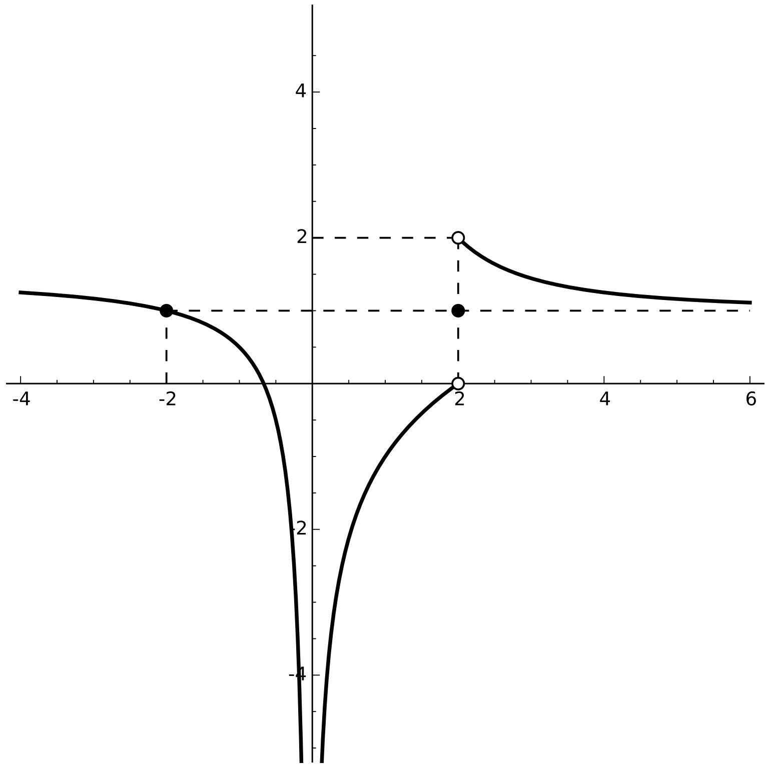
1) [12 points] Consider the graph $y = f(x)$ below:



Find [no need to justify]:

(i) $f(2) = \boxed{1}$

(ii) $\lim_{x \to -2} f(x) = \boxed{1}$

(iii) $\lim_{x \to 0} f(x) = \boxed{-\infty}$

(iv) $\lim_{x \to 2^-} f(x) = \boxed{0}$

(v) $\lim_{x \to 2^+} f(x) = \boxed{2}$

(vi) $\lim_{x \to \infty} f(x) = \boxed{1}$

2) [28 points] Compute the following limits. [Show work in all!]

(a) $\lim_{x\to 1} \dfrac{x^2+x-2}{x^2-2x+1}$

Solution.

$$\begin{aligned}\lim_{x\to 1} \frac{x^2+x-2}{x^2-2x+1} &= \lim_{x\to 1} \frac{(x-1)(x+2)}{(x-1)^2} \\ &= \lim_{x\to 1} \frac{x+2}{x-1}\end{aligned}$$

This gives us "3/0", some some kind of infinite limit. Analyzing the signs we see that $x+2$ is positive on both sides of 1, while $x-1$ is positive on let left and negative on the right of one. This gives,

$$\lim_{x\to 1^+} \frac{x^2+x-2}{x^2-2x+1} = \lim_{x\to 1^+} \frac{x+2}{x-1} = +\infty$$

and

$$\lim_{x\to 1^-} \frac{x^2+x-2}{x^2-2x+1} = \lim_{x\to 1^-} \frac{x+2}{x-1} = -\infty.$$

Thus, $\lim_{x\to 1} \dfrac{x^2+x-2}{x^2-2x+1}$ does not exist and is neither $+\infty$ nor $-\infty$. [It is a *split* infinite limit.] □

(b) $\lim_{x\to 4} \dfrac{x-4}{\sqrt{x}-\sqrt{8-x}}$

Solution.

$$\begin{aligned}\lim_{x\to 4} \frac{x-4}{\sqrt{x}-\sqrt{8-x}} &= \lim_{x\to 4} \frac{x-4}{\sqrt{x}-\sqrt{8-x}} \cdot \frac{\sqrt{x}+\sqrt{8-x}}{\sqrt{x}+\sqrt{8-x}} \\ &= \lim_{x\to 4} \frac{(x-4)(\sqrt{x}+\sqrt{8-x})}{x-(8-x)} \\ &= \lim_{x\to 4} \frac{(x-4)(\sqrt{x}+\sqrt{8-x})}{2(x-4)} \\ &= \lim_{x\to 4} \frac{\sqrt{x}+\sqrt{8-x}}{2} \\ &= 2\end{aligned}$$

