Introductory Lecture 1

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Example

$x(t)$ Number of cancer cells at time t
(exponential growth) **State**

$u(t)$ Drug concentration **Control**

$$\frac{dx}{dt} = \alpha x(t) - u(t)$$

$x(0) = x_0$ known initial data

$$\text{minimize} \left\{ x(T) + \int_0^T u^2(t)\, dt \right\}$$

where the first term represents number of cancer cells and the second term represents harmful effects of drug on body.

Optimal Control

Adjust controls in a system to achieve a goal

System:

- Ordinary differential equations
- Partial differential equations
- Discrete equations
- Stochastic differential equations
- Integro-difference equations

Control

- Controllability
  (Use controls to steer system from one position to another)
- Observability
  (deduce system information from control input and observation output)
- Stabilization
  (implement controls to force stability)

Notation

$$x' \qquad \frac{dx}{dt}$$

$$f_x \qquad \frac{\partial f}{\partial x}$$

$$f = x^2 + xu$$

$$f_x = 2x + u$$

$$x' = kx$$ exponential growth

$$x(t) = x_0 e^{kt}$$

time is underlying variable.

Deterministic Optimal Control

Control of Ordinary Differential Equations (DE)

$u(t)$ control

$x(t)$ state

State function satisfies DE

Control affects DE

$$x'(t) = g(t, x(t), u(t))$$

$u(t) \to x(t)$ Goal (objective functional)

Deterministic Optimal Control- ODEs

Find piecewise continuous control $u(t)$ and associated state variable $x(t)$ to maximize

$$\max \int_0^T f(t, x(t), u(t))\, dt$$

subject to

$$x'(t) = g(t, x(t), u(t))$$

$x(0) = x_0$ and $x(T)$ $free$

Contd.

- Optimal Control $u^*(t)$ achieves the maximum
- Put $u^*(t)$ into state DE and obtain $x^*(t)$
- $x^*(t)$ corresponding optimal state
- $u^*(t)$, $x^*(t)$ optimal pair

Necessary and Sufficient Conditions

Necessary Conditions

If $u^*(t)$, $x^*(t)$ are optimal, then the following conditions hold

$\vdots$

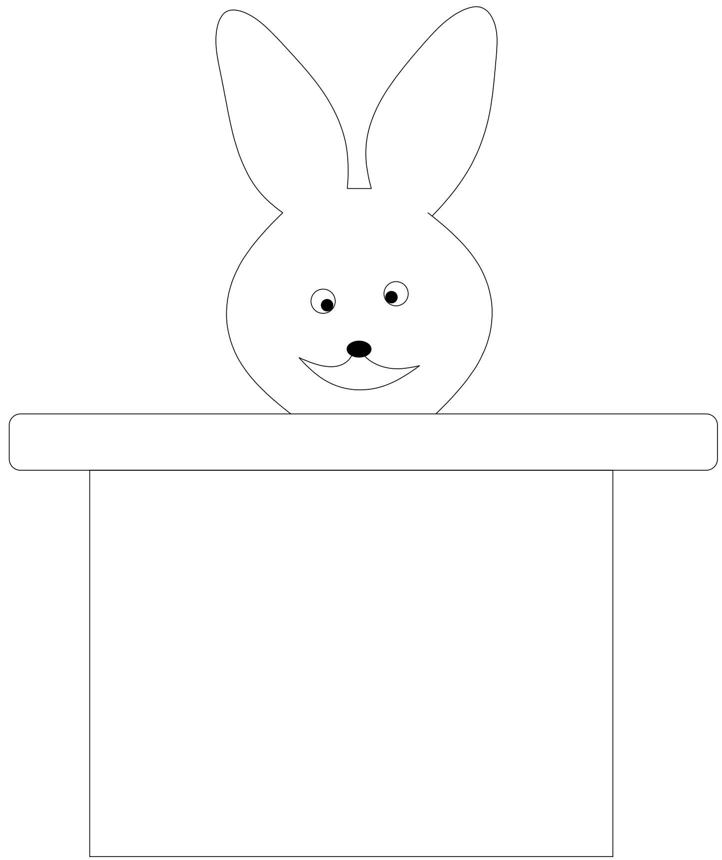
Sufficient Conditions

If $u^*(t)$, $x^*(t)$ and λ (adjoint) satisfy the conditions

$\vdots$

then $u^*(t)$, $x^*(t)$ are optimal.

Adjoint



like Lagrange multipliers to attach DE to objective functional.

