma$ the cone of symmetric $n \times n$ matrices with n-tuple of eigenvalues, $\lambda := (\lambda_1, \dots, \lambda_n)$, lying in Γ. A result of Glaeser [Gl63] implies that there is a smooth, $GL(n)$ invariant function $F : \mathcal{S}_\Gamma \to \mathbb{R}$ such that $f(\lambda(A)) = F(A)$. The invariance of F under similarity transformations implies that the speed $s(x,t) = f(\kappa_1(x,t), \dots, \kappa_n(x,t))$ is a well-defined, smooth function of the Weingarten map $\mathcal{W}$, that is, $s(x,t) = F(\mathcal{W}(x,t)) := F(W(x,t))$, where $W(x,t)$ is the component matrix of $\mathcal{W}(x,t)$ with respect to some basis for $T_x^*M \otimes T_xM$. If we restrict attention to orthonormal bases, then $W_i{}^j = h_{ij}$, where h_{ij} are the components of the second fundamental form.

We shall use dots to indicate derivatives of f and F as follows:

$$\begin{aligned} \dot f^i(\lambda) v_i &:= \frac{d}{ds}\bigg|_{s=0} f(\lambda + sv)\,, \qquad \ddot f^{ij}(\lambda) v_i v_j := \frac{d^2}{ds^2}\bigg|_{s=0} f(\lambda + sv)\,, \\ \dot F^{ij}(A) B_{ij} &:= \frac{d}{ds}\bigg|_{s=0} F(A + sB)\,, \qquad \ddot F^{pq,rs}(A) B_{pq} B_{rs} := \frac{d^2}{ds^2}\bigg|_{s=0} F(A + sB)\,. \end{aligned} \tag{2.1}$$

The derivatives of f and F are related in the following way (c.f. [Ge90, An94a, An07]):

Lemma 2.1. *Suppose that the function f satisfies Condition (i). Define the function $F : \mathcal{S}_\Gamma :\to \mathbb{R}$ by $F(A) := f(\lambda(A))$ as above. Then for any diagonal $A \in \mathcal{S}_\Gamma$ we have*

$$\dot F^{kl}(A) = \dot f^k(\lambda(A))\delta^{kl}\,, \tag{2.2}$$

and for any diagonal $A \in \mathcal{S}_\Gamma$ and symmetric $B \in GL(n)$ we have

$$\ddot F^{pq,rs}(A)B_{pq}B_{rs} = \ddot f^{pq}(\lambda(A))B_{pp}B_{qq} + 2\sum_{p>q}\frac{\dot f^p(\lambda(A)) - \dot f^q(\lambda(A))}{\lambda_p(A) - \lambda_q(A)}\left(B_{pq}\right)^2. \tag{2.3}$$

Note that (2.3) *holds (as a limit) even if A has eigenvalues of multiplicity greater than one.*

In particular, in an orthonormal frame of eigenvectors of $\mathcal{W}$, we have

$$\begin{aligned}\dot F^{kl}(\mathcal{W}) =& \dot f^k(\kappa)\delta^{kl}\\ \ddot F^{pq,rs}(\mathcal{W})B_{pq}B_{rs} =& \ddot f^{pq}(\kappa)B_{pp}B_{qq} + 2\sum_{p>q}\frac{\dot f^p(\kappa) - \dot f^q(\kappa)}{\kappa_p - \kappa_q}\left(B_{pq}\right)^2.\end{aligned}$$

Observe that, by (2.2), Conditions (i)–(ii) imply that (1.1)–(1.2) is parabolic. The methods of [Ge06, Section 2.5] (see also [GG92] and [Ba10]) then imply short time existence of solutions, so long as the principal curvatures of the initial immersion lie in Γ.

It follows from (2.3) that the function F is convex if and only if the function f is convex and satisfies $(\dot f^p - \dot f^q)(z_p - z_q) \geq 0$. We now show that in most cases of interest the second condition is automatic.

Lemma 2.2. *Suppose that f satisfies one of the Ancillary Conditions (xi), (xii) or (xiii). Then f satisfies Condition (vii).*

Proof. Suppose first that Condition (xi) is satisfied, so that Γ is convex. If $\Gamma = \Gamma_+$ then the claim is proved in [An94a, Lemma 2.2] (see also [EH89]). However, the proof applies to any convex cone: Consider an arbitrary point $z \in \Gamma$. Since f is smooth and convex, for any $v \in \mathbb{R}^n$ and any $s \in \mathbb{R}$ such that $z + sv \in \Gamma$ we have

$$0 \leq \frac{d^2}{ds^2}f(z+sv) = \frac{d}{ds}\dot f^i(z+sv)v_i\,.$$

Therefore, if $s > 0$,

$$\dot f^i(z+sv)v_i \geq \dot f^i(z)v_i\,.$$

Setting $v = -(e_p - e_q)$, where e_i is the basis vector in the direction of z_i, we obtain

$$\left(\dot f^p - \dot f^q\right)\Big|_z \geq \left(\dot f^p - \dot f^q\right)\Big|_{z-s(e_p-e_q)}\,.$$

If $z_p \geq z_q$ then there is some $s_0 > 0$ such that $\left(z - s_0(e_p-e_q)\right)_p = \left(z - s_0(e_p - e_q)\right)_q$. By the symmetry and convexity of Γ, this point is in Γ. Since f is symmetric, $\dot f^p = \dot f^q$ at this point and the claim follows.

Now suppose that (xii) is satisfied, so that f extends to a convex, symmetric function on $\mathbb{R}^n$. If the extension is smooth, then the claim follows as above. If not, then we need to be more careful; we make use of the fact that the difference quotient $\left(f(\gamma(s)) - f(\gamma(t))\right)/(s-t)$ is non-decreasing in both s and t along all lines $\gamma(s) = z + sv$.

Consider a point $z \in \Gamma$ and a direction $v \in \mathbb{R}^n$. Then, for any $s \in \mathbb{R}$ and any $s_0 > 0$, we have

$$\frac{f(z+sv) - f(z+s_0v)}{s - s_0} \geq \frac{f(z+sv) - f(z)}{s} \geq \lim_{s\searrow 0}\frac{f(z+sv)-f(z)}{s} = \dot f^i\big|_z v_i\,.$$

Setting $v = -(e_p - e_q)$, it follows that

$$-\left(\dot f^p - \dot f^q\right)\Big|_z = \dot f^i\Big|_z v_i \leq \frac{f(z+sv) - f(z+s_0v)}{s-s_0} \leq \lim_{s\nearrow s_0}\frac{f(z+sv) - f(z+s_0v)}{s-s_0} = \psi'_-(0)\,,$$

where we have defined $\psi(\sigma) := f(z + (\sigma + s_0)v)$. We note that the left derivative $\psi'_-(0)$ exists, and is no greater than the right derivative ψ'_+, by convexity of ψ. Supposing without loss of generality that $z_p \geq z_q$, we may choose s_0 such that $z_p - s_0 = z_q + s_0$. With this choice, it is easily checked that ψ is an even function. Since ψ is convex, we have

$$\psi'_-(0) \leq \psi'_+(0) = \lim_{s\searrow 0} \frac{\psi(s)-\psi(0)}{s}$$
$$= -\lim_{s\nearrow 0} \frac{\psi(-s)-\psi(0)}{s} = -\lim_{s\nearrow 0} \frac{\psi(s)-\psi(0)}{s} = -\psi'_-(0)\,.$$

It follows that $\psi'_-(0) \leq 0$ and we obtain $\left(\dot f^p - \dot f^q\right)\Big|_z \geq 0$ as required.

Finally, suppose that (xiii) is satisfied, so that $\Gamma \subset \mathbb{R}^2$. Consider some point $z \in \Gamma$ and suppose $p \neq q$ are such that $z_p \geq z_q$. Since f is homogeneous of degree one, we have $f = \dot f^1 z_1 + \dot f^2 z_2$. Then, since f, $\dot f^1$ and $\dot f^2$ are positive on Γ, we must have $z_p > 0$. Now,

$$2f = 2\left(\dot f^p z_p + \dot f^q z_q\right) = \left(\dot f^p - \dot f^q\right)(z_p - z_q) + \left(\dot f^p + \dot f^q\right)(z_p + z_q)\ ,$$

so that

$$\left(\dot f^p - \dot f^q\right)(z_p - z_q) = 2f - \left(\dot f^p + \dot f^q\right)(z_p + z_q)\ .$$

If $z_p + z_q \leq 0$, then we are done (since f, $\dot f^1$ and $\dot f^2$ are positive). Otherwise, z lies in the open, symmetric, convex cone $\{z \in \mathbb{R}^2 : z_1 + z_2 > 0\}$. But we have just proved that the claim already holds in this case. This completes the proof. □

In the following, we are interested in the behaviour of solutions of the flow equation (1.1)–(1.2). We consider speeds $s = f(\kappa)$ such that f satisfies Condition (i), and denote the corresponding function of $\mathcal{W}$ by F. We will use the following convention in order to simplify notation: If g satisfies Condition (i), and $G(A) = g(\lambda(A))$ is the corresponding function on $\mathcal{S}_\Gamma$, then we write $g(x,t) \equiv g(\kappa(x,t))$ and $G(x,t) \equiv G(\mathcal{W}(x,t))$. Similarly, $\dot G(x,t) \equiv \dot G(\mathcal{W}(x,t))$ and $\ddot G(x,t) \equiv \ddot G(\mathcal{W}(x,t))$. This convention makes the notation s for the speed unnecessary, and from here on the speed will be denoted by F.

