**1)** Compute the derivatives of the following functions. [**No need to simplify your answers!**]

(a) [6 points] $f(x) = \sqrt{x} \cdot \ln(\cos(x))$

*Solution.*

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{x}} \cdot \ln(\cos(x)) + \sqrt{x} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(\cos(x))\right) \\ &= \frac{1}{2\sqrt{x}} \cdot \ln(x \cdot \cos(x)) + \sqrt{x}\left[\frac{1}{\cos(x)} \cdot (-\sin(x))\right]. \end{aligned}$$

□

(b) [7 points] $f(x) = \dfrac{x^2}{x \cdot \mathrm{e}^x + 1}$

*Solution.*

$$\begin{aligned} f'(x) &= \frac{2x \cdot (x \cdot \mathrm{e}^x + 1) - x^2 \cdot \frac{\mathrm{d}}{\mathrm{d}x}(x \cdot \mathrm{e}^x + 1)}{(x \cdot \mathrm{e}^x + 1)^2} \\ &= \frac{2x \cdot (x \cdot \mathrm{e}^x + 1) - x^2 \cdot (1 \cdot \mathrm{e}^x + x \cdot \mathrm{e}^x)}{(x \cdot \mathrm{e}^x + 1)^2} \end{aligned}$$

□

(c) [7 points] $f(x) = x^{\arcsin(x)}$. [**Note:** $\arcsin(x)$ is the same as $\sin^{-1}(x)$.]

*Solution.*

$$\begin{aligned} f'(x) &= f(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(f(x))\right) && \text{[logarithmic derivative]} \\ &= x^{\arcsin(x)} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(x^{\arcsin(x)})\right) \\ &= x^{\arcsin(x)} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\arcsin(x) \cdot \ln(x)\right) \\ &= x^{\arcsin(x)} \cdot \left(\frac{1}{\sqrt{1-x^2}} \cdot \ln(x) + \arcsin(x) \cdot \frac{1}{x}\right) \end{aligned}$$

□

**2)** [15 points] Find the equation of the line tangent to the curve given by

$$x^2 + y^2 = xy^3 + 1$$

at the point $(0, -1)$.

*Solution.* Taking derivatives [with respect to $x$] on both sides:

$$2x + 2yy' = 1 \cdot y^3 + x \cdot 3y^2 \cdot y'.$$

Solving for $y'$, we get:

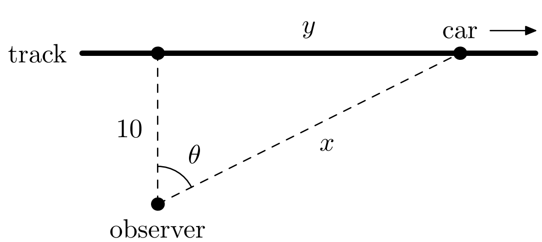
$$y' = \frac{y^3 - 2x}{2y - 3xy^2}.$$

So, at $(0, -1)$ we have $y' = 1/2$. So, the equation of the tangent line is:

$$y - (-1) = \frac{1}{2} \cdot (x - 0) \qquad \text{or} \qquad y = \frac{x}{2} + 1.$$

□

**3)** [15 points] An observer watches a racing car pass by on a straight track 10 meters away. [See picture below.] If the observer is turning his head at a rate of 1 radian per second when his distance to the car [labeled $x$ in the picture below] is 20 meters, how fast is the car moving at that instant?




*Solution.* [We know $\theta'$ and we want $y'$.] We have that $\tan(\theta) = y/10$ [for $\theta$ and $y$ as in the picture]. Taking derivatives [with respect to time], we get

$$\sec^2(\theta) \cdot \theta' = y'/10 \qquad \text{and so} \qquad y' = 10 \cdot \sec^2(\theta) \cdot \theta'.$$

When $x = 20$, we have $\cos(\theta) = 1/2$, and so $\sec^2(\theta) = 4$, and $\theta' = 1$. So, then we have $y' = 10 \cdot 4 \cdot 1 = 40$. So, the speed of the car at that instant is 40 meters per second. □

**4)** [15 points] Use the tangent line approximation to estimate ln(1.1). Also, give the percentage error for the approximation. [**Note:** You can leave computations and log's indicated in the second part, but the first part must be a simple number.]

