1) Give the conjugacy classes and the class equation for Q_8. [**Hint:** Let Q_8 act on itself by *conjugation*. Then the conjugacy classes are the distinct orbits, and the class equation is given by the orders of these classes. The class equation is something like: "$8 = 1+1+1+2+3$".]

Solution. Since $Z(Q_8) = \{1, -1\}$, we have $O_1 = \{1\}$ and $O_{-1} = \{-1\}$. [Moreover, these are the only orbits, or conjugacy classes in this case, that have only one element.]

Observe that for all $x, y \in Q_8$, we have

$$\begin{aligned}(-x) \cdot y \cdot (-x)^{-1} &= -1 \cdot x \cdot y \cdot (-1 \cdot x)^{-1} \\ &= -1 \cdot x \cdot y \cdot x^{-1} \cdot (-1)^{-1} \\ &= -1 \cdot x \cdot y \cdot x^{-1} \cdot -1 \\ &= x \cdot y \cdot x^{-1}\end{aligned}$$

[since $-1 \in Z(Q_8)$]. This makes things easier to compute, and one gets:

$$O_i = \{i, -i\}, \qquad O_j = \{j, -j\}, \qquad O_k = \{k, -k\},$$

Hence the class equation is:

$$8 = 1 + 1 + 2 + 2 + 2$$

□

2) Let R be a ring [with identity, as usual]. Prove that $R^\times$, with the operation of multiplication, is a group.

Solution. [Note that we cannot prove it is a *subgroup*, since R is *not* a group with respect to multiplication!]

(0) *Law of composition:* Let $x, y \in R^\times$. Hence, there are $x^{-1}, y^{-1} \in R$ such that $xx^{-1} = x^{-1}x = 1_R$ and $yy^{-1} = y^{-1}y = 1_R$. So,

$$(y^{-1}x^{-1})xy = y^{-1}y = 1_R \qquad xy(y^{-1}x^{-1}) = xx^{-1} = 1_R.$$

Hence, [since xy has a multiplicative inverse in R] $xy \in R^\times$.

(1) *Identity:* We have that $1_R \cdot 1_R = 1_R$, so $1_R \in R^\times$. Since $x \cdot 1_R = 1_R \cdot x$ for all $x \in R$ [from the definition of a *ring*], we have that 1_R is the [multiplicative] identity of $R^\times$.

(2) *Associativity:* Since $R^\times \subset R$ and R is associative with respect to multiplication, then so is $R^\times$.

(3) *Inverses:* Let $x \in R^\times$. By definition, there is $x^{-1} \in R$ [not, a priori, in R^{times}] such that $x^{-1}x = xx^{-1} = 1_R$. But this equation tells us that $x^{-1} \in R^\times$ and is the multiplicative inverse of x.

Hence, $R^\times$ is a group.

$\square$

3) Let R be a ring. An element $a \in R$ is a *zero-divisor* if $a \neq 0_R$ and there exists $b \neq 0_R$ in R such that $a \cdot b = 0_R$. Prove that if R is a *field* [i.e., $1_R \neq 0_R$, and every element but zero has a *multiplicative* inverse], then it has no zero divisors. [Note that, by definition, 0_R is *not* a zero divisor.]

Solution. Assume that R is a field and that we have $a, b \in R - \{0\}$ such that $a \cdot b = 0$. Since $a \neq 0$ [and R is a field], there is a multiplicative inverse a^{-1}. Thus

$$\begin{aligned} a \cdot b = 0 &\Rightarrow a^{-1} \cdot (a \cdot b) = a^{-1} \cdot 0 && \text{[multiply by } a^{-1}\text{]} \\ &\Rightarrow (a^{-1} \cdot a) \cdot b = 0 && \text{[rings are associative, and we proved that } x \cdot 0 = 0 \cdot x = 0\text{]} \\ &\Rightarrow b = 0 \end{aligned}$$

But we assumed that $b \neq 0$, hence we get a contradiction and R as no zero divisors.

□

4) Prove that the dihedral group D_{2n} [for $n \geq 3$] is never simple.

Solution. Remember that

$$D_{2n} = \left\langle x, y \ : \ x^n = 1, y^2 = 1, yx = x^{-1}y \right\rangle .$$

Let $H \stackrel{\text{def}}{=} \langle x \rangle$. Hence, $|H| = n$. So, $[D_{2n} : H] = |D_{2n}| \, / \, |H| = (2n)/n = 2$, and thus $H \triangleleft G$. Since $1 < n < 2n$, H is a proper normal subgroup.

□

5) Let $G \stackrel{\text{def}}{=} \langle x, y, z \ : \ yxyz^{-2} = 1 \rangle$. Prove that $G = \langle y, z \rangle$, i.e., that G can be generated by y and z only.

Solution. We have that

$$yxyz^{-2} = 1,$$

and solving for x [in the group], we obtain

$$x = y^{-1}z^2y^{-1}.$$

Hence $x \in \langle y, z \rangle$. Since, clearly also $y, z \in \langle y, z \rangle$, and x, y and z generate G, we have that $\langle y, z \rangle = G$.

$\square$

6) Prove that if $|G| = 8$ and $|G'| = 25$, then the only
homomorphism $\phi : G \to G'$ is the one that takes every element of G to the identity of G'.

Solution. Since G and G' are finite, we have that $|\operatorname{im} \phi|$ divides both $|G| =$ and $|G'| = 25$. [This is Corollary 2.6.15 in Artin, and is a consequence of the facts that $\operatorname{im} \phi < G'$ and $|G| = |\ker \phi| \cdot |\operatorname{im} \phi|$.] Since the only [positive] common divisor of 8 and 25 is 1, we must have $|\operatorname{im} \phi| = 1$, i.e., there is only one element in the image, i.e., all elements of G are sent to the same element of G'. Since $\phi(1_G) = 1_{G'}$ [since ϕ is a homomorphism], we have that ϕ takes all elements of G to $1_{G'}$.

$\square$