1) [10 points] Give examples of functions $f : \mathbb{R} \to \mathbb{R}$ such that:

(a) f is one-to-one, but not onto.

Solution. There are many examples, for instance, $f(x) = \mathrm{e}^x$. We know that it is one-to-one and onto $(0, \infty)$, so it is one-to-one, but not onto all of $\mathbb{R}$.

□

(b) f is onto, but not one-to-one.

Solution. There are many examples, for instance,

$$f(x) = \begin{cases} \ln(x), & \text{if } x > 0, \\ 0, & \text{if } x \leq 0. \end{cases}$$

We know that $\ln(x)$ is onto, as it is the inverse of $\mathrm{e}^x : \mathbb{R} \to (0, \infty)$. But it's domain is not $\mathbb{R}$. We make the domain $\mathbb{R}$ by "attaching" the half-line from $(-\infty, 0]$ at $y = 0$. Then, its not one-to-one, as $f(-1) = f(-2) = 0$.

□

(c) f is neither one-to-one nor onto.

Solution. There are many examples, for instance, $f(x) = x^2$. Not onto, since the image of $f(x)$ is $[0, \infty)$, and not one-to-one, since $f(-1) = f(1)$.

□

2) [10 points] Prove that

$$\sum_{k=1}^{n}\left(k^2 - \frac{k}{3}\right) = \frac{n^2(n+1)}{3},$$

for all integers $n \geq 1$.

Solution. See Example 4.6 on pg. 43 from the textbook.

$\square$

3) [10 points] Show that for all integers $n \geq 1$, we have that 5 divides $4^{2n-1} + 1$.

Solution. [Compare with Problem 9 from pg. 46 from our solutions!]
We prove it by induction on n. For $n = 1$, we have $4^{2\cdot 1-1} + 1 = 5$, which is divisible by 5. Now, assume that $4^{2n-1} + 1$ is divisible by 5. Then,

$$\begin{aligned} 4^{2(n+1)-1} + 1 &= 4^{2n+1} + 1 \\ &= 4^2 \cdot 4^{2n-1} + 1 \\ &= (15 + 1) \cdot 4^{2n-1} + 1 \\ &= 15 \cdot 4^{2n-1} + (4^{2n-1} + 1) \end{aligned}$$

Since 5 divides 15, it clearly divides $15 \cdot 4^{2n-1}$. Also, by the induction hypothesis, 5 divides $4^{2n-1} + 1$. Thus, 5 divides $4^{2(n+1)-1} + 1 = 15 \cdot 4^{2n-1} + (4^{2n-1} + 1)$.

□

4) [15 points] Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} -x, & \text{if } x \in \mathbb{Q}, \\ x, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

So, for instance, $f(1/2) = -1/2$ and $f(\sqrt{2}) = \sqrt{2}$. Is f onto $\mathbb{R}$? Is it one-to-one? If both, find its inverse. [As always, justify your answers!]

Solution. By Theorem 3.24 [on pg. 36 from the textbook], we see that it suffices to show that f is invertible to show that it is also one-to-one and onto.

[Inverse:] We have that f is its own inverse: If $x \in \mathbb{Q}$, then also $-x \in \mathbb{Q}$. Hence $f \circ f(x) = f(f(x)) = f(-x) = -(-x) = x$. If $x \notin \mathbb{Q}$, then $f \circ f(x) = f(f(x)) = f(x) = x$. So, for all $x \in \mathbb{R}$, we have that $f \circ f(x) = x$, and hence $f^{-1}(x) = f(x)$.

[If you want to see how does one show that it is one-to-one and onto directly, here it is:

[Onto:] Let $b \in \mathbb{R}$. [We need $a \in \mathbb{R}$ such that $f(a) = b$.] If $b \notin \mathbb{Q}$, then let $a = b$. Then, $f(a) = a = b$. If $b \in \mathbb{Q}$, then let $a = -b$. Since $b \in \mathbb{Q}$, then $-b = a \in \mathbb{Q}$, and hence $f(a) = -a = -(-b) = b$. So, in either case, there is $a \in \mathbb{R}$ such that $f(a) = b$.

