THE OPTIMAL INTERIOR BALL ESTIMATE FOR A k-CONVEX MEAN CURVATURE FLOW

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Abstract. In this note, we prove that at a singularity of an $(m+1)$-convex mean curvature flow, Andrews' non-collapsing ratio [An12] improves as much as is allowed by the example of the shrinking cylinder $\mathbb{R}^m \times S^{n-m}$. More precisely, we show that for any $\varepsilon > 0$ we have $\overline{k} \leq (1+\varepsilon)\frac{1}{n-m}H$ wherever the mean curvature H is sufficiently large, where $\overline{k}$ is the interior ball curvature. When $(m+1) < n$, this estimate improves the inscribed radius estimate of Brendle [Br], which was subsequently proved much more directly by Haslhofer-Kleiner in [HKb] using the powerful new local blow-up method they developed in [HKa]. Our estimate is also based on their local blow-up method, but we do not require the structure theorem for ancient flows, instead making use of the gradient term which appears in the evolution equation of the two-point function which defines the interior and exterior ball curvatures. We also obtain an optimal exterior ball estimate for flows of convex hypersurfaces.

1. Introduction

Recently, Haslhofer-Kleiner have developed a powerful new approach to the study of mean convex mean curvature flow [HKa]. In particular, making use of Andrews' non-collapsing estimate [An12], they have obtained local curvature estimates which allow them to extract a blow-up limit from a sequence whose curvature is normalised only at a single point. This observation is invaluable for many applications of the strong maximum principle which were previously impossible. The power of their technique is vividly illustrated by their short proofs of the crucial curvature estimates for mean convex mean curvature flow, and, more recently, their short proof of Brendle's inscribed radius estimate [HKa, Br].

Let $X : M \times [0,T) \to \mathbb{R}^{n+1}$ be an embedded solution of the mean curvature flow, and consider the function which at each point gives the boundary curvature of the largest ball which is enclosed by the solution, and touches it at that point. We call this quantity the *interior ball curvature*, and denote it by $\overline{k}$. Similarly, we refer to the function which at each point gives the boundary curvature of the smallest ball, halfspace or ball compliment which encloses the hypersurface and touches it at that point as the *exterior ball curvature*[1], and denote it by $\underline{k}$. We recall [ALM13, Proposition 4] that

$$\overline{k}(x,t) = \sup_{y \in M\setminus\{x\}} k(x,y,t)$$

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[1] With the convention that the boundary curvature of an enclosing ball is positive, and the boundary curvature of an enclosing ball-compliment is negative. Although we have stated the definitions for closed hypersurfaces, they may be easily modified to include non-closed hypersurfaces equipped with a choice of 'outer' unit normal field.

and

$$\underline{k}(x,t) = \inf_{y \in M \setminus \{x\}} k(x,y,t)\,,$$

where

$$k(x,y,t) := \frac{2\left\langle X(x,t) - X(y,t), \nu(x,t)\right\rangle}{||X(x,t) - X(y,t)||^2}\,.$$

We note that $\kappa_{\max} \leq \overline{k}$ and $\kappa_{\min} \geq \underline{k}$, where $\kappa_{\max}$ and $\kappa_{\min}$ are, respectively, the largest and smallest principal curvatures of the solution.

The main assertion of [An12] is that there are constants $k_0 \in \mathbb{R}$ and $K_0 > 0$ such that

$$\overline{k} \leq K_0 H$$

and

$$\underline{k} \geq k_0 H\,.$$

That is, the collapsing ratios $\overline{k}/H$ and $\underline{k}/H$ do not deteriorate under the flow. Brendle was able to prove, using a weak version of Huisken's Stampacchia iteration scheme and the Huisken-Sinestrari convexity estimates [HS99], that in fact these ratios improve under the flow. In terms of the interior ball curvatures, Brendle proved the following two statements:

Theorem 1.1 (Interior ball estimate for mean convex flows [Br, HK*b*])**.** *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a smooth, embedded, closed mean convex solution of the mean curvature flow. Then for every $\varepsilon > 0$ there exists $H_\varepsilon < \infty$ such that*

$$\overline{k} \leq (1+\varepsilon)H$$

wherever $H \geq H_\varepsilon$.

Theorem 1.2 (Exterior ball estimate for mean convex flows [Br, HK*b*])**.** *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a smooth, embedded, closed, mean convex solution of the mean curvature flow. Then for every $\varepsilon > 0$ there exists $H_\varepsilon < \infty$ such that*

$$\underline{k} \geq -\varepsilon H$$

wherever $H \geq H_\varepsilon$.

Haslhofer-Kleiner subsequently produced a very simple proof of these estimates [HK*b*] based on the new methods they develpoed in [HK*a*]. Importantly, their proof does not require the convexity estimates, which then immediately follow from Theorem 1.2.

Our goal in this note is to improve these estimates for $(m+1)$-convex flows. The interior ball estimate is as follows:

Theorem 1.3 (Interior ball estimate for $(m+1)$-convex flows)**.** *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a smooth, embedded, $(m+1)$-convex solution of the mean curvature flow. Then for every $\varepsilon > 0$ there exists $H_\varepsilon < \infty$ such that*

$$\overline{k} \leq (1+\varepsilon)\frac{H}{n-m}$$

wherever $H \geq H_\varepsilon$.

