1) Let $G \stackrel{\text{def}}{=} \mathbb{Z}/36\mathbb{Z}$ and $H \stackrel{\text{def}}{=} \langle \bar{2} \rangle \cap \langle \bar{3} \rangle$. [As usual, $\bar{a}$ represents the coset $(a + 36\mathbb{Z})$ of $\mathbb{Z}/36\mathbb{Z}$.]

(a) Describe G/H as a set. [In other words, give its elements.]

Solution. We have that:

$$\langle \bar{2} \rangle = \{\bar{0}, \bar{2}, \bar{4}, \ldots, \bar{32}\}$$
$$\langle \bar{3} \rangle = \{\bar{0}, \bar{3}, \bar{6}, \ldots, \bar{33}\}$$

Thus,

$$\langle \bar{2} \rangle \cap \langle \bar{3} \rangle = \langle \bar{6} \rangle = \{\bar{0}, \bar{6}, \bar{12}, \ldots, \bar{30}\},$$

and

$$G/H = \{H, (\bar{1} + H), (\bar{2} + H), (\bar{3} + H), (\bar{4} + H), (\bar{5} + H)\}.$$

□

(b) To what group is G/H isomorphic? [Give a precise description, like S_3, Q_8, C_7, $C_2 \times C_2, \mathbb{Z}$, etc.]

Solution. Since $\mathbb{Z}/36\mathbb{Z}$ is cyclic [$\mathbb{Z}/36\mathbb{Z} = \langle \bar{1} \rangle$], we know that G/H is cyclic. [We proved in class that every quotient group of a cyclic group is also cyclic.] Since $|G/H| = 6$, we have $G/H \cong C_6$.

[Even if you did not remember that, it would be easy to verify: just note that G/H is Abelian (and so not isomorphic to S_3) or that $G/H = \langle \bar{1} + H \rangle$, since $(\bar{1} + H)$ has order 6.]

□

2) Let $G \overset{\text{def}}{=} \mathbb{R}^\times \times \mathbb{R}^\times$ act on $S \overset{\text{def}}{=} \mathbb{R}^2$ by: given $(a, b) \in \mathbb{R}^\times \times \mathbb{R}^\times$, and $(x, y) \in \mathbb{R}^2$,

$$f_{(a,b)}(x, y) \overset{\text{def}}{=} (ax, by).$$

(a) Prove that this indeed defines a group action.

Proof. (i) The identity of $\mathbb{R}^\times \times \mathbb{R}^\times$ is $(1, 1)$ and

$$f_{(1,1)}(x, y) = (x, y).$$

Hence $f_{(1,1)}$ is the identity function.

(ii) Let $(a, b), (c, d) \in \mathbb{R}^\times \times \mathbb{R}^\times$. Then

$$\begin{aligned} f_{(a,b)} \circ f_{(c,d)}(x, y) &= f_{(a,b)}(f_{(c,d)}(x, y)) = f_{(a,b)}(cx, dy) \\ &= (acx, bdy) = f_{(ac,bd)}(x, y) \\ &= f_{(a,b)(c,d)}(x, y). \end{aligned}$$

So, $f_{(a,b)} \circ f_{(c,d)} = f_{(a,b)(c,d)}$.

□

(b) Describe the orbits of $(1, -3)$ and $(-\pi, 0)$ *geometrically*. [Like, "the circle of radius 3 and center at the origin", or "the vertical line passing though -2", or "the line $x = y$ minus the point $(1, 1)$", etc.]

Solution. We have:

$$\begin{aligned} O_{(1,-3)} &= \{f_{(a,b)}(1, -3) \ : \ (a, b) \in \mathbb{R}^\times \times \mathbb{R}^\times\} \\ &= \{(a, -3b) \ : \ (a, b)\} \\ &= \{(x, y) \in \mathbb{R}^2 \ : \ x, y \neq 0\}. \end{aligned}$$

To see the last equality notice that: for all $(a, b) \in \mathbb{R}^\times \times \mathbb{R}^\times$, $a, -3b \neq 0$ [and hence we have "$\subseteq$"], and for all (x, y) with $x, y \neq 0$, we have $f_{(x,-y/3)}(1, -3) = (x, y)$ [and hence we have "$\supseteq$"]. Thus, $O_{(1,-3)}$ is *the plane minus the x and y-axes.*

Also:

$$\begin{aligned} O_{(-\pi,0)} &= \{f_{(a,b)}(-\pi, 0) \ : \ (a, b) \in \mathbb{R}^\times \times \mathbb{R}^\times\} \\ &= \{(-a\pi, 0 \cdot b) \ : \ (a, b)\} \\ &= \{(x, 0) \in \mathbb{R}^2 \ : \ x \in \mathbb{R}^\times\}. \end{aligned}$$

To see the last equality notice that: for all $(a, b) \in \mathbb{R}^\times \times \mathbb{R}^\times$, $-a\pi \neq 0$ [and hence we have "$\subseteq$"], and for all $(x, 0)$ with $x \neq 0$, we have $f_{(-x/\pi,1)}(-\pi, 0) = (x, 0)$ [and hence we have "$\supseteq$"]. Thus, $O_{(-\pi,0)}$ is *the x-axis minus the origin.*

□

(c) Describe the stabilizers of $(1, -3)$ and $(-\pi, 0)$.