□

(c) $\displaystyle\lim_{x\to\infty} \frac{2e^{3x}-1}{1-e^{x}-3e^{3x}}$

Solution.

$$\begin{aligned}
\lim_{x\to\infty} \frac{2e^{3x}-1}{1-e^{x}-3e^{3x}} &= \lim_{x\to\infty} \frac{e^{3x}(2-\frac{1}{e^{3x}})}{e^{3x}(\frac{1}{e^{3x}}-\frac{e^{x}}{e^{3x}}-3)} \\
&= \lim_{x\to\infty} \frac{2-\frac{1}{e^{3x}}}{\frac{1}{e^{3x}}-\frac{1}{e^{2x}}-3} \\
&= -\frac{2}{3}
\end{aligned}$$

□

(d) $\displaystyle\lim_{x\to 1^+} \frac{x+2}{x^2-1}$

Solution. Trying to evaluate, we get "3/0", so it is [again] some kind of infinite limit. On the left of 1 [i.e., for $x>1$ but close to 1] we have that both $x+2$ and x^2-1 are positive, so

$$\lim_{x\to 1^+} \frac{x+2}{x^2-1} = +\infty$$

□

3) [15 points] Give the equation of the line tangent to the graph of $f(x) = \cos(2x)$ at $x = 0$. [You *cannot* use any derivative formula we haven't seen in class yet!]

Solution. We have

$$\begin{aligned}
f'(0) &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \\
&= \lim_{h \to 0} \frac{\cos(2h) - \cos(0)}{h} \\
&= \lim_{h \to 0} \frac{\cos(2h) - 1}{h} \\
&= \lim_{k \to 0} \frac{\cos(k) - 1}{k/2} && [\text{use subst. } k = 2h] \\
&= \lim_{k \to 0} \left(2 \cdot \frac{\cos(k) - 1}{k} \right) \\
&= 2 \cdot 0 = 0.
\end{aligned}$$

Now the equation of the tangent line at $x = c$ is

$$y - f(c) = f'(c)(x - c),$$

so, in this case

$$y - 1 = 0 \cdot (x - 0),$$

or

$$y = 1.$$

□

4) [15 points] Let

$$\lim_{x\to 1} f(x) = 3, \qquad \lim_{x\to 1} g(x) = -2, \qquad \lim_{x\to 1} h(x) = +\infty.$$

Compute the following limits. [If a limit does not exist and is neither $+\infty$ nor $-\infty$, write DNE. You do not need to show work here.]

(a) $\lim_{x\to 1} f(x) - g(x) = \boxed{3 - (-2) = 5}$

(b) $\lim_{x\to 1} g(x) \cdot h(x) = \boxed{-\infty}$

(c) $\lim_{x\to 1} f(x)/h(x) = \boxed{0}$

(d) $\lim_{x\to 1} h(x)/g(x) = \boxed{-\infty}$

(e) $\lim_{x\to 1} \arctan(x - h(x)) = \boxed{-\pi/2}$

5) [15 points] Give a [finite] closed interval in which we have a solution to $3^x = x^2$. [Justify!]

Solution. We use the *Intermediate Value Theorem.*
Let $f(x) = 3^x - x^2$. Then, [by trial an error], we have

$$f(-1) = 1/3 - 1 = -2/3 < 0$$

and

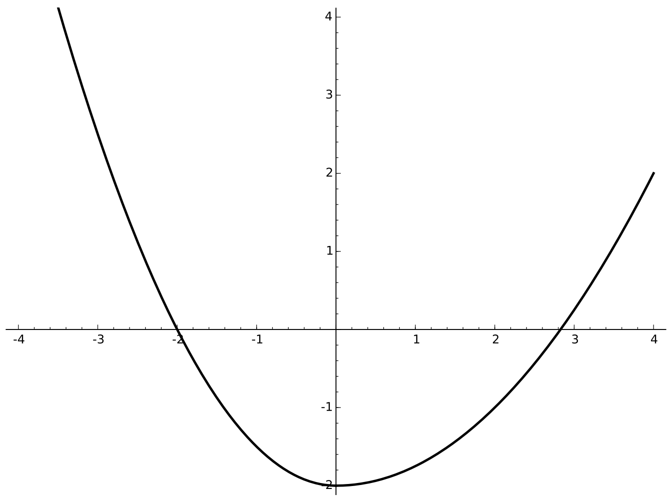
$$f(0) = 1 - 0 = 1 > 0.$$

Since $f(x)$ is continuous for all x [and so, in particular on $[-1, 0]$], we have, by the *Intermediate Value Theorem*, that there is a $c \in [-1, 0]$ such that $f(c) = 0$, i.e., $3^c = c^2$.
In other words, the equation has a solution [namely $x = c$] in the closed interval $[-1, 0]$.

