**1)** [20 points] Fill in the blanks [no need to justify]:

(a) $\dim(\mathbb{R}^5) = \boxed{5}$

(b) $\dim(P_7) = \boxed{8}$

(c) $\dim(M_{3\times 2}) = \boxed{6}$

(d) If $\operatorname{rank}(A) = 4$ and the system $A\mathbf{x} = \mathbf{b}$ is consistent, then $\operatorname{rank}([A|\mathbf{b}]) = \boxed{4}$

(e) If $A$ is an $3 \times 4$ matrix for which $T_A$ [the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$] is one-to-one, then $\operatorname{rank}(A) = \boxed{3}$

(f) If $T : \mathbb{R}^7 \to \mathbb{R}^5$ is an onto linear transformation, then $\operatorname{rank}([T]) = \boxed{5}$

(g) If $A$ is a $4 \times 6$ matrix with $\text{nulltiy}(A) = 3$, then:

$$\begin{aligned} \text{dim. of row sp of } A &= \boxed{3} \\ \text{dim. of col. sp of } A &= \boxed{3} \\ \operatorname{rank}(A) &= \boxed{3} \\ \operatorname{rank}(A^{\mathrm{T}}) &= \boxed{3} \\ \text{nulltiy}(A^{\mathrm{T}}) &= \boxed{1} \end{aligned}$$

**2)** [15 points] Let $\mathbf{v}_1 = (1, -1, 0, 3)$ and $\mathbf{v}_2 = (1, 0, -1, 0)$. Is $\mathbf{v} = (-1, -2, 3, 6)$ a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$? If so, write $\mathbf{v}$ as such linear combination. [Show work!]

*Solution.*

$$\left[\begin{array}{rr|r} 1 & 1 & -1 \\ -1 & 0 & 2 \\ 0 & -1 & 3 \\ 3 & 0 & 6 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 1 & -1 \\ 0 & 1 & -3 \\ 0 & -1 & 3 \\ 0 & 3 & 3 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The system has solution $x_1 = 2$, $x_2 = -3$, so it is a linear combination, namely

$$\mathbf{v} = 2 \cdot \mathbf{v}_1 + (-3) \cdot \mathbf{v}_2.$$

□

**3)** [15 points] Let $W = \text{span}\{(1,0,2,1),(0,1,1,1)\}$. Find a basis for the orthogonal complement $W^\perp$.

*Solution.*

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

is already in reduced row echelon form giving solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s - t \\ -s - t \\ s \\ t \end{bmatrix} = s \cdot \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

So, a basis for $W^\perp$ is $\{(-2,-1,1,0),(-1,-1,0,1)\}$. □

**4)** [15 points] Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation for which $T(\mathbf{x})$ is given by:

(i) Rotate $\mathbf{x}$ by 45 degrees [counter-clockwise];

(ii) Reflect the resulting vector about the $y$-axis;

(iii) Project this last vector onto the $x$-axis.

Find $[T]$ and $T(-2,1)$.

*Solution.* There are two ways to find $[T]$. First you can multiply the matrices of the given linear transformations [from right to left]:

$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ 0 & 0 \end{bmatrix}$$

Alternatively, we have $[T] = [T(\mathbf{e}_1)\ \ T(\mathbf{e}_2)]$:

$$\mathbf{e}_1 = (1,0) \xrightarrow{\text{(i)}} (\sqrt{2}/2, \sqrt{2}/2) \xrightarrow{\text{(ii)}} (-\sqrt{2}/2, \sqrt{2}/2) \xrightarrow{\text{(iii)}} (-\sqrt{2}/2, 0)$$

$$\mathbf{e}_2 = (0,1) \xrightarrow{\text{(i)}} (-\sqrt{2}/2, \sqrt{2}/2) \xrightarrow{\text{(ii)}} (\sqrt{2}/2, \sqrt{2}/2) \xrightarrow{\text{(iii)}} (\sqrt{2}/2, 0)$$

So,

$$[T] = \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ 0 & 0 \end{bmatrix}.$$

In either case,

$$T(-2,1) = [T] \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3\sqrt{2}/2 \\ 0 \end{bmatrix}$$

. □

**5)** [15 points] Let $B$ be the standard basis of $P_2$ and $B' = \{1, 1+x, 1+x+x^2\}$. You may assume [without proving] that $B'$ is also a basis of $P_2$.