We recall the following evolution equations (see [An94a, An07, AMZ13, Ge06, Mc05]):

Lemma 2.3. *Let $X : M \times [0,T) \to \mathbb{R}^{n+1}$ be a solution of the flow (1.1)–(1.2) such that f satisfies Conditions (i)–(iii). Then the following evolution equations hold along X:*

(1) $(\partial_t - \mathcal{L})h_i{}^j = (\nabla_i dF)^j + F h_i{}^k h_k{}^j = \ddot F^{pq,rs}\nabla_i h_{pq}\nabla^j h_{rs} + \dot F^{kl} h^2_{kl} h_i{}^j$;
(2) $(\partial_t - \mathcal{L})F = F\dot F^{kl} h^2_{kl}$;
(3) $\partial_t\, d\mu = -HF\, d\mu$; and
(4) $(\partial_t - \mathcal{L})G = \left(\dot G^{kl}\ddot F^{pq,rs} - \dot F^{kl}\ddot G^{pq,rs}\right)\nabla_k h_{pq}\nabla_l h_{rs} + \dot G^{pq} h_{pq}\dot F^{kl} h^2_{kl}$,

where $\mathcal{L}$ is the elliptic operator $\dot F^{ij}\nabla_i\nabla_j$, $h^2_{ij} = h_i{}^k h_{kj}$, $\mu(t)$ is the measure induced on M by the immersion $X(\cdot,t)$, and G is any function given by $G(x,t) := g(\kappa_1(x,t),\ldots,\kappa_n(x,t))$ for some smooth, symmetric $g : \Gamma \to \mathbb{R}$.

Applying the maximum principle to Lemma 2.3 (2), we see that F remains positive for all $t \in [0,T)$ whenever it is initially positive. It then follows from Euler's theorem and the monotonicity of f that the largest principal curvature also remains positive.

In the case that g is homogeneous of degree one, Euler's theorem simplifies Lemma 2.3 (4) to

$$(\partial_t - \mathcal{L})G = (\dot G^{kl}\ddot F^{pq,rs} - \dot F^{kl}\ddot G^{pq,rs})\nabla_k h_{pq}\nabla_l h_{rs} + \dot F^{kl} h^2_{kl} G\,. \tag{2.4}$$

It follows that

$$(\partial_t - \mathcal{L})\left(\frac{G}{F}\right) = \frac{1}{F}\left(\dot G^{kl}\ddot F^{pq,rs} - \dot F^{kl}\ddot G^{pq,rs}\right)\nabla_k h_{pq}\nabla_l h_{rs} - \frac{2}{F}\dot F^{kl}\nabla_k F\,\nabla_l\left(\frac{G}{F}\right)\,. \tag{2.5}$$

Therefore $\max_{M\times\{t\}}(G/F)$ will be non-increasing in t whenever G satisfies the condition

$$(\dot G^{kl}\ddot F^{pq,rs} - \dot F^{kl}\ddot G^{pq,rs})\nabla_k h_{pq}\nabla_l h_{rs} \leq 0\,. \tag{2.6}$$

These observations help us to find preserved cones for the flow: Suppose that f satisfies Conditions (i)–(iii). If there is a smooth, non-negative, symmetric, homogeneous degree one function $g : \Gamma \to \mathbb{R}$ such that

$$(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})T_{kpq}T_{lrs} \leq 0$$

for any totally symmetric $T \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n$, where G is the corresponding function on $\mathcal{S}_\Gamma$, then any solution of the corresponding flow admits a preserved cone. Namely, the cone $\Gamma_0 := \{z \in \mathbb{R}^n : g(z) < \max_{M\times\{0\}} \left(\frac{G}{F}\right) f(z)\}$ is preserved.

In general, finding such a function g will be highly specific to the choice of flow speed f, however, in many cases we can be sure preserved cones exists:

Lemma 2.4. *Suppose f satisfies one of the Ancillary Conditions (viii), (ix), or (x). Then f satisfies Condition (v).*

Proof. Suppose that Condition (viii) holds, so that the cone Γ is convex. It follows from Lemma 2.2 that Condition (vii) holds, so that $\ddot{F} \geq 0$ by Lemma 2.1. Let X be a solution of (1.1)–(1.2). Then the Weingarten map of X satisfies

$$(\partial_t - \mathcal{L})h_i{}^j \geq \dot{F}^{kl}h^2_{kl}h_i{}^j\,. \tag{2.7}$$

Let Γ_0 be the interior of the symmetrised convex conic hull in $\mathbb{R}^n$ of the principal curvatures of X_0. Then $\bar{\Gamma}_0 \setminus \{0\} \subset \Gamma$. The preservation of Γ_0 by the flow follows by applying a slight modification of Hamilton's tensor maximum principle [Ham84, Section 3] to (2.7) (c.f. [An07, Theorem 3.2] and [AnHo, Chapter 6]).

Now suppose that (ix) is satisfied, so that f vanishes on $\partial\Gamma$. If $X : M \times [0,T) \to \mathbb{R}^{n+1}$ is a solution of the corresponding flow, then F is initially positive, and the maximum principle implies that it remains so. Then we may consider the function $G_1(x,t) := g_1(\kappa_1(x,t),\ldots,\kappa_n(x,t))$, where g_1 is the function defined by equation (3.1) of the following section. Observe that f extends to a convex function on $\mathbb{R}^n$ by setting $f = 0$ outside Γ, so that, by Lemma 2.2, Condition (vii) holds. Then we may proceed as in Lemma 3.2 to obtain

$$Z := \left(\dot{G}_1^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}_1^{pq,rs}\right)\nabla_k h_{pq}\nabla_l h_{rs} \leq 0\,, \tag{2.8}$$

and it follows that $G_1/F \leq c_0 := \max_{M\times\{0\}} G_1/F$. So consider $\Gamma_0 := \{z \in \mathbb{R}^n : g_1(z) < c_0 f(z)\}$. Since $g_1(z) = 0$ iff $z \in \bar{\Gamma}_+ \cap \Gamma$ and, by convexity of the extension of f, $\{z \in \mathbb{R}^n : z_1 + \cdots + z_n > 0\} \subset \Gamma$, we have $(\partial\Gamma \cap \partial\Gamma_0) \setminus \{0\} = \emptyset$. It follows that Γ_0 is a preserved cone.

Finally, consider the case that Condition (x) holds, so that $\Gamma \subset \mathbb{R}^2$. Observe that, in this case, it is sufficient to obtain an estimate on the pinching ratio of the solution (which in this case follows from an estimate on G_1/F), since any open, connected, symmetric cone Γ in $\mathbb{R}^2$ that contains the positive ray is of the form $\{z \in \mathbb{R}^2 : z_{\min} > \varepsilon z_{\max}\}$. However, we can no longer use any convexity properties of f to control G_1/F, and the above proof that $Z \leq 0$ no longer applies. On the other hand, by carefully analysing each of the terms in the expression for Z, it is possible to write the terms involving second derivatives of the speed as gradient terms, and the remaining terms turn out to be automatically favourable for obtaining the desired estimate on Z. We refer the reader to the papers [An07, ALM12b] for the proof of this assertion. □

The existence of a preserved cone ensures that the flow is uniformly parabolic:

Lemma 2.5. *Let $X : M \times [0,T) \to \mathbb{R}^{n+1}$ be a solution of* (1.1)*, with an admissible speed F. Then there is a constant $c_1 > 0$ such that for all $(x,t) \in M \times [0,T)$ it holds that*

$$\frac{1}{c_1}|v|^2 \;\leq \dot{F}^{kl}(x,t)v_kv_l \;\leq\; c_1|v|^2$$

for all $v \in T_xM$, where $|\cdot|$ is the norm induced on TM by the immersion $X(\cdot,t)$.

Proof. In an orthonormal frame of eigenvectors of the Weingarten map, we have, by (2.2), that $\dot{F}^{kl} = \dot{f}^k\delta^{kl}$. Let Γ_0 be a preserved cone for the flow. Since $\bar{\Gamma}_0 \setminus \{0\} \subset \Gamma$, and $\dot{f}^k > 0$ on Γ for all k, we see that the derivatives $\dot{f}^k$ are bounded by positive constants on the compact set $K := \{z \in \bar{\Gamma}_{c_0} : |z| = 1\}$. Since the derivatives $\dot{f}^k$ are homogeneous of degree zero, these bounds extend to the cone $\bar{\Gamma}_{c_0}\setminus\{0\}$, which completes the proof. □

The following long time existence result then follows using standard methods (c.f. [ALM12b]).

Proposition 2.6. *Let $X : M \times [0, T) \to \mathbb{R}^{n+1}$ be a maximally extended solution of* (1.1), *with an admissible speed. Then $T < \infty$, and $\max_{M\times\{t\}} |\mathcal{W}| \to \infty$ as $t \to T$.*

We now focus on the proof of Theorem 1.1 and Corollary 1.2, so for the rest of the paper we will assume that f defines an admissible speed, and $X : M \times [0, T) \to \mathbb{R}^{n+1}$ is a maximally extended solution of the corresponding flow (1.1).

3. The Pinching Function

In this section, we carefully construct an appropriate curvature pinching function to be used in the proof of Theorem 1.1. That is, a smooth, symmetric, homogeneous (degree one, say) function $G(x, t) = g(\kappa_1(x, t), \ldots, \kappa_n(x, t))$ of the principal curvatures that vanishes only if the hypersurface is weakly convex. Our goal is to show that the ratio G/F vanishes asymptotically along the flow. In particular, this ratio should be non-increasing. In view of (2.5) we would therefore like G to satisfy

$$(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})\nabla_k h_{pq}\nabla_l h_{rs} \leq 0\,.$$

In fact, as we shall see, the following two estimates will be essential

Properties 1.

(1) for all $\varepsilon > 0$, there is a constant $c_\varepsilon > 0$ such that

$$(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})\nabla_k h_{pq}\nabla_l h_{rs} \leq -c_\varepsilon \frac{|\nabla\mathcal{W}|^2}{F}$$

whenever $G > \varepsilon F$; and,

(2) for all $\varepsilon > 0$, there is a constant $\gamma_\varepsilon > 0$ such that

$$(F\dot{G}^{kl} - G\dot{F}^{kl})h^2_{kl} \leq -\gamma_\varepsilon F|\mathcal{W}|^2$$

whenever $G > \varepsilon F$.

These estimates are needed to show that the positive part of the function $G_{\varepsilon,\sigma} := (G/F - \varepsilon)F^\sigma$ is bounded in $L^p(M\times[0, T))$ for any $\varepsilon > 0$, so long as σ is sufficiently small. This is done in Section 4. The proof of Theorem 1.1 then follows from standard arguments, which we recall in Section 5. But first, we construct our pinching function. We first try a smoothed out version of the natural choice, $\max\{-\kappa_1, 0\}$. The function we obtain possesses the second of the above properties, but the first property only weakly (that is, with $c_\varepsilon = 0$). By making this function slightly more convex (namely, strictly convex in non-radial directions) we are able to obtain a function satisfying both estimates uniformly (without harming the other properties).