*Solution.* We have that

$$\ln(1.1) \approx \ln(1) + \frac{\mathrm{d}}{\mathrm{d}x}(\ln(x))\bigg|_{x=1} \cdot (1.1-1) = 0 + \frac{1}{1}\cdot 0.1 = 0.1.$$

The percentage error is

$$\frac{|\ln(1.1) - 0.1|}{|\ln(1.1)|}\cdot 100\%.$$

[You could not compute it in the exam without a calculator, but this is approximately 4.92%.]

□

**5)** [15 points] Find [absolute] maximum and minimum [*both* $x$-coordinate and corresponding value of the function] of $f(x) = -x + 3\sqrt[3]{x}$ in the interval $[-1, 8]$.

*Solution.* First, we observe that $f(x)$ is continuous at the given *closed* interval. So, we can apply the method of Section 4.2.

We have

$$f'(x) = -1 + \frac{1}{\sqrt[3]{x^2}}.$$

Then, $f'(x)$ is not defined at $x = 0$, but $f(x)$ is, so $x = 0$ is a critical point. Also, $f'(x) = 0$ gives us

$$\frac{1}{\sqrt[3]{x^2}} = 1 \quad \Rightarrow \quad \sqrt[3]{x^2} = \frac{1}{1} = 1 \quad \Rightarrow \quad x^2 = 1^3 = 1 \quad \Rightarrow \quad x = \pm 1.$$

So, we have three critical points: 0, 1 and −1. Thus:

| $x$ | $f(x)$ |
|---|---|
| −1 | −2 |
| 0 | 0 |
| 1 | 2 |
| 8 | −2 |

So, the maximum is 2 and occurs at $x = 1$ and the minimum is −2 and occurs at $x = -1, 8$.

□

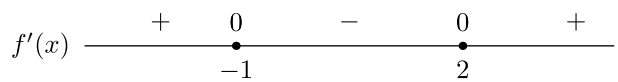
**6)** [20 points] Let $f(x) = 2x^3 - 3x^2 - 12x + 6$. [**Note:** In all items below you can use "DNE" for "does not exist".]

(a) Give the intervals in which $f(x)$ is increasing and the intervals in which it is decreasing.

*Solution.* We have

$$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x+1)(x-2).$$

So, $f'(x) = 0$ at $x = -1, 2$. We then have:




So, $f(x)$ is increasing on $(-\infty, -1)$ and $(2, \infty)$ and decreasing on $(-1, 2)$. □

(b) Give all critical points [$x$-coordinate only] and classify them as local maximum, local minimum or neither.

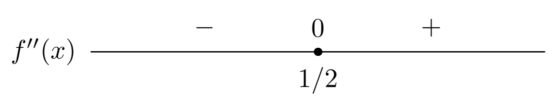
*Solution.* From the work above we see that the critical points occur at $x = -1$ and $x = 2$. Also, we see that we have a local maximum at $x = -1$ and a local minimum at $x = 2$. □

(c) Give all intervals in which the graph of the function $f(x)$ is concave up and all intervals in which it is concave down.

*Solution.* We have

$$f''(x) = 6(2x - 1)$$

So, $f''(x) = 0$ at $x = 1/2$. We then have:




So, $f(x)$ is concave up $(1/2, \infty)$ and concave down on $(-\infty, 1/2)$. □

(d) Give all inflection points [$x$-coordinate only] of $f(x)$.

*Solution.* From the work above we have that $f(x)$ has an inflection point $x = 1/2$. □