Deterministic Optimal Control- ODEs

Find piecewise continuous control $u(t)$ and associated state variable $x(t)$ to maximize

$$\max \int_0^T f(t, x(t), u(t))\, dt$$

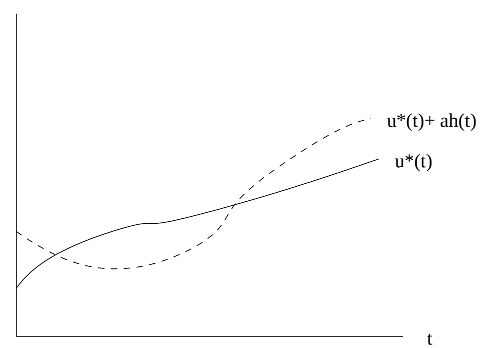
subject to

$$x'(t) = g(t, x(t), u(t))$$

$x(0) = x_0$ and $x(T)$ $free$

Quick Derivation of Necessary Condition

Suppose u^* is an optimal control and x^* corresponding state. $h(t)$ variation function, $a \in \mathbb{R}$.

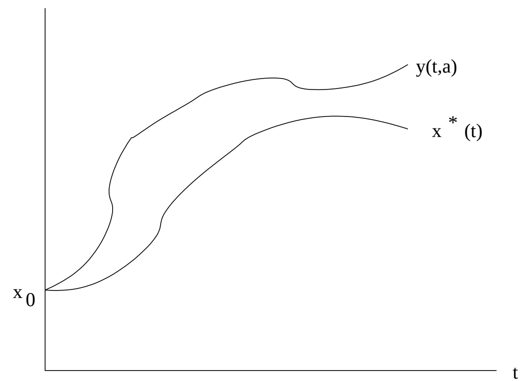


$u^*(t) + ah(t)$ another control.
$y(t,a)$ state corresponding to $u^* + ah$,

$$\frac{dy(t,a)}{dt} = g\left(t, y(t,a), (u^* + ah)(t)\right)$$

Contd.

At $t = 0$, $y(0, a) = x_0$



all trajectories start at same position

$$y(t, 0) = x^*(t) \qquad \text{when } a = 0, \text{ control } u^*$$

$$J(a) = \int_0^T f\left(t, y(t, a), u^*(t) + ah(t)\right) dt$$

Maximum of J w.r.t. a occurs at $a = 0$.

Contd.

$$\left.\frac{dJ(a)}{dt}\right|_{a=0} = 0$$

$$\int_0^T \frac{d}{dt}\left(\lambda(t)y(t,a)\right)\, dt = \lambda(T)y(T,a) - \lambda(0)y(0,a)$$

$$\Rightarrow \int_0^T \frac{d}{dt}\left(\lambda(t)y(t,a)\right)\, dt + \lambda(0)y(0,a) - \lambda(T)y(T,a) = 0.$$

Adding 0 to our $J(a)$ gives

Contd.

$$J(a) = \int_0^T \left[f(t, y(t,a), u^* + ah) + \frac{d}{dt}(\lambda(t) y(t,a)) \right] dt$$
$$+ \lambda(0) y(0,a) - \lambda(T) y(T,a)$$
$$= \int_0^T [f(t, y(t,a), u^* + ah) + \lambda'(t) y(t,a)$$
$$+ \lambda(t) g(t, y, u^* + ah)] \, dt + \lambda(0) x_0 - \lambda(T) y(T,a)$$

here we used product rule and $g = dy/dt$.

Contd.

$$\frac{dJ}{da} = \int_0^T \left[f_x \frac{\partial y}{\partial a} + f_u \frac{\partial (u^* + ah)}{\partial a} + \lambda'(t) \frac{\partial y}{\partial a} \right.$$
$$\left. + \lambda(t) \left(g_x \frac{\partial y}{\partial a} + g_u \frac{\partial (u^* + ah)}{\partial a} \right) \right] dt - \lambda(T) \frac{\partial}{\partial a} y(T, a).$$

Arguments of f, g terms are $(t, y(t,a), u^* + ah(t))$.

$$0 = \frac{dJ}{da}(0) = \int_0^T \left[(f_x + \lambda g_x + \lambda') \left. \frac{dy}{da} \right|_{a=0} + (f_u + \lambda g_u)\, h \right] dt$$
$$- \lambda(t) \frac{\partial y}{\partial a}(T, 0).$$

Arguments of f, g terms are $(t, x^*(t), u^*(t))$.

Contd.

Choose $\lambda(t)$ s.t.

$$\lambda'(t) = -\left[f_x(t, x^*, u^*) + \lambda(t) g_x(t, x^*, u^*)\right] \quad \text{adjoint equation}$$

$$\lambda(T) = 0 \quad \text{transversality condition}$$

$$0 = \int_0^T \left(f_u + \lambda g_u\right) h(t)\, dt$$

$h(t)$ arbitrary variation

$$\Rightarrow f_u(t, x^*, u^*) + \lambda(t) g_u(t, x^*, u^*) = 0 \quad \text{for all } 0 \leq t \leq T.$$

Optimality condition.