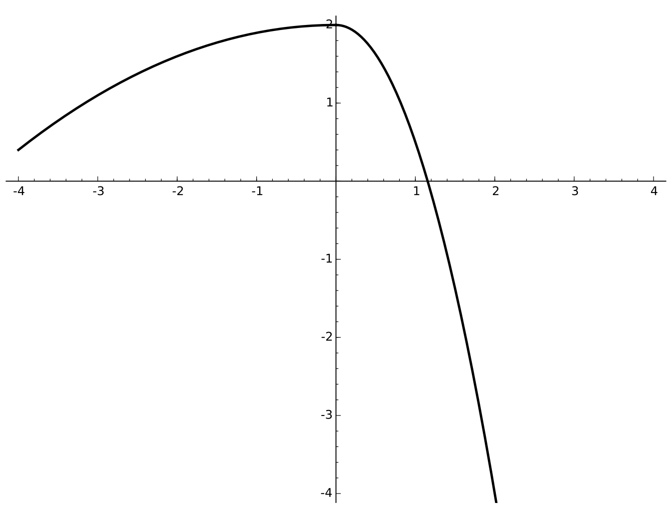
$\square$

6) [15 points] Let $f(x)$ be a function for which $f'(-2) = 2$, $f'(0) = 0$ and $f'(2) = -1$. Mark the option below that could represent the graph of this function. [You do not *have* to justify, but in that case, we *cannot* give partial credit! If you do write an explanation for your choice, we *can*.]

(a)



(b)



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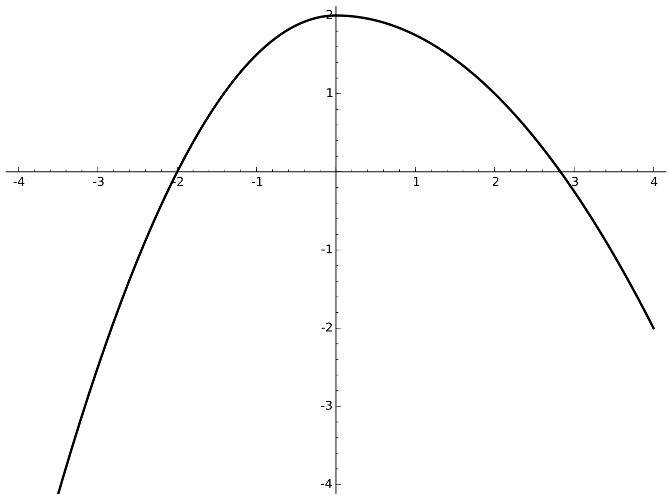
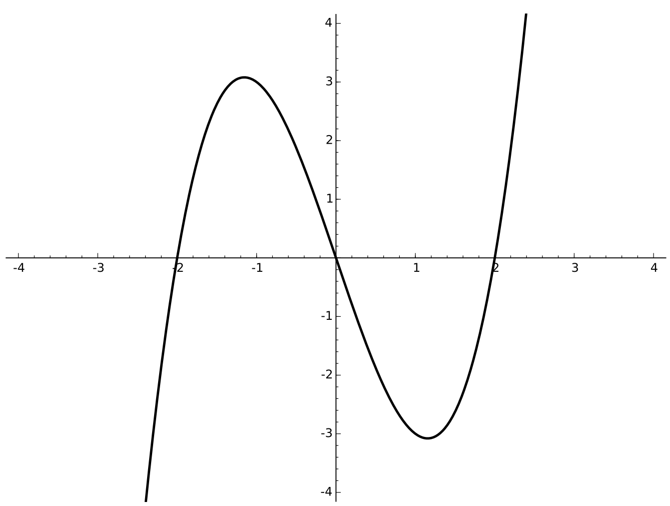
(c)



(d)



(e) None of the above.

Solution. Only (b) and (c) have positive slope [i.e., increasing] at $x = -2$, slope zero [i.e., horizontal] at $x = 0$ and negative slope [i.e., decreasing] at $x = 2$. But (b) has a small [i.e., close to zero] slope at $x = -2$, so different from 2, and a very steep negative slope at $x = 2$, so different from -1. So, (c) is the correct answer. □