1) [16 points] Let $a, b \in \mathbb{Z}$. Prove that $(a, b) = (a, a + b)$.

Proof. Suffices to prove that the set of common divisors of a and b is the same as the set of common divisors of a and $a + b$ [as then the maximum of those, i.e., the GCD, must be the same.]
Suppose that $d \mid a, b$. Then, $a = a_1 d$ and $b = b_1 d$, with $a_1, b_1 \in \mathbb{Z}$, and hence $a+b = (a_1+b_1)d$, and so $d \mid a, (a + b)$.
Conversely, if $d \mid a, (a+b)$, then $a = a_1 d$ and $(a+b) = c_1 d$. Then, $b = (a+b)-a = (c_1-a_1)d$, and thus $d \mid a, b$. □

2) [17 points] Show that for any positive integer n, the number $n^2 + 3n + 2$ is never prime. [**Hint:** If it is not prime, then it factors!]

Proof. We have that $n^2 + 3n + 2 = (n + 2)(n + 1)$. So, if $n \geq 1$, then $(n + 1), (n + 2) > 1$, and thus $n^2 + 3n + 2$ has two proper factors greater than one, and thus it is not prime. □

3) [17 points] Prove that if n and is composite and not a perfect square, then $(n - 1)! \equiv 0 \pmod{n}$. [**Hint:** If it is composite, then it factors! Then use the fact it is not a perfect square.]

Proof. Since n is composite, we have that $n = ab$, with $1 < a, b < n$. Since n is not a perfect square, we have that $a \neq b$. Now, $(n - 1)! = 1 \cdot 2 \cdot 3 \cdots (n - 2) \cdot (n - 1)$. Since $a, b \in \{1, 2, \ldots, (n - 1)\}$, we have that a and b appear in the factors of $(n - 1)!$ and since they are different, each one of them appears. [If, say $1 < a < b < (n - 1)$, we would have $(n-1)! = 1 \cdot 2 \cdots a \cdots b \cdots (n-2) \cdot (n-1)$.] Hence, $ab = n \mid (n-1)!$ and therefore, $(n-1) \equiv 0 \pmod{n}$.

□