We begin with a smooth function $\phi : \mathbb{R} \to \mathbb{R}$ which is strictly convex and positive, except on $\mathbb{R}_+$, where it vanishes identically. Such a function is easily constructed; for example, we could use

$$\phi(r) = \begin{cases} r^4 e^{-\frac{1}{r^2}} & \text{if} \quad r < 0\,; \\ 0 & \text{if} \quad r \geq 0\,. \end{cases}$$

Now consider the following function, defined on Γ:

$$g_1(z) := f(z)\sum_{i=1}^n \phi\left(\frac{z_i}{f(z)}\right)\,. \tag{3.1}$$

Observe that g_1 is non-negative and vanishes on (and only on) $\bar{\Gamma}_+ \cap \Gamma$. Furthermore, g_1 is clearly smooth, symmetric, and homogeneous of degree one. We now calculate:

$$\begin{aligned} \dot{g}_1^k &= \dot{f}^k \sum_{i=1}^n \phi\left(\frac{z_i}{f}\right) + \sum_{i=1}^n \dot{\phi}\left(\frac{z_i}{f}\right)\left(\delta_i^k - \frac{z_i}{f}\dot{f}^k\right) \\ &= \dot{\phi}\left(\frac{z_k}{f}\right) + \dot{f}^k\sum_{i=1}^n\left[\phi\left(\frac{z_i}{f}\right) - \frac{z_i}{f}\dot{\phi}\left(\frac{z_i}{f}\right)\right]\,. \end{aligned}$$

It follows easily from the convexity of ϕ that $\phi(r) - r\dot{\phi}(r) \leq \phi(0) = 0$. Since ϕ is positive and $\dot{\phi}$ vanishes on $\mathbb{R}_+$, we must also have $\dot{\phi}(r) \leq 0$ for all $r \in \mathbb{R}$. Moreover, equality holds in the above inequalities only if $r \geq 0$. Therefore $\dot{g}_1^k(z) \leq 0$ for each k, with equality if and only if $z \in \bar{\Gamma}_+ \cap \Gamma$.

Now compute

$$\ddot{g}_1^{pq} = \ddot{f}^{pq} \sum_{i=1}^n \left[\phi\left(\frac{z_i}{f}\right) - \frac{z_i}{f}\dot{\phi}\left(\frac{z_i}{f}\right)\right] + \frac{1}{f}\sum_{i=1}^n \ddot{\phi}\left(\frac{z_i}{f}\right)\left(\delta_i{}^p - \frac{z_i}{f}\dot{f}^p\right)\left(\delta_i{}^q - \frac{z_i}{f}\dot{f}^q\right).$$

and

$$\dot{g}_1^k \ddot{f}^{pq} - \dot{f}^k \ddot{g}_1^{pq} = \dot{\phi}\left(\frac{z_k}{f}\right)\ddot{f}^{pq} - \frac{\dot{f}^k}{f}\sum_{i=1}^n \ddot{\phi}\left(\frac{z_i}{f}\right)\left(\delta_i{}^p - \frac{z_i}{f}\dot{f}^p\right)\left(\delta_i{}^q - \frac{z_i}{f}\dot{f}^q\right). \tag{3.2}$$

This forms a non-positive definite matrix for each k. Finally, consider

$$\dot{g}_1^k \frac{\dot{f}^p - \dot{f}^q}{z_p - z_q} - \dot{f}^k \frac{\dot{g}_1^p - \dot{g}_1^q}{z_p - z_q} = \dot{\phi}\left(\frac{z_k}{f}\right)\frac{\dot{f}^p - \dot{f}^q}{z_p - z_q} - \dot{f}^k \frac{\dot{\phi}\left(\frac{z_p}{f}\right) - \dot{\phi}\left(\frac{z_q}{f}\right)}{z_p - z_q}. \tag{3.3}$$

This is also non-positive for each k, since convexity of ϕ implies $\frac{\dot{\phi}(r)-\dot{\phi}(s)}{r-s} \geq 0$. Putting (3.2) and (3.3) together using Lemma 2.1, we see that

$$\left(\dot{G}_1^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}_1^{pq,rs}\right)\nabla_k h_{pq}\nabla_l h_{rs} \leq 0\,.$$

To obtain the uniform estimate, we modify the function g_1 to introduce a slightly stronger convexity property. We utilise the good convexity properties of the Euclidean norm: Consider the function g defined by

$$g := K(g_1, g_2) \;:=\; \frac{g_1^2}{g_2}\,, \tag{3.4}$$

where g_2 is a positive, monotone, degree one homogeneous function of the principle curvatures which is *strictly convex in non-radial directions.* The function defined by

$$g_2(z) := Rf(z) + \sum_{i=1}^n z_i - |z|$$

has the properties we require, so long as the constant $R > 0$ may be chosen such that $g_2 > 0$ (at least along the flow). Let's first show that such a choice is possible.

Lemma 3.1. *There exists a constant $R > 0$ such that*

$$RF(x,t) + H(x,t) - |\mathcal{W}(x,t)| > 0$$

for all $(x,t) \in M \times [0,T)$.

Proof. Define $G_2(x,t) := g_2(\kappa_1(x,t),\ldots,\kappa_n(x,t))$. Since $F(\cdot,0) > 0$ and M is compact, we may choose $R > 0$ such that $G_2(\cdot,0) > 0$. By (2.4), it suffices to show that

$$\left(\dot{G}_2^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}_2^{pq,rs}\right)\nabla_k h_{pq}\nabla_l h_{rs} \geq 0\,.$$

First calculate

$$\dot{g}_2^k = R\dot{f}^k + 1 - \frac{z_k}{|z|}$$

and

$$\ddot{g}_2^{pq} = R\ddot{f}^{pq} - \frac{1}{|z|^3}\left(|z|^2\delta_{pq} - z_p z_q\right).$$

It follows that

$$\dot{g}_2^k \ddot{f}^{pq} - \dot{f}^k \ddot{g}_2^{pq} = \left(1 - \frac{z_k}{|z|}\right)\ddot{f}^{pq} + \frac{\dot{f}^k}{|z|^3}\left(|z|^2\delta_{pq} - z_p z_q\right), \tag{3.5}$$

which, by the Cauchy-Schwarz inequality, is non-negative definite for each k.

Finally,

$$\dot{g}_2^k\frac{\dot{f}^p-\dot{f}^q}{z_p-z_q}-\dot{f}^k\frac{\dot{g}_2^p-\dot{g}_2^q}{z_p-z_q}=\left(1-\frac{z_k}{|z|}\right)\frac{\dot{f}^p-\dot{f}^q}{z_p-z_q}+\frac{1}{|z|}\dot{f}^k\,,$$

which is also non-negative definite for each k. It now follows from (2.2) and (2.3) that

$$\left(\dot{G}_2^{kl}\ddot{F}^{pq,rs}-\dot{F}^{kl}\ddot{G}_2^{pq,rs}\right)\nabla_k h_{pq}\nabla_l h_{rs}\geq 0$$

as required. □

So the function G is well defined. Let us first observe that it also satisfies Properties 1 (i) weakly:

Lemma 3.2. *There is a constant $c_0>0$ such that*

$$G(x,t)\leq c_0F(x,t)\,.$$

for all $(x,t)\in M\times[0,T)$.

Proof. By a straightforward calculation, we find

$$\begin{aligned}\left(\dot{G}^{kl}\ddot{F}^{pq,rs}-\dot{F}^{kl}\ddot{G}^{pq,rs}\right)=&\dot{K}^1\left(\dot{G}_1^{kl}\ddot{F}^{pq,rs}-\dot{F}^{kl}\ddot{G}_2^{pq,rs}\right)+\dot{K}^2\left(\dot{G}_2^{kl}\ddot{F}^{pq,rs}-\dot{F}^{kl}\ddot{G}_2^{pq,rs}\right)\\&-\dot{F}^{kl}\ddot{K}^{\alpha\beta}\dot{g}_\alpha^p\dot{g}_\beta^q\end{aligned}$$

at any diagonal matrix. Noting that $\dot{K}^1(x,y)>0$, $\dot{K}^2(x,y)<0$ and $\ddot{K}(x,y)\geq 0$ whenever x and y are positive, we see that

$$\left(\dot{G}^{kl}\ddot{F}^{pq,rs}-\dot{F}^{kl}\ddot{G}^{pq,rs}\right)\nabla_k\mathcal{W}_{pq}\nabla_l\mathcal{W}_{rs}\leq 0\,. \tag{3.6}$$

In view of (2.5), the claim now follows from the maximum principle. □

We now show that G satisfies both of our required estimates (Properties 1) uniformly:

Lemma 3.3. *For all $\varepsilon>0$ there exist constants $c_2>0$ and $\gamma>0$ such that*

$$-c_2\frac{|\nabla\mathcal{W}|^2}{F}\leq(\dot{G}^{kl}\ddot{F}^{pq,rs}-\dot{F}^{kl}\ddot{G}^{pq,rs})\nabla_k h_{pq}\nabla_l h_{rs}\leq-\frac{1}{c_2}\frac{|\nabla\mathcal{W}|^2}{F} \tag{3.7}$$

and

$$(F\dot{G}^{kl}-G\dot{F}^{kl})h_{kl}^2\leq-\gamma F|\mathcal{W}|^2 \tag{3.8}$$

whenever $G>\varepsilon F$.

Proof. Let $A\in GL(n)$ be a diagonal matrix and $T\in\mathbb{R}^n\otimes\mathbb{R}^n\otimes\mathbb{R}^n$ be a totally symmetric tensor. Define

$$Q(A,T):=-\left(\dot{G}^{kl}\ddot{F}^{pq,rs}-\dot{F}^{kl}\ddot{G}^{pq,rs}\right)\Big|_A T_{kpq}T_{lrs}\geq 0\,. \tag{3.9}$$

Recalling the application of the Cauchy-Schwarz inequality to (3.5) reveals that equality occurs in (3.9) only if T is radial, that is, if for each k we have $T_{kpq}=\mu_kA_{pq}$ for some constant μ_k.