[One-to-one:] Suppose that $f(a) = f(b)$. [We need to prove that $a = b$.]
If $a \notin \mathbb{Q}$, then $f(a) = a = f(b)$. If $b \in \mathbb{Q}$, then $f(b) = -b \in \mathbb{Q}$. But then, $a = f(b) = -b$ cannot hold, as $a \notin Q$ and $-b \in \mathbb{Q}$. So, we must have that if $a \notin \mathbb{Q}$, then $b \notin \mathbb{Q}$. So, we would have $a = f(a) = f(b) = b$.
Now, if $a \in \mathbb{Q}$, then $f(a) = -a = f(b)$. If $b \notin \mathbb{Q}$, then $f(b) = b \notin \mathbb{Q}$. But then, $-a = f(b) = b$ cannot hold, as $-a \in Q$ and $b \notin \mathbb{Q}$. So, we must have that if $a \in \mathbb{Q}$, then $b \in \mathbb{Q}$. So, we would have $a = -f(a) = -f(b) = b$.]

$\square$

5) [15 points] Prove that for all integers $n \geq 4$, we have:

(a) $2n + 1 < 2^n$

Solution. We prove it by induction on n. For $n = 4$, we have $2 \cdot 4 + 1 = 9 < 16 = 2^4$. Now, assume that $2n + 1 < 2^n$. Then,

$$\begin{aligned} 2(n+1) + 1 &= 2n + 1 + 2 \\ &< 2^n + 2 && \text{[by the IH]} \\ &< 2^n + 2^n && \text{[as } n \geq 4,\ 2^n > 2\text{]} \\ &= 2^{n+1}. \end{aligned}$$

□

(b) $n^2 \leq 2^n$

Solution. We prove it by induction on n. For $n = 4$, we have $4^2 = 16 = 2^4$. Now, assume that $n^2 \leq 2^n$. Then,

$$\begin{aligned} (n+1)^2 &= n^2 + 2n + 1 \\ &\leq 2^n + 2n + 1 && \text{[by the IH]} \\ &\leq 2^n + 2^n && \text{[by part (a)]} \\ &= 2^{n+1}. \end{aligned}$$

□

6) [20 points] Let $f : X \to Y$ be a function, and $A, B \subseteq X$. [You cannot quote previous work on this question, as we've done all these questions already.]

(a) Prove that $f(A \setminus B) \supseteq f(A) \setminus f(B)$.

Solution. [Done in class.] Let $y \in f(A) \setminus f(B)$. Hence, $y \in f(A)$, but $y \notin f(B)$. Since $y \in f(A)$, there exists $x \in A$ such that $f(x) = y$. If $x \in B$, then $f(x) \in f(B)$, and since $y = f(x)$, we would have $y \in f(B)$, which is a contradiction. So, x cannot be in B. Hence, [since $x \in A$], $x \in A \setminus B$. Since $y = f(x)$, we have that $y \in f(A \setminus B)$.

$\square$

(b) Disprove that $f(A \setminus B) \subseteq f(A) \setminus f(B)$ [in general].

Solution. [Done in class.] Let $f(x) = x^2$, $A = [-1, 1]$, $B = (0, 1]$. Thus, $A \setminus B = [-1, 0]$. Using the graph we see that $f(A) = [0, 1] = f(A \setminus B)$, and $f(B) = (0, 1]$. Thus, $f(A \setminus B) = [0, 1] \neq \{0\} = f(A) \setminus f(B)$.

$\square$

(c) Prove that if f is one-to-one, then we do have $f(A \setminus B) \subseteq f(A) \setminus f(B)$.

Solution. See our solutions for Problem 9(b) on pg. 38.

$\square$

7) [20 points] Let $f : X \to Y$ be a function and $A \subseteq Y$. Show that if f is onto, then $f(f^{-1}(A)) = A$. Show that this is not necessarily true if f is not onto. [Again, this was done before, so you cannot just quote the result.]

Solution. For $f(f^{-1}(A)) = A$, note that since f is onto, then $A \subseteq f(X) = Y$. Then, the proof is the proof of Theorem 3.15 on pg. 33. [As I said, you'd have to repeat that proof here, not just quote it.]

[Done in class.] To show that it fails if f is not onto, take $f(x) = x^2$, and $A = \mathbb{R}$. Then, $f^{-1}(\mathbb{R}) = \mathbb{R}$ and $f(\mathbb{R}) = [0, \infty)$. Thus, $f(f^{-1}(\mathbb{R})) = f(\mathbb{R}) = [0, \infty) \subsetneqq \mathbb{R}$.

□