Using Hamiltonian

Generate these Necessary conditions from Hamiltonian

$$H(t,x,u,\lambda) = f(t,x,u) + \lambda g(t,x,u)$$

integrand $+$ (adjoint) (RHS of DE)

maximize H w.r.t. u at u^*

$$\frac{\partial H}{\partial u} = 0 \Rightarrow f_u + \lambda g_u = 0 \quad \text{optimality eq.}$$

$$\lambda' = -\frac{\partial H}{\partial x} \Rightarrow \lambda' = -\left(f_x + \lambda g_x\right) \quad \text{adjoint eq.}$$

$$\lambda(T) = 0 \quad \text{transversality condition}$$

$$\text{Given} \quad x' = g(t, x, u) \quad \text{DE}$$

$$x(t_0) = x_0 \qquad \text{IC.}$$

Use Hamiltonian to get other conditions

$$\frac{\partial H}{\partial u} = 0$$

$$\lambda' = -\frac{\partial H}{\partial x}$$

$$\lambda(T) = 0.$$

Converted problem of finding control to maximize objective functional subject to DE, IC to using Hamiltonian pointwise.

For maximization

$$\frac{\partial^2 H}{\partial u^2} \leq 0 \quad \text{at } u^* \quad \bigcap H(u) \quad \text{as a function of } u$$

For minimization

$$\frac{\partial^2 H}{\partial u^2} \geq 0 \quad \text{at } u^* \quad \bigcup H(u) \quad \text{as a function of } u$$

Two unknowns u^* and x^*
introduce adjoint λ (like a Lagrange multiplier)

Three unknowns u^*, x^* and λ

H nonlinear w.r.t. u

Eliminate u^* by setting $H_u = 0$
and solve for u^* in terms of x^* and λ

Two unknowns x^* and λ
with 2 ODEs (2 point BVP)
+ 2 boundary conditions.

Pontryagin Maximum Principle

If $u^*(t)$ and $x^*(t)$ are optimal for above problem, then there exists adjoint variable $\lambda(t)$ s.t.

$$H(t, x^*(t), u(t), \lambda(t)) \leq H(t, x^*(t), u^*(t), \lambda(t)),$$

at each time, where Hamiltonian H is defined by

$$H(t, x(t), u(t), \lambda(t)) = f(t, x(t), u(t)) + \lambda g(t, x(t), u(t)).$$

and

$$\lambda'(t) = -\frac{\partial H(t, x(t), u(t), \lambda(t))}{\partial x}$$

$$\lambda(T) = 0 \qquad \text{transversality condition}$$

Hamiltonian

$$H = f(t, x, u) + \lambda(t) f(t, x, u)$$

u^* maximizes H w.r.t. u, H is linear w.r.t. u

$$H = h(t, x, \lambda) u(t) + k(t, x, \lambda)$$

bounded controls, $a \le u(t) \le b$.
Bang-bang control or singular control

Example: $H = 2u + \lambda u + x - \lambda x^2$

$$\frac{\partial H}{\partial u} = 2 + \lambda \stackrel{?}{=} 0 \quad \text{cannot solve for } u$$

H is nonlinear w.r.t. u, set $H_u = 0$ and solve for u^*
optimality equation.

Example 1

$$\min \int_0^1 u^2(t)\,dt$$

$$x' = x + u, \quad x(0) = 1$$

$$H = \text{ integrand } + \lambda \text{ RHS of DE } = u^2 + \lambda(x+u)$$

$$\frac{\partial H}{\partial u} = 2u + \lambda = 0 \Rightarrow u^* = -\frac{\lambda}{2} \quad \text{at } u^*$$

$$\frac{\partial^2 H}{\partial u^2} = 2$$

$$\lambda' = -\frac{\partial H}{\partial x} = -\lambda \quad \lambda(1) = 0$$

$$\lambda = \lambda_0 e^{-t} \to 0 = \lambda_0 e^{-1} \Rightarrow \lambda_0 = 0$$

$$\lambda \equiv 0, u^* \equiv 0, x^* = e^t$$

Example 2

$$\int_1^5 (xu - u^2 - x^2)\, dt$$

$$x' = x + u, \qquad x(1) = 2$$

$$H = xu - u^2 - x^2 + \lambda(x + u)$$

$$\frac{\partial H}{\partial u} = x - 2u + \lambda = 0 \quad \text{at } u^* \Rightarrow u^* = \frac{x + \lambda}{2}$$

$$\lambda' = -\frac{\partial H}{\partial x} = -(u - 2x + \lambda), \qquad \lambda(5) = 0$$

$$\lambda' = -\left(\frac{x + \lambda}{2} - 2x + \lambda\right)$$

$$x' = x + \frac{x + \lambda}{2}$$

Contd.

$$x' = \frac{3}{2}x + \frac{\lambda}{2}$$
$$\lambda' = \frac{3}{2}x - \frac{3}{2}\lambda$$
$$x(1) = 2, \quad \lambda(5) = 0$$

Solve for x^*, λ and then get u^*.

Do numerically with Matlab or by hand

$$\begin{pmatrix} x \\ \lambda \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 2\sqrt{3} - 3 \end{pmatrix} e^{\sqrt{3}t} + C_2 \begin{pmatrix} 1 \\ -2\sqrt{3} - 3 \end{pmatrix} e^{-\sqrt{3}t}$$