Define the set $\Gamma_\varepsilon:=\{x\in\Gamma:\varepsilon f(z)\leq g(z)\leq c_0f(z)\}$. Then, to prove (3.7), we need to demonstrate uniform positive bounds for $FQ(A,T)$ whenever A has eigenvalues in Γ_ε and $|T|\neq 0$. Since Q is homogeneous of degree two with respect to T, we may assume without loss of generality that $|T|=1$. Moreover, since Q is homogeneous of degree -1 with respect to A, it suffices to obtain the required bounds on the compact slice $K:=\{A\in\mathcal{S}_\Gamma:\varepsilon F(A)\leq G(A)\leq c_0F(A),\,|A|=1\}$. The upper bound now follows immediately from the continuity of Q.

To prove the lower bound, it suffices to show that $Q(A,T)=0$ for $A\in K$ only if $|T|=0$. We have seen that $Q(A,T)$ can only vanish if T is radial. Then, since A is diagonal, it follows that T is also diagonal: $T_{klm}\neq 0$ only if $k=l=m$. Since $A\neq 0$, there is some p for which $\lambda_p(A)\neq 0$. But, since $T_{klm}=\mu_kA_{lm}=\mu_k\lambda_l(A)\delta_{lm}$, we have for any k

$$T_{kkk}=\frac{\lambda_k(A)}{\lambda_p(A)}T_{kpp}\,.$$

But T_{kpp} vanishes unless $k = p$. Thus T has at most one non-zero component: T_{ppp}. It follows that A has at most one non-zero eigenvalue: If instead we had $\lambda_q > 0$ for some $q \neq p$, then we could obtain the contradiction $T_{ppp} = \frac{\lambda_p}{\lambda_q} T_{qpp} = 0$. Since $A \in \mathcal{S}_{\Gamma_\varepsilon} \subset \mathcal{S}_\Gamma$, we must have $\lambda_p(A) > 0$. But this implies that that $G(A) = 0$, so that $A \notin K$, a contradiction. Therefore Q can only vanish if T vanishes. This completes the proof of (3.7).

For the second estimate, we observe that, in an orthonormal basis of eigenvectors of $\mathcal{W}$,

$$(F\dot{G}^{kl} - G\dot{F}^{kl}) \leq F\dot{G}^{kl} = F\dot{g}^k\delta^{kl} \leq 2F\frac{g_1}{g_2}\dot{g}_1^{kl}\delta^{kl}\,.$$

Now g_1/g_2 is positive on Γ_ε and therefore has a strictly positive lower bound on the compact slice $\Gamma_\varepsilon \cap \{|z| = 1\}$. Similarly, $\dot{g}_1^k < 0$ on Γ_ε, and therefore has a strictly negative upper bound on $\Gamma_\varepsilon \cap \{|z| = 1\}$. Since both terms are homogeneous of degree zero, these bounds extend unharmed to Γ_ε, and the claim follows. □

Now consider, for some positive constants ε and σ, the function

$$G_{\varepsilon,\sigma} \;:=\; \left(\frac{G}{F} - \varepsilon\right) F^\sigma\,.$$

Observe that the upper bound $G/F < c_0$ implies

$$G_{\varepsilon,\sigma} < c_0 F^\sigma\,. \tag{3.10}$$

We have the following evolution equation for $G_{\varepsilon,\sigma}$:

Lemma 3.4. *The function $G_{\varepsilon,\sigma}$ satisfies the following evolution equation:*

$$\begin{aligned}(\partial_t - \mathcal{L})G_{\varepsilon,\sigma} = F^{\sigma-1}(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})\nabla_k h_{pq}\nabla_l h_{rs} + \frac{2(1-\sigma)}{F}\langle\nabla G_{\varepsilon,\sigma}, \nabla F\rangle_F \\ - \frac{\sigma(1-\sigma)}{F^2}|\nabla F|_F^2 + \sigma G_{\varepsilon,\sigma}|\mathcal{W}|_F^2\,,\end{aligned} \tag{3.11}$$

where we have introduced the notation $\langle u, v\rangle_F := \dot{F}^{kl}u_k u_l$ and $|\mathcal{W}|_F^2 := \dot{F}^{kl}h_{kl}^2$.

Proof. We first compute

$$\nabla G_{\varepsilon,\sigma} = F^{\sigma-1}\left(\nabla G - \frac{G}{F}\nabla F\right) + \frac{\sigma}{F}G_{\varepsilon,\sigma}\nabla F\,.$$

It follows that

$$\begin{aligned}\mathcal{L}G_{\varepsilon,\sigma} = F^{\sigma-1}\left(\mathcal{L}G - \frac{G}{F}\mathcal{L}F\right) + \frac{\sigma}{F}G_{\varepsilon,\sigma}\mathcal{L}F - 2\frac{\sigma-1}{F}\langle\nabla G_{\varepsilon,\sigma}, \nabla F\rangle_F \\ + \frac{\sigma(1-\sigma)}{F^2}G_{\varepsilon,\sigma}|\nabla F|_F^2\,.\end{aligned} \tag{3.12}$$

Therefore,

$$\begin{aligned}(\partial_t - \mathcal{L})G_{\varepsilon,\sigma} &= F^{\sigma-1}\left((\partial_t - \mathcal{L})G - \frac{G}{F}(\partial_t - \mathcal{L})F\right) + \frac{\sigma}{F}G_{\varepsilon,\sigma}(\partial_t - \mathcal{L})F \\ &\quad + 2\frac{1-\sigma}{F}\langle\nabla G_{\varepsilon,\sigma}, \nabla F\rangle_F - \frac{\sigma(1-\sigma)}{F^2}G_{\varepsilon,\sigma}|\nabla F|_F^2 \\ &= F^{\sigma-1}(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})\nabla_k h_{pq}\nabla_l h_{rs} + \sigma G_{\varepsilon,\sigma}|h|_F^2 \\ &\quad + 2\frac{1-\sigma}{F}\langle\nabla G_{\varepsilon,\sigma}, \nabla F\rangle_F - \frac{\sigma(1-\sigma)}{F^2}G_{\varepsilon,\sigma}|\nabla F|_F^2\end{aligned}$$

as required.

□

Just as in for the the mean curvature flow, it is the final two terms of the evolution equation (3.11) which obstruct the application of the maximum principle. We will proceed by the Stampacchia-De Giorgi iteration method as applied in [Hu84, HS99a]. The first step is to show that the spatial L^p norms of the positive part, $(G_{\varepsilon,\sigma})_+ := \max\{G_{\varepsilon,\sigma}, 0\}$, of $G_{\varepsilon,\sigma}$ are non-increasing in t, so long as σ is sufficiently small. As in [Hu84, HS99a, HS99b], this leads to a uniform upper bound on $G_{\varepsilon,\sigma}$ for small, non-zero σ.

4. The Integral Estimates

Proposition 4.1. *For all $\varepsilon > 0$ there exist constants $\ell, L > 0$ such that for all $p > L$ and $0 < \sigma < \ell p^{-\frac{1}{2}}$, the $L^p(M, \mu(t))$ norm of $(G_{\varepsilon,\sigma})_+$ is non-increasing in t.*

To simplify notation somewhat, we fix $\varepsilon > 0$ and denote $E := (G_{\varepsilon,\sigma})_+$. Then E^p is C^1 in t for $p > 1$, with $\partial_t E^p = pE^{p-1}\partial_t G_{\varepsilon,\sigma}$. The evolution equation (3.11) for $G_{\varepsilon,\sigma}$ then implies

$$\begin{aligned}\frac{d}{dt}\int E^p\,d\mu = p\int E^{p-1}\mathcal{L}G_{\varepsilon,\sigma}\,d\mu - p\int E^{p-1}F^{\sigma-1}Q(\nabla\mathcal{W})\,d\mu\\ + 2(1-\sigma)p\int E^{p-1}\frac{\langle\nabla G_{\varepsilon,\sigma},\nabla F\rangle_F}{F}\,d\mu - \sigma(1-\sigma)p\int E^p\frac{|\nabla F|_F^2}{F^2}\,d\mu\\ + \sigma p\int E^p|\mathcal{W}|_F^2\,d\mu - \int E^p HF\,d\mu\,, \qquad (4.1)\end{aligned}$$

where $Q(\nabla\mathcal{W}) = -\big(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs}\big)\nabla_k h_{pq}\nabla_l h_{rs}$. It will be useful to estimate $|\nabla F|_F$ in terms of $|\nabla\mathcal{W}|$ as follows

Lemma 4.2. *There is a constant, $c_3 > 0$ for which $|\nabla F|_F^2 \le c_3|\nabla\mathcal{W}|^2$.*

Proof. Since $\nabla_k F = \dot{f}^p\nabla_k h_{pp}$ in an orthonormal basis of eigenvectors of $\mathcal{W}$, the claim follows from the uniform positive bounds on $\dot{f}^i$ along the flow. □

For $p > 2$, we can integrate the first term of (4.1) by parts:

$$\int E^{p-1}\mathcal{L}G_{\varepsilon,\sigma}\;d\mu = \;-(p-1)\int E^{p-2}|\nabla G_{\varepsilon,\sigma}|_F^2\,d\mu - \int E^{p-1}\ddot{F}^{kl,rs}\nabla_k h_{rs}\nabla_l G_{\varepsilon,\sigma}\,d\mu\,.$$

The first term on the right will be useful. We estimate the second term (when $G_{\varepsilon,\sigma} > 0$) using Young's inequality as follows:

$$\begin{aligned}-\ddot{F}^{kl,rs}\nabla_k h_{rs}\nabla_l G_{\varepsilon,\sigma} &\le \frac{2c_4}{F}\sum_{k,l,r,s}\big|\nabla_k h_{rs}\nabla_l G_{\varepsilon,\sigma}\big|\\ &\le c_4 E\sum_{k,l,r,s}\left(\frac{(\nabla_k h_{rs})^2}{p^{\frac{1}{2}}F^2} + \frac{p^{\frac{1}{2}}(\nabla_l G_{\varepsilon,\sigma})^2}{E^2}\right)\\ &= c_4 E\left(p^{-\frac{1}{2}}\frac{|\nabla\mathcal{W}|^2}{F^2} + p^{\frac{1}{2}}\frac{|\nabla G_{\varepsilon,\sigma}|^2}{E^2}\right)\,, \qquad (4.2)\end{aligned}$$

where we have estimated each of the homogeneous terms $\ddot{F}^{kl,rs}$ above by $2c_4/F$.