(a) Find the transition matrix $P_{B \to B'}$.

*Solution.* We have $(1)_B = (1,0,0)$, $(1+x)_B = (1,1,0)$, $(1+x+x^2)_B = (1,1,1)$. Thus,

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right].$$

So,

$$P_{B \to B'} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

□

(b) Find $(1 - 2x + 3x^2)_{B'}$.

*Solution.* We have that $(1 - 2x + 3x^2)_B = (1, -2, 3)$. So,

$$(1 - 2x + 3x^2)_{B'} = P_{B \to B'} \cdot (1 - 2x + 3x^2)_B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 3 \end{bmatrix}.$$

□

**6)** Let $S = \{(1,0,1,2,1),(0,1,1,-1,2),(-1,2,1,-4,3),(2,1,2,-1,1),(0,0,1,4,3)\}$ and let $V = \text{span}(S)$ [the subspace of $\mathbb{R}^5$ spanned by the set $S$]. Given that

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & -1 & 2 \\ -1 & 2 & 1 & -4 & 3 \\ 2 & 1 & 2 & -1 & 1 \\ 0 & 0 & 1 & 4 & 3 \end{bmatrix} \xrightarrow{\text{red. ech. form}} \begin{bmatrix} 1 & 0 & 0 & -2 & -2 \\ 0 & 1 & 0 & -5 & -1 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 2 & 1 \\ 2 & -1 & -4 & -1 & 4 \\ 1 & 2 & 3 & 1 & 3 \end{bmatrix} \xrightarrow{\text{red. ech. form}} \begin{bmatrix} 1 & 0 & -1 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

answer the following. [No need to justify.]

(a) [5 points] Find a basis of $V$ made of vectors in $S$.

*Solution.* Use columns of the matrix where the elements of $S$ are put into *columns* [the on in the bottom] that correspond to the columns of its reduced row echelon form with leading ones. So,

$$B = \{(1,0,1,2,1),(0,1,1,-1,2),(2,1,2,-1,1)\}$$

[first, second and fourth vectors]. □

(b) [5 points] If $B$ is the basis you've found in part (a), express the vectors in $S$ that are not in $B$ as a linear combination of vectors in $B$.

*Solution.* [You can use the reduced row echelon form, which makes it easy!] If $\mathbf{v}_i$ is the $i$-th vector of $S$, we have:

$$\mathbf{v}_3 = -1 \cdot \mathbf{v}_1 + 2 \cdot \mathbf{v}_2, \qquad \mathbf{v}_5 = 2 \cdot \mathbf{v}_1 + 1 \cdot \mathbf{v}_2 + (-1) \cdot \mathbf{v}_4.$$

□

(c) [5 points] Find a second basis $B'$ for $V$ [with $B \neq B'$].

*Solution.* We can now put the vectors of $S$ as *rows* [top matrix] to find a new basis, made of non-zero vectors of its reduced row echelon form. So, we get

$$B' = \{(1,0,0,-2,-2),(0,1,0,-5,-1),(0,0,1,4,3)\}.$$

□

(d) [5 points] Find the coordinates of the first vector of $B$ with respect to $B'$.

*Solution.* The nature of basis $B'$ [the "simplest" basis for span$(S)$], makes it very easy:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -2 & -5 & 4 & 2 \\ 2 & -1 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system has solution $(x_1, x_2, x_3) = (1, 0, 1)$ and so $(\mathbf{v}_1)_{B'} = (1, 0, 1)$.

[Note that the second step to put the matrix of the system in reduced row echelon form is not necessary! We *know* $\mathbf{v}_1$ is a linear combination of elements of $B'$, and hence the system does have a solution, and this solution can be seen straight from the first matrix.] □