Final (In Class Part)

M551 – Abstract Algebra

December 13th, 2007

1. Let p be a prime and G be a *non-abelian* group of order p^3. Prove that $Z(G)$ [the center of G] has order p and that it is equal to the commutator subgroup G' [also denoted by $[G, G]$].

2. Let p, q, r be three primes such that $p < q < r$ and G be a group with $|G| = pqr$. Prove that G is solvable. [You can use neither Feit-Thompson's nor Burnside's Theorems, which we did not prove in class.]

3. Let R be a DVR with field of fractions F. [You can use any theorem proved in class, but state it clearly.]

(a) Is $\mathbb{Q}[x, y]$ a DVR?

(b) Show that if $a \in F$ and $f \in R[x]$ is *monic* polynomial such that $f(a) = 0$, then $a \in R$. [This says that R is *integrally closed*.]

(c) Show that F is not *algebraically closed*, i.e., that there exists a non-constant polynomial $g \in F[x] - F$ that has no roots in F.

4. Let R be a UFD.

(a) Prove that $R[x_1, x_2, \ldots]$ is also a UFD. [So, this ring is a non-Noetherian UFD.]

(b) Prove that if for all $a, b \in R$, there is $c \in R$ such that $(a, b) = (c)$ [i.e., R is a Bezout domain], then R is a PID. [We are still assuming that R is a UFD!]