A useful term is obtained from the second term of (4.1) using the first estimate of Lemma 3.3. We estimate the third term using Young's inequality as follows:

$$\int E^p\left\langle\frac{\nabla G_{\varepsilon,\sigma}}{E},\frac{\nabla F}{F}\right\rangle_F d\mu \le \frac{p^{\frac{1}{2}}}{2}\int E^{p-2}|\nabla G_{\varepsilon,\sigma}|_F^2\,d\mu + \frac{p^{-\frac{1}{2}}}{2}\int E^p\frac{|\nabla F|_F^2}{F}\,d\mu\,. \qquad (4.3)$$

Putting this back together, we obtain the following Lemma:

Lemma 4.3. *For all $\sigma \in (0,1)$ it holds that*

$$\begin{aligned}\frac{d}{dt}\int E^p d\mu \le \left((c_1+c_4)p^{\frac{3}{2}} - \frac{1}{c_1}p(p-1)\right)\int E^{p-2}|\nabla G_{\varepsilon,\sigma}|^2\;d\mu\\ + \left((c_3+c_4)p^{\frac{1}{2}} - \frac{1}{c_0c_2}p\right)\int E^p\frac{|\nabla\mathcal{W}|^2}{F^2}\,d\mu + c_5\,(\sigma p+1)\int E^p|\mathcal{W}|^2 d\mu\,. \qquad (4.4)\end{aligned}$$

Proof. Since $-HF/|\mathcal{W}|^2$ is homogeneous of degree zero in the principal curvatures, it may be estimated above by some constant, which allows us to estimate the final term in (4.1). Now apply the estimates of Lemmata 2.5, 4.2 and 3.3, and the inequalities (3.10), (4.2) and (4.3) to the remaining terms. □

Notice that for sufficiently large p the first two terms of (4.4) become negative. We now show that the final term may similarly be controlled by the good negative terms, so long as σ is also sufficiently small.

Proposition 4.4. *There are positive constants A_1, A_2, A_3, B_1, B_2 which are independent of p and σ such that:*

$$\int E^p|\mathcal{W}|^2 \leq \big(A_1p^{\frac{3}{2}} + A_2p^{\frac{1}{2}} + A_3\big)\int E^{p-2}|\nabla G_{\varepsilon,\sigma}|^2\, d\mu + \big(B_1p^{\frac{1}{2}} + B_2\big)\int E^p\frac{|\nabla\mathcal{W}|^2}{F^2}\, d\mu\,. \tag{4.5}$$

Proof. We begin with the commutation formula (c.f. [AnBa, Proposition 5])

$$\nabla_k\nabla_l h_{pq} = \nabla_p\nabla_q h_{kl} + h_{kl}{h^2}_{pq} - h_{pq}{h^2}_{kl} + h_{kq}{h^2}_{pl} - h_{pl}{h^2}_{kq}\,,$$

which holds on a general hypersurface of $\mathbb{R}^{n+1}$. This contracts to the following Simons type identity:

$$\mathcal{L}h_{pq} = \dot{F}^{kl}\nabla_p\nabla_q h_{kl} + F{h^2}_{pq} - \dot{F}^{kl}h_{pq}{h^2}_{kl} + \dot{F}^{kl}h_{kq}{h^2}_{pl} - \dot{F}^{kl}h_{pl}{h^2}_{kq}\,.$$

Contracting further with $\dot{G}$ yields

$$\dot{G}^{pq}\mathcal{L}h_{pq} = \dot{G}^{pq}\dot{F}^{kl}\nabla_p\nabla_q h_{kl} + (F\dot{G}^{kl} - G\dot{F}^{kl}){h^2}_{kl}\,.$$

On the other hand, we have that

$$\dot{F}^{kl}\nabla_p\nabla_q h_{kl} = \nabla_p\nabla_q F - \ddot{F}^{kl,rs}\nabla_p h_{rs}\nabla_q h_{kl}\,,$$

so that

$$\dot{G}^{pq}\mathcal{L}h_{pq} = \dot{G}^{pq}\nabla_p\nabla_q F - \dot{G}^{pq}\ddot{F}^{kl,rs}\nabla_p h_{rs}\nabla_q h_{kl} + (F\dot{G}^{kl} - G\dot{F}^{kl}){h^2}_{kl}\,. \tag{4.6}$$

We now recall (3.12):

$$\begin{aligned}\mathcal{L}G_{\varepsilon,\sigma} &= F^{\sigma-1}\left(\mathcal{L}G - \frac{G}{F}\mathcal{L}F\right) + \frac{\sigma}{F}G_{\varepsilon,\sigma}\mathcal{L}F - 2\frac{1-\sigma}{F}\left\langle\nabla G_{\varepsilon,\sigma}, \nabla F\right\rangle_F - \frac{\sigma(1-\sigma)}{F^2}G_{\varepsilon,\sigma}|\nabla F|_F^2\\ &= F^{\sigma-1}\left(\dot{F}^{kl}\dot{G}^{pq}\nabla_k\nabla_l h_{pq} + \dot{F}^{kl}\ddot{G}^{pq,rs}\nabla_k h_{pq}\nabla_l h_{rs} - \frac{G}{F}\mathcal{L}F\right) + \frac{\sigma}{F}G_{\varepsilon,\sigma}\mathcal{L}F\\ &\quad - 2\frac{1-\sigma}{F}\left\langle\nabla G_{\varepsilon,\sigma}, \nabla F\right\rangle_F + \frac{\sigma(1-\sigma)}{F^2}G_{\varepsilon,\sigma}|\nabla F|_F^2\,. \end{aligned} \tag{4.7}$$

Putting (4.6) and (4.7) together, we obtain

$$\begin{aligned}\mathcal{L}G_{\varepsilon,\sigma} &= F^{\sigma-1}\big(\dot{F}^{kl}\ddot{G}^{pq,rs} - \dot{G}^{kl}\ddot{F}^{pq,rs}\big)\nabla_k h_{pq}\nabla_l h_{rs} + F^{\sigma-2}(F\dot{G}^{kl} - G\dot{F}^{kl})\nabla_k\nabla_l F\\ &\quad + F^{\sigma-1}\big(F\dot{G}^{kl} - G\dot{F}^{kl}\big){h^2}_{kl} + \frac{\sigma}{F}G_{\varepsilon,\sigma}\mathcal{L}F - 2\frac{(1-\sigma)}{F}\langle\nabla F, \nabla G_{\varepsilon,\sigma}\rangle_F\\ &\quad + \frac{\sigma(1-\sigma)}{F^2}G_{\varepsilon,\sigma}|\nabla F|_F^2\,. \end{aligned} \tag{4.8}$$

The first and third terms on the right may be estimated from below using Lemma 3.3.

Applying Young's inequality to the term involving the inner product, we obtain

$$-2\frac{(1-\sigma)}{F}\langle\nabla F, \nabla G_{\varepsilon,\sigma}\rangle_F \leq (1-\sigma)E\left(\frac{|\nabla F|_F^2}{F^2} + \frac{|\nabla G_{\varepsilon,\sigma}|_F^2}{E^2}\right)$$

wherever $G_{\varepsilon,\sigma} > 0$. Recalling the estimates of Lemmata 2.5, 3.3 and 4.2, and equation (3.10), we arrive at

$$\begin{aligned}\mathcal{L}G_{\varepsilon,\sigma} &\leq (c_2 + c_0c_3 + c_0c_1)F^\sigma\frac{|\nabla\mathcal{W}|^2}{F^2} + F^{\sigma-2}(F\dot{G}^{kl} - G\dot{F}^{kl})\nabla_k\nabla_l F - \gamma F^\sigma|\mathcal{W}|^2\\ &\quad + \frac{\sigma}{F}G_{\varepsilon,\sigma}\mathcal{L}F + c_0c_1F^\sigma\frac{|\nabla G_{\varepsilon,\sigma}|^2}{E^2}\,.\end{aligned}$$

Now put the $\gamma F^\sigma|\mathcal{W}|^2$ term on the left, multiply the equation by $E^pF^{-\sigma}$, and integrate over M to obtain

$$\begin{aligned}\gamma\int E^p|\mathcal{W}|^2\,d\mu \leq &-\int E^pF^{-\sigma}\mathcal{L}G_{\varepsilon,\sigma}\,d\mu+(c_2+c_0c_3+c_0c_1)\int E^p\frac{|\nabla\mathcal{W}|^2}{F^2}\,d\mu\\ &+\int E^pF^{-2}(F\dot{G}^{kl}-G\dot{F}^{kl})\nabla_k\nabla_lF\,d\mu+\sigma\int E^{p+1}F^{-1-\sigma}\mathcal{L}F\,d\mu\\ &+c_0c_1\int E^{p-2}|\nabla G_{\varepsilon,\sigma}|^2\,d\mu\,.\end{aligned}\tag{4.9}$$

Integrating the first term on the right by parts, we obtain the following estimate:

Lemma 4.5. *If $\sigma\in(0,1)$ and $p>2$, there are constants $C_1,C_2,D_1>0$, independent of σ and p, such that*

$$-\int E^pF^{-\sigma}\mathcal{L}G_{\varepsilon,\sigma}\,d\mu\leq\big(C_1p+C_2\big)\int E^{p-2}|\nabla G_{\varepsilon,\sigma}|^2\,d\mu+D_1\int E^p\frac{|\nabla\mathcal{W}|^2}{F^2}\,d\mu\,.$$

Proof. Integrating by parts, we find

$$\begin{aligned}-\int E^pF^{-\sigma}\mathcal{L}G_{\varepsilon,\sigma}\,d\mu=&\,p\int E^{p-1}F^{-\sigma}|\nabla G_{\varepsilon,\sigma}|_F^2\,d\mu-\sigma\int E^pF^{-\sigma-1}\langle\nabla G_{\varepsilon,\sigma},\nabla F\rangle_F\,d\mu\\ &+\int E^pF^{-\sigma}\ddot{F}^{kl,rs}\nabla_kh_{rs}\nabla_lG_{\varepsilon,\sigma}\,d\mu\,.\end{aligned}$$