The exterior ball estimate is already optimal unless the flow becomes convex (i.e. $m = 0$). In that case we obtain the following stronger estimate:

Theorem 1.4 (Exterior ball estimate for convex flows)**.** *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a smooth, embedded, convex solution of the mean curvature flow. Then for every $\varepsilon > 0$ there exists $H_\varepsilon < \infty$ such that*

$$\underline{k} \geq (1-\varepsilon)\frac{H}{n}$$

wherever $H \geq H_\varepsilon$.

We note that our proof does not require any a priori estimates for the principal curvatures. We therefore immediately obtain the corresponding cylindrical and pinching estmates for the second fundamental form. For flows of convex hypersurfaces, the interior and exterior ball estimates yield a very direct proof of Huisken's Theorem [Hu84] (cf. [ALb]).

2. Proof of the Theorems

Proof of Theorem 1.3. Suppose the claim were false and let $\varepsilon_0 > 0$ be the infimum over all $\varepsilon > 0$ such that the estimate of the theorem holds. By the interior non-collapsing estimate, $\varepsilon_0 < \infty$. As in [HKa, Theorem 1.10] and [HKb], we may blow the solution up at the singular time such that the maximum value of $\overline{k}/H$ occurs at the origin, where $H = 1$. More explicitly, let $(x_i, t_i) \in M \times [0,T)$ be a sequence satisfying $\overline{k}/H(x_i,t_i) \to \frac{1+\varepsilon_0}{n-m}$ and set $\lambda_i = \overline{k}(x_i,t_i)\frac{n-m}{1+\varepsilon_0}$. Then the blow-up sequence given by

$$X_i(x,t) := \lambda_i \left(X(x, \lambda_i^{-2}t + t_i) - X(x_i,t_i) \right)$$

satisfies $X_i(x_i,0) = 0$, $\overline{k}_i(x_i,0) = \frac{1+\varepsilon_0}{n-m}$ and $H_i(x_i,0) \to 1$. By the Haslhofer-Kleiner global convergence theorem [HKa, Theorem 1.12], after passing to a subsequence, X_i converges smoothly to a convex mean curvature flow $X_\infty : M_\infty \times (-\infty, 0] \to \mathbb{R}^{n+1}$. This limit attains the maximum value of $\overline{k}/H$ at the origin at time 0. But $\overline{k}/H$ is a viscosity subsolution of the equation [ALM13]

$$\partial_t u = \Delta u + \frac{2}{H}\left\langle \nabla H, \nabla u \right\rangle .$$

Therefore, by the strong maximum principle, $\overline{k}/H$ must be constant on the limit flow. Thus $\overline{k} \equiv K_0 H$ with $K_0 = \frac{1+\varepsilon_0}{n-m}$.

Now consider the set $U := \{x \in M_\infty : \overline{k}(x,0) > \kappa_{\max}(x,0)\}$. Then for any $x_0 \in U$ there is a point $y_0 \in M_\infty \setminus \{x_0\}$ such that $\overline{k}(x_0,0) = k(x_0,y_0,0)$. Since $k \leq \overline{k} \equiv K_0 H$, we have $k(x,y,t) - K_0 H(x,t) \leq 0$, with equality at $(x_0,y_0,0)$. Thus, computing as in [ALb] we have at $(x_0,y_0,0)$

$$0 \geq (\partial_t - \Delta^{M\times M})(K_0 H - k) \geq \sum_{i=1}^{n} \frac{(\partial_{x^i} k)^2}{k - \kappa_i^x} \geq 0$$

in local orthonormal coordinates on $M \times M$ about (x_0,y_0) which diagonalise the Weingarten map at x_0. It follows that $0 = \nabla\overline{k} = K_0 \nabla H$ at $(x_0,0)$. Since K_0 is positive and $x_0 \in U$ was arbitrary, we find $\nabla H \equiv 0$ on U. Since U is open and has constant mean curvature, it must be a part of a cylinder $\mathbb{R}^k \times S^{n-k}$ (with $k \leq m$ since the limit is $(m+1)$-convex). Since a complete cylinder $\mathbb{R}^k \times S^{n-k}$ satisfies $\overline{k} \equiv \frac{1}{n-k}H < \frac{1+\varepsilon_0}{n-m}H$, we must have $U \subsetneq M_\infty$. Thus either U is empty or has a non-empty boundary in M_∞. Suppose that U is non-empty, so that there is a point $x_0 \in \partial U$. By continuity, x_0 is also a cylindrical point, so that $\kappa_{\max}(x_0,0) = \overline{k}(x_0,0) = \frac{1+\varepsilon_0}{n-m}H(x_0,0) = \frac{1+\varepsilon_0}{n-m}(n-k)\kappa_{\max}(x_0,0) > \kappa_{\max}(x_0,0)$. This is a contradiction. Thus U is empty, in which case $\kappa_{\max} \equiv \overline{k}$. But in this case, Brendle's trick [Br13, Proposition 8] implies $\nabla\kappa_{\max} \equiv 0$, so that again $\nabla H \equiv 0$, which leads to a contradiction just as above. We have now exhausted all possibilities and are therefore forced to conclude that $\varepsilon_0 = 0$. □

For the exterior ball estimate we perform a similar blow-up and use the exterior non-collapsing estimate to deduce that the limit is compact (and therefore cannot be a cylinder). The rest of the proof is similar.

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