Solution. We have:

$$\begin{aligned}
(\mathbb{R}^\times \times \mathbb{R}^\times)_{(1,-3)} &= \{(a, b) \in \mathbb{R}^\times \times \mathbb{R}^\times) \;:\; f_{(a,b)}(1, -3) = (1, -3)\} \\
&= \{(a, b) \in \mathbb{R}^\times \times \mathbb{R}^\times) \;:\; (a, -3b) = (1, -3)\} \\
&= \{(1, 1)\}
\end{aligned}$$

and

$$\begin{aligned}
(\mathbb{R}^\times \times \mathbb{R}^\times)_{(-\pi,0)} &= \{(a, b) \in \mathbb{R}^\times \times \mathbb{R}^\times) \;:\; f_{(a,b)}(-\pi, 0) = (-\pi, 0)\} \\
&= \{(a, b) \in \mathbb{R}^\times \times \mathbb{R}^\times) \;:\; (-a\pi, 0) = (-\pi, 0)\} \\
&= \{(1, b) \;:\; b \in \mathbb{R}^\times\} \\
&= \{1\} \times \mathbb{R}^\times.
\end{aligned}$$

□

3) Prove the following:

(a) Let G be a *finite* group. Prove that for all $a \in G$, we have $a^{|G|} = 1_G$. [**Note:** You **cannot** use item (b) in this proof!]

Proof. Let n be the order of a. Hence, $|\langle a \rangle| = n$, and $a^n = 1_G$. By Lagrange's Theorem, n divides $|G|$ [since $\langle a \rangle < G$]. Thus $|G| = n \cdot k$, for some integer k. So,

$$a^{|G|} = a^{nk} = (a^n)^k = 1_G^k = 1_G.$$

□

(b) Let $H \triangleleft G$ with $[G : H] = n$. Prove that for all $a \in G$, we have $a^n \in H$. [**Note:** You **can** use item (a) in this proof, even if you didn't do it.]

Proof. Let $a \in G$. Then, since $H \triangleleft G$, we have a quotient group G/H and $|G/H| = [G : H] = n$. Thus, by item (a), $(aH)^{|G/H|} = (aH)^n = H$ [since $H = 1H$ is the unit of G/H]. On the other hand $(aH)^n = a^n H$. Thus $a^n H = H$, and therefore, $a^n \in H$.

□

4) Let G be an **Abelian** group and

$$\Delta \stackrel{\text{def}}{=} \{(g,g) \ : \ g \in G\}.$$

Prove that $\Delta \triangleleft G \times G$ and $(G \times G)/\Delta \cong G$.

Proof. [We will use the *First Isomorphism Theorem.*] Let

$$\phi : G \times G \to G$$

defined by $\phi(g,h) = gh^{-1}$. Then,

$$\begin{aligned}
\phi((g_1,h_1)(g_2,h_2)) &= \phi(g_1g_2, h_1h_2) && \text{[prod. in } G \times G\text{]} \\
&= (g_1g_2)(h_1h_2)^{-1} && \text{[defn. of } \phi\text{]} \\
&= g_1h_1^{-1}g_2h_2^{-1} && \text{[}G\text{ is Abelian]} \\
&= \phi(g_1,h_1)\phi(g_2,h_2) && \text{[defn. of } \phi\text{],}
\end{aligned}$$

and so ϕ is a homomorphism.

Now,

$$\begin{aligned}
(g,h) \in \ker \phi &\Leftrightarrow gh^{-1} = 1_G \\
&\Leftrightarrow g = h \\
&\Leftrightarrow (g,h) \in \Delta.
\end{aligned}$$

So, $\Delta = \ker \phi$. Therefore $\Delta \triangleleft G \times G$ [the kernel of a homomorphism is always a normal subgroup].

Moreover ϕ is onto, since given $g \in G$, $\phi(g, 1_G) = g$. Thus, by the First Isomorphism Theorem, $(G \times G)/\Delta \cong G$.

□