Estimating each of the coefficients of $\ddot{F}$ above by $\frac{2c_4}{F}$ and applying Young's inequality to the second and third terms, we obtain

$$\begin{aligned}-\int E^pF^{-\sigma}\mathcal{L}G_{\varepsilon,\sigma}\,d\mu\leq&\,c_0p\int E^{p-2}|\nabla G_{\varepsilon,\sigma}|_F^2\,d\mu+\frac{c_0\sigma}{2}\int E^p\left(\frac{|\nabla G_{\varepsilon,\sigma}|_F^2}{E^2}+\frac{|\nabla F|_F^2}{F^2}\right)d\mu\\ &+\frac{c_0c_4}{2}\int E^p\left(\frac{|\nabla\mathcal{W}|^2}{F^2}+\frac{|\nabla G_{\varepsilon,\sigma}|^2}{E^2}\right)d\mu\,.\end{aligned}$$

Therefore,

$$\begin{aligned}-\int E^pF^{-\sigma}\mathcal{L}G_{\varepsilon,\sigma}\,d\mu\leq&\left(c_0c_1p+\frac{c_0c_1\sigma}{2}+\frac{c_0c_4}{2}\right)\int E^{p-2}|\nabla G_{\varepsilon,\sigma}|^2\,d\mu\\ &+\left(\frac{c_0c_1c_2\sigma}{2}+\frac{c_0c_4}{2}\right)\int E^p\frac{|\nabla\mathcal{W}|^2}{F^2}\,d\mu\,.\end{aligned}$$

□

In the same way, we obtain the following estimate on the third term of (4.9):

Lemma 4.6. *There are constants $C_3,C_4,D_3,D_4>0$, independent of $p>2$ and $\sigma\in(0,1)$, such that*

$$\begin{aligned}\int E^pF^{-2}(F\dot{G}^{kl}-G\dot{F}^{kl})\nabla_k\nabla_lF\,d\mu\leq&\big(C_3p^{\frac{3}{2}}+C_4\big)\int E^{p-2}|\nabla G_{\varepsilon,\sigma}|^2\,d\mu\\ &+\big(D_3p^{\frac{1}{2}}+D_4\big)\int E^p\frac{|\nabla\mathcal{W}|^2}{F^2}\,d\mu\,.\end{aligned}$$

And the fourth term:

Lemma 4.7. *There are constants $C_5,C_6,D_5,D_6>0$, independent of p and σ, such that*

$$\int E^{p+1}F^{-1-\sigma}\mathcal{L}F\,d\mu\leq\big(C_5p^{\frac{3}{2}}+C_6\big)\int E^{p-2}|\nabla G_{\varepsilon,\sigma}|^2\,d\mu+\big(D_5p^{\frac{1}{2}}+D_6\big)\int E^p\frac{|\nabla\mathcal{W}|^2}{F^2}\,d\mu\,.$$

This completes the proof of Proposition 4.4.

□

Combining Proposition 4.4 with Lemma 4.3, we obtain

$$\frac{d}{dt}\int E^p\,d\mu \le -\Big(c_1p^2-\alpha_1\sigma p^{\frac{5}{2}}-\alpha_2\sigma p^2-\alpha_3p^{\frac{3}{2}}-\alpha_4p\Big)\int E^{p-2}|G_{\varepsilon,\sigma}|^2\;d\mu$$
$$-\Big(\beta_1p-\beta_2\sigma p^{\frac{1}{2}}-\beta_3\sigma p-\beta_4p^{\frac{1}{2}}-\beta_5\Big)\int E^p\frac{|\nabla\mathcal{W}|^2}{F^2}\,d\mu\,.$$

for some constants $\alpha_i, \beta_i > 0$, which are independent of σ and p. Proposition 4.1 follows easily.

5. Proof of Theorem 1.1

We are now able to proceed just as in [Hu84, Section 5] and [HS99a, Section 3], using Proposition 4.1 and the following lemma to derive the desired bound on $G_{\varepsilon,\sigma}$.

Lemma 5.1 (Stampacchia [St66])**.** *Let $\varphi : [k_0,\infty) \to \mathbb{R}$ be a non-negative, non-increasing function satisfying*

$$\varphi(h) \le \frac{C}{(h-k)^\alpha}\varphi(k)^\beta, \qquad h > k > k_0\,, \tag{5.1}$$

for some constants $C > 0$, $\alpha > 0$ and $\beta > 1$. Then

$$\varphi(k_0+d) = 0\,,$$

where $d^\alpha = C\varphi(k_0)^{\beta-1}2^{\frac{\alpha\beta}{\beta-1}}$.

Now, given any $k \ge k_0$, where $k_0 := \sup_{\sigma\in(0,1)}\sup_M G_{\varepsilon,\sigma}(\cdot,0)$, set

$$v_k(x,t) := \big(G_{\varepsilon,\sigma}(x,t)-k\big)_+^{\frac{p}{2}} \quad\text{and}\quad A_k(t) := \{x\in M : v_k(x,t) > 0\}\,.$$

We will show that $\varphi(k) = |A_k| := \int_0^T\int_{A_k(t)} d\mu(\cdot,t)\,dt$ satisfies the conditions of Stampacchia's Lemma for some $k_1 \ge k_0$. This provides us with a constant d for which $|A_{k_1+d}|$ vanishes. Theorem 1.1 then follows. Observe that $|A_k|$ is non-negative and non-increasing with respect to k. Then we only need to demonstrate that an inequality of the form (5.1) holds.

We begin by noting that

Lemma 5.2. *There are constants $L_1 \ge L$ and $c_6 > 0$ such that for all $p > L_1$ we have*

$$\frac{d}{dt}\int v_k^2\,d\mu + \frac{1}{c_1}\int|\nabla v_k|^2\,d\mu \le c_6(\sigma p+1)\int_{A_k}F^2G_{\varepsilon,\sigma}^p\,d\mu\,. \tag{5.2}$$

Proof. Observe that

$$\frac{d}{dt}\int v_k^2\,d\mu = \int_{A_k}p(G_{\varepsilon,\sigma}-k)_+^{p-1}\partial_tG_{\varepsilon,\sigma}\;d\mu - \int v_k^2HF\,d\mu\,.$$

The result is then obtained by proceeding as in Lemma (4.3), applying

$$|\nabla v_k|^2 = \frac{p^2}{4}(G_{\varepsilon,\sigma}-k)_+^{p-2}|\nabla G_{\varepsilon,\sigma}|^2\,,$$

and estimating $|\mathcal{W}|^2 \le CF^2$ using the degree zero homogeneity of $|\mathcal{W}|^2/F^2$. □

Now set $\sigma' = \sigma + \frac{n}{p}$. Then

$$\int_{A_k}F^n\,d\mu \le \int_{A_k}F^n\frac{(G_{\varepsilon,\sigma})_+^p}{k^p}\,d\mu = k^{-p}\int_{A_k}(G_{\varepsilon,\sigma'})_+^p\;d\mu \le k^{-p}\int(G_{\varepsilon,\sigma'})_+^p\;d\mu\,. \tag{5.3}$$

If $p \ge \max\left\{L_1, \frac{4n^2}{\ell^2}\right\}$ and $\sigma \le \frac{\ell}{2}p^{-\frac{1}{2}}$, then $p \ge L_1$ and $\sigma' \le \ell p^{-\frac{1}{2}}$, so that, by Proposition 4.1,

$$\int_{A_k}F^n\,d\mu \le k^{-p}\int(G_{\varepsilon,\sigma'}(\cdot,0))_+^p\;d\mu_0 \le \mu_0(M)\left(\frac{k_0}{k}\right)^p. \tag{5.4}$$

Choosing k sufficiently large, the right hand side of this inequality can be made arbitrarily small. We will use this fact in conjunction with the following Sobolev inequality to exploit the good gradient term in (5.2).

Lemma 5.3 (Huisken [Hu84]). *There is a constant c_S (independent of $\sigma, p,$ and ε) such that*

$$\left(\int v_k^{2q}\,d\mu\right)^{\frac{1}{q}} \le c_S\left(\int |\nabla v_k|^2\,d\mu + \left(\int_{A_k} F^n\,d\mu\right)^{\frac{2}{n}}\left(\int v_k^{2q}\,d\mu\right)^{\frac{1}{q}}\right), \tag{5.5}$$

where q is equal to $\frac{n}{n-2}$ if $n>2$, or any positive number if $n=2$.

Proof. Since we have the estimate $H^2 < CF^2$ (by degree zero homogeneity of the quantity H^2/F^2) this follows from the Michael-Simon Sobolev inequality [MS73] and the Hölder inequality just as in [Hu84]. □

It follows from (5.4) and (5.5) that there is some $k_1 > k_0$ such that for all $k > k_1$ we have

$$\left(\int v_k^{2q}\,d\mu\right)^{\frac{1}{q}} \le 2c_S\int |\nabla v_k|^2\,d\mu\,.$$

Therefore, from (5.2), we have for all $k > k_1$

$$\frac{d}{dt}\int v_k^2\,d\mu + \frac{1}{2c_1c_S}\left(\int v^{2q}\,d\mu\right)^{\frac{1}{q}} \le c_6(\sigma p+1)\int_{A_k} F^2 G_{\varepsilon,\sigma}^p\,d\mu\,.$$

Integrating this over time, and noting that $A_k(0)=\emptyset$, we find

$$\sup_{[0,T)}\left(\int_{A_k} v_k^2\,d\mu\right) + \frac{1}{2c_1c_S}\int_0^T\left(\int v^{2q}\,d\mu\right)^{\frac{1}{q}} dt \le 2c_6(\sigma p+1)\int_0^T\int_{A_k} F^2G_{\varepsilon,\sigma}^p\,d\mu\,dt\,. \tag{5.6}$$

We now exploit the interpolation inequality for L^p spaces:

$$|f|_{q_0} \le |f|_r^{1-\theta}|f|_q^{\theta}\,, \tag{5.7}$$

where $\theta\in(0,1)$ and $\frac{1}{q_0} = \frac{\theta}{q} + \frac{1-\theta}{r}$. Setting $r=1$ and $\theta = \frac{1}{q_0}$, we may assume $1<q_0<q$. Then applying (5.7) we find

$$\int_{A_k} v_k^{2q_0}\,d\mu \le \left(\int_{A_k} v_k^2\,d\mu\right)^{q_0-1}\left(\int_{A_k} v^{2q}\,d\mu\right)^{\frac{1}{q}}.$$

Now, applying the Hölder inequality, we find,

$$\left(\int_0^T\int_{A_k} v_k^{2q_0}\,d\mu\,dt\right)^{\frac{1}{q_0}} \le \left(\sup_{[0,T)}\int_{A_k} v_k^2\,d\mu\right)^{\frac{q_0-1}{q_0}}\left(\int_0^T\left(\int_{A_k} v^{2q}\,d\mu\right)^{\frac{1}{q}}dt\right)^{\frac{1}{q_0}}.$$

Using Young's inequality, $ab \le \left(1-\frac{1}{q_0}\right)a^{\frac{q_0}{q_0-1}} + \frac{1}{q_0}b^{q_0}$, on the right hand side, we obtain

$$\begin{aligned}\left(\int_0^T\int_{A_k} v_k^{2q_0}\,d\mu dt\right)^{\frac{1}{q_0}} &\le \left(1-\frac{1}{q_0}\right)\sup_{[0,T)}\int_{A_k} v_k^2\,d\mu + \frac{1}{q_0}\int_0^T\left(\int_{A_k} v^{2q}\,d\mu\right)^{\frac{1}{q}} dt\\ &\le \sup_{[0,T)}\int_{A_k} v_k^2\,d\mu + \int_0^T\left(\int_{A_k} v^{2q}\,d\mu\right)^{\frac{1}{q}} dt\,.\end{aligned}$$

Recalling (5.6), we arrive at

$$\left(\int_0^T\int_{A_k} v_k^{2q_0}\,d\mu\,dt\right)^{\frac{1}{q_0}} \le 2c_6(\sigma p+1)\int_0^T\int_{A_k} F^2G_{\varepsilon,\sigma}^p\,d\mu\,dt\,. \tag{5.8}$$

Application of the Hölder inequality yields the inequalities

$$\int_0^T\int_{A_k} F^2G_{\varepsilon,\sigma}^p\,d\mu\,dt \le |A_k|^{1-\frac{1}{r}}\left(\int_0^T\int_{A_k} F^{2r}G_{\varepsilon,\sigma}^{pr}\,d\mu\,,dt\right)^{\frac{1}{r}} \le c_7|A_k|^{1-\frac{1}{r}} \tag{5.9}$$

$$\text{and}\quad \int_0^T\int_{A_k} v_k^2\,d\mu\,dt \le |A_k|^{1-\frac{1}{q_0}}\left(\int_0^T\int_{A_k} v_k^{2q_0}\,d\mu\,dt\right)^{\frac{1}{q_0}}, \tag{5.10}$$

where the integral on the right hand side of (5.9) was estimated in a similar manner to (5.4), with $c_7 := k_0^2\,(T\mu_0(M))^{\frac{1}{r}}$ (so long as $\sigma \le \frac{l}{4}p^{-\frac{1}{2}}$, and $2r > L_2 := \max\{L_1, \frac{4n^2}{l^2}, \frac{64}{l^2}\}$, say). Finally, for $h > k \ge k_1$ we may estimate

$$|A_h| := \int_0^T \int_{A_h} d\mu\,dt = \int_0^T \int_{A_h} \frac{(G_{\varepsilon,\sigma}-k)_+^p}{(G_{\varepsilon,\sigma}-k)_+^p}\,d\mu\,dt \le \int_0^T \int_{A_h} \frac{(G_{\varepsilon,\sigma}-k)_+^p}{(h-k)^p}\,d\mu\,dt\,.$$

Since $A_h(t) \subset A_k(t)$ for all $t \in [0.T)$, and $v_k^2 := (G_{\varepsilon,\sigma}-k)_+^p$, we obtain

$$(h-k)^p|A_h| \le \int_0^T \int_{A_k} v_k^2\,d\mu\,dt\,. \tag{5.11}$$

Putting together estimates (5.8), (5.9), (5.10) and (5.11), we arrive at

$$|A_h| \le \frac{2c_6c_7(\sigma p+1)}{(h-k)^p}|A_k|^\gamma$$

for all $h > k \ge k_1$, where $\gamma := 2 - \frac{1}{q_0} - \frac{1}{r}$. Now fix $p := 2L_2$ and choose $\sigma < \frac{\ell}{4}p^{-\frac{1}{2}}$ sufficiently small that $\sigma p < 1$. Then, choosing $r > \max\{\frac{q_0}{q_0-1}, L_2\}$, so that $\gamma > 1$, we may apply Stampacchia's Lemma. We conclude that $|A_k| = 0$ for all $k > k_1 + d$, where $d^p = c_6c_7 2^{\frac{\gamma p}{\gamma-1}+2}|A_{k_1}|^{\gamma-1}$. We note that d is finite, since T is finite and

$$\int_{A_{k_1}} d\mu \le \int_{A_{k_1}} \frac{(G_{\varepsilon,\sigma})_+^p}{k_1^p}\,d\mu \le k_1^{-p}\int (G_{\varepsilon,\sigma})_+^p\,d\mu \le k_1^{-p}\int (G_{\varepsilon,\sigma}(\cdot,0))_+^p\,d\mu_0\,,$$

where the final estimate follows from Proposition 4.1.

It follows that

$$G \le \varepsilon F + (k_1+d)F^{1-\sigma} \le 2\varepsilon F + C_\varepsilon$$

for some suitably large constant $C_\varepsilon > 0$. Theorem 1.1 follows.

6. Rescaling about type-II singularities

We now analyse the structure of fast forming singularities. Let $X : M \times [0,T) \to \mathbb{R}^{n+1}$ be a smooth, compact solution of (1.1) satisfying the following ansatz: For all $C > 0$ there is a time $t_C \in [0,T)$ such that

$$\max_{x\in M} |\mathcal{W}(x,t)|^2 \ge \frac{C}{T-t} \tag{6.1}$$

for all $t \in [t_C, T)$. We say that the flow undergoes a *type-II singularity.* To analyse the shape of type-II singularities, we consider, following Hamilton [Ham95b] and Huisken-Sinestrari [HS99a], the following sequence of parabolic rescalings: For each $k \in \mathbb{N}$, choose a sequence (t_k) of times $t_k \in [0, T-1/k]$, and a sequence (x_k) of points $x_k \in M$ such that

$$|\mathcal{W}(x_k,t_k)|^2\left(T-\frac{1}{k}-t_k\right) = \max_{(x,t)\in M\times[0,T-1/k]} |\mathcal{W}(x,t)|^2\left(T-\frac{1}{k}-t\right)\,.$$

Now set

$$L_k := |\mathcal{W}(x_k,t_k)|^2\,, \quad \alpha_k := -L_kt_k\,, \quad \text{and} \quad \sigma_k := L_k\left(T-\frac{1}{k}-t_k\right)\,.$$

Lemma 6.1. *As $k \to \infty$, we have*

$$t_k \to T\,, \quad L_k \to \infty\,, \quad \alpha_k \to -\infty\,, \quad \textit{and} \quad \sigma_k \to \infty.$$

Proof. By the ansatz (6.1), for all $R > 0$ there exists $t_R \in [0,T)$ and $x_R \in M$ such that

$$|\mathcal{W}(x_R,t_R)|^2(T-t_R) > 2R\,.$$

On the other hand, there is some sufficiently large $k_R \in \mathbb{N}$ such that

$$t_R < T - \frac{1}{k}\,, \qquad |\mathcal{W}(x_R,t_R)|^2\left(T-\frac{1}{k}-t_R\right) > R$$

for all $k > k_R$. Therefore, by definition,

$$\sigma_k \geq |\mathcal{W}(x_R, t_R)|^2 \left(T - \frac{1}{k} - t_R\right) > R$$

for all $k > k_R$. Since R was arbitrary, we find $\sigma_k \to \infty$ as $k \to \infty$.

Since $\left(T - \frac{1}{k} - t_k\right)$ is bounded, it follows from the definition of σ_k that $L_k \to \infty$ as $k \to \infty$. Therefore, since $|\mathcal{W}|$ remains bounded whilst $t < T$, we must have $t_k \to T$. It follows that $\alpha_k \to -\infty$. □

Now consider the rescalings

$$X_k(x,t) = \sqrt{L_k}\left(X\left(x, \frac{t}{L_k} + t_k\right) - X(x_k, t_k)\right); \quad \text{for} \quad t \in [\alpha_k, \sigma_k].$$

It is straightforward to compute

$$\frac{\partial X_k}{\partial t}(x,t) = -L_k^{-\frac{1}{2}} F\left(x, \frac{t}{L_k} + t_k\right) \nu\left(x, \frac{t}{L_k} + t_k\right) ;$$

$$\frac{\partial X_k}{\partial x^i}(x,t) = \sqrt{L_k}\frac{\partial X}{\partial x^i}\left(x, \frac{t}{L_k} + t_k\right) \quad \Rightarrow \quad (g_k)_{ij}(x,t) = L_k g_{ij}\left(x, \frac{t}{L_k} + t_k\right)$$

$$\Rightarrow \quad (g_k)^{ij}(x,t) = \frac{1}{L_k} g^{ij}\left(x, \frac{t}{L_k} + t_k\right) ;$$

and

$$\nu_k(x,t) = \nu\left(x, \frac{t}{L_k} + t_k\right) \quad \Rightarrow \quad {}^kD_i\nu_k(x,t) = {}^kD_i\nu\left(x, \frac{t}{L_k} + t_k\right)$$

$$\Rightarrow \quad \mathcal{W}_k(x,t) = L_k^{-\frac{1}{2}}\mathcal{W}\left(x, \frac{t}{L_k} + t_k\right)$$

$$\Rightarrow \quad F_k(x,t) = L_k^{-\frac{1}{2}} F\left(x, \frac{t}{L_k} + t_k\right),$$

where we used the script k to distinguish quantities related to the rescaling X_k (in particular, ${}^kD$ is the pullback of the Euclidean connection along X_k). We refer to the sequence (X_k) as a *blow-up sequence*. Observe that the rescalings satisfy the flow equation (1.1). We also note the following properties (c.f. [HS99a, Lemma 4.4]):

Lemma 6.2.

(i) For each $k \in \mathbb{N}$, $X_k(x_k, 0) = 0$ and $|\mathcal{W}(x_k, 0)| = 1$

(ii) For any $\varepsilon > 0$ and $\Sigma > 0$ there exists $k_0 \in \mathbb{N}$ such that $\sigma_k > \Sigma$ and

$$\max_{M \times [\alpha_{k_0}, \Sigma]} |\mathcal{W}_k|^2 \leq 1 + \varepsilon \tag{6.2}$$

for all $k \geq k_0$.

(iii) For any $\varepsilon > 0$ there exists C_ε such that

$$-\kappa_1^{(k)}(x,t) \leq \varepsilon F_k(x,t) + \frac{C_\varepsilon}{\sqrt{L_k}} \tag{6.3}$$

for all $(x,t) \in M \times [\alpha_k, \sigma_k]$, where $\kappa_1^{(k)}$ is the smallest principal curvature of X_k.

Proof. Part (i) is immediate from the definitions and our calculation of $\mathcal{W}_k$.

To prove part (ii), first note that

$$|\mathcal{W}_k(x,t)|^2 = L_k^{-1}|\mathcal{W}(x, L_k^{-1}t + t_k)|^2 .$$

By the definition of L_k and the choice of (x_k, t_k) we also have

$$|\mathcal{W}(x, L_k^{-1}t + t_k)|^2 \left(T - \frac{1}{k} - (L_k^{-1}t + t_k)\right) \leq L_k\left(T - \frac{1}{k} - t_k\right) .$$

Therefore:

$$|\mathcal{W}_k(x,t)|^2 \le \frac{T-\frac{1}{k}-t_k}{T-\frac{1}{k}-t_k-L_k^{-1}t} = \frac{\sigma_k}{\sigma_k - t} = 1 + \frac{t}{\sigma_k - t}\,.$$

Since $\sigma_k \to \infty$, the claim follows.

For part (iii), we have

$$\kappa_1^k(x,t) = \frac{1}{\sqrt{L_k}}\kappa_1(x, L_k^{-1}t + t_k)\,.$$

Therefore, by Theorem 1.1, for all $\varepsilon > 0$ there exists C_ε such that

$$-\kappa_1^k(x,t) \le \frac{1}{\sqrt{L_k}}\left(\varepsilon F(x, L_k^{-1}t + t_k) + C_\varepsilon\right) = \varepsilon F_k(x,t) + \frac{C_\varepsilon}{\sqrt{L_k}}$$

for all $(x,t) \in M \times [-\alpha_k, \sigma_k]$. □

We now prove Corollary 1.2.

Proof of Corollary 1.2. Since the flow speed is a convex function of the Weingarten map, the flow admits second derivative Hölder estimates, and we may proceed as in [Ba11, Section 3], using Lemma 6.2, to obtain a sublimit $X_\infty : M_\infty \times I_\infty \to \mathbb{R}^{n+1}$ of the blow-up sequence. Since for each k the rescaled immersion X_k is a solution of the flow on the time interval $[\alpha_k, \sigma_k]$, we deduce from Lemma 6.1 that X_∞ is an eternal solution of the flow (1.1) (that is, $I_\infty = \mathbb{R}$). Part (iii) of Lemma 6.2 implies that X_∞ is weakly convex. Applying the strong tensor maximum principle [Ham82] (c.f. [An07, Theorem 3.1]) to the evolution equation for the Weingarten map

$$\partial_t h_i{}^j = \mathcal{L} h_i{}^j + \ddot{F}^{pq,rs}\nabla_i h_{pq}\nabla^j h_{rs} + \dot{F}^{kl}h_{kl}^2 h_i{}^j\,,$$

we deduce, just as in [HS99b, Theorem 4.1], that the rank of $\mathcal{W}$ is constant and its null-space is invariant under parallel transport. The same use of Frobenius' Theorem as in [Hu93, Theorem 5.1] (c.f. [Ham84]) then implies that M_∞ splits isometrically as a product $\mathbb{R}^{n-k} \times \Sigma_\infty^k$ for some $1 \le k \le n$, where Σ_∞^k is strictly convex. Moreover, $X_\infty\big|_{\Sigma_\infty^k}$ solves the flow (1.1) in $\mathbb{R}^{k+1}$.

Now observe that, by Lemma 6.2 (i) and (ii), the maximum value of $|\mathcal{W}_\infty|$ is 1, and occurs at $(x_\infty, 0)$; it follows that the maximum value of F is also attained here. We complete the proof by applying the differential Harnack inequality of [An94b] to deduce that $X_\infty\big|_{\Sigma_\infty^k}(\Sigma_\infty^k)$ moves by translation (c.f. [Ham95a]).

Proposition 6.3. *Let* $X : \Sigma^k \times \mathbb{R} \to \mathbb{R}^{k+1}$ *be a strictly convex, eternal solution of* (1.1) *with admissible speed* F *such that* $\sup_{\Sigma\times\mathbb{R}} F$ *is attained. Then* X *moves by translation.*

Proof. Consider the function $\Phi(A) = -F(A^{-1})$, where $F : \mathcal{S}_+ \to \mathbb{R}$ gives the flow speed as a function of the Weingarten map (here, $\mathcal{S}_+$ is the cone of symmetric, positive definite matrices). For any $A \in \mathcal{S}_+$, $B \in GL(n)$, we have

$$\dot{\Phi}\big|_A(B) = \frac{d}{ds}\bigg|_{s=0}\Phi(A+sB) = -\frac{d}{ds}\bigg|_{s=0} F\left([A+sB]^{-1}\right) = \dot{F}\big|_A\left(A^{-1}BA^{-1}\right)\,,$$

and

$$\ddot{\Phi}\big|_A(B,B) = \frac{d^2}{ds^2}\bigg|_{s=0}\Phi(A+sB) = -\ddot{F}\big|_A\left(A^{-1}BA^{-1}, A^{-1}BA^{-1}\right) - 2\dot{F}\big|_A\left(A^{-1}BA^{-1}BA^{-1}\right)\,.$$

Since $\ddot{F} \ge 0$, $\dot{F} > 0$, and $F > 0$, it follows that

$$\ddot{\Phi} + \frac{1-\alpha}{\alpha}\frac{\dot{\Phi}\otimes\dot{\Phi}}{\Phi} \le 0$$

for all $\alpha \in (0,1)$. That is, Φ is α-concave for all $\alpha \in (0,1)$. Thus Corollary 5.11 of [An94b] may be applied. We deduce that any strictly convex solution of (1.1) satisfies

$$\partial_t \overline{F} - g\left(\mathcal{W}^{-1}(\operatorname{grad}\overline{F}), \operatorname{grad}\overline{F}\right) + \frac{(\alpha-1)\overline{F}}{\alpha(t-t_0)} \ge 0 \tag{6.4}$$

for all $t > t_0$, where t_0 is the initial time, grad is the gradient operator on M, and $\overline{F}$ gives the speed along the flow in the Gauss map parametrisation. It follows that any strictly convex, eternal solution of (1.1) satisfies

$$P := \partial_t \overline{F} - g\left(\mathcal{W}^{-1}(\operatorname{grad}\overline{F}), \operatorname{grad}\overline{F}\right) \geq 0\,.$$

Moreover, (6.4) is deduced from the maximum principle applied to the time evolution of P, such that equality is attained at a space-time point only if equality holds identically. Since by assumption $\sup_{\Sigma\times\mathbb{R}} F$ is attained, P vanishes identically.

We now recall the evolution equation [An94b, Equation 5.2] for the Harnack quantity P:

$$\left(\partial_t - \bar{\mathcal{L}}\right) P = \dot{\Phi}(\mathrm{Id})P + \ddot{\Phi}(\overline{Q}, \overline{Q})\,,$$

where $\overline{Q}$ is the time derivative of the inverse of the Weingarten map in the Gauss map parametrisation, and $\bar{\mathcal{L}}$ is the elliptic operator corresponding to $\mathcal{L}$ in the Gauss map parametrisation. Since P is identically zero, this simply says $\ddot{\Phi}(\overline{Q}, \overline{Q}) = 0$. Recalling the equation for $\ddot{\Phi}$, positive definiteness of $\dot{F}$ and strict convexity of Σ imply that $\overline{Q}$ must vanish. Returning to the standard parametrisation (e.g. using [An94b, Lemma 3.10]), we find $0 = Q = \mathcal{W}^{-1} \circ (\partial_t \mathcal{W} - \nabla_V \mathcal{W}) \circ \mathcal{W}^{-1}$, where we have defined the vector field $V := -\mathcal{W}^{-1}(\operatorname{grad} F)$. Substituting $\partial_t \mathcal{W} = \nabla \operatorname{grad} F + F\mathcal{W}^2$, we have, for all $u \in T\Sigma$,

$$\begin{aligned} 0 &= \nabla_u \operatorname{grad} F + F\mathcal{W}^2(u) - \nabla_u \mathcal{W}(V) \\ &= \nabla_u(\operatorname{grad} F + \mathcal{W}(V)) + \mathcal{W}(F\mathcal{W}(u) - \nabla_u V)\,. \end{aligned}$$

It follows that $\nabla V - F\mathcal{W} = 0$.

Now define the Euclidean vector $T := V^i \frac{\partial X}{\partial x^i} - F\nu$. Then, for all $u \in T\Sigma$,

$${}^X\!D_u T = \left(\nabla_u V - F\mathcal{W}(u)\right) - g\left(\mathcal{W}(V) + \operatorname{grad} F, u\right)\nu = 0\,.$$

Thus T is parallel. Now set $\widetilde{X}(x,t) := X(\phi(x,t),t)$, where ϕ is the solution of $\frac{d\phi^i}{dt} = V^i$ with initial condition $\phi(x,0) = x$. Then

$$\frac{\partial \widetilde{X}}{\partial t} = \frac{\partial X}{\partial x^i}\frac{d\phi^i}{dt} + \frac{\partial X}{\partial t} = T\,.$$

□

This completes the proof of Corollary 